INTRODUCTION TO LINEAR ALGEBRA

Sixth Edition

SOLUTIONS TO PROBLEM SETS

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Problem Set 6.1, page 226

1 The eigenvalues of A are $\lambda = 1$ and 0.5 $\left(\text{or } \frac{1}{2}\right)$.

The eigenvalues of A^n are $\lambda = 1$ and $\left(\frac{1}{2}\right)^n$.

The eigenvalues of A^∞ are $\lambda = 1$ and 0.

(a) A row exchange leaves this A with $\lambda = 1$ and -0.5 $\left(\text{or } -\frac{1}{2}\right)$.

(b) Every A has $n - r$ zero eigenvalues ($r =$ rank) : not changed by elimination.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to $\mathbf{0}$ and $\mathbf{6}$. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 $\det(A - \lambda I) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. Then A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $\boldsymbol{x}_1 = (3, -2)$ and $\boldsymbol{x}_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3 (their diagonal entries : triangular matrices). $A + B$ has $\lambda^2 + 8\lambda + 15 = 0$ and $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda^2 - 4\lambda + 1 = 0$ and the quadratic formula gives $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. ***Eigenvalues of AB and BA are equal*** (this is proved at the end of Section 6.2).

7 The eigenvalues of U (on its diagonal) are the *pivots* of A. The eigenvalues of L (on its diagonal) are all 1's. The eigenvalues of A *are not* the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply $A\boldsymbol{x} = \lambda\boldsymbol{x}$ by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$

(b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$

(c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 $\det(A - \lambda I) = \lambda^2 - 1.4\lambda + 0.4$ so A has $\lambda_1 = 1$ and $\lambda_2 = 0.4$ with $\boldsymbol{x}_1 = (1, 2)$ and $\boldsymbol{x}_2 = (1, -1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors as A). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (0.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Proof 1. $A - \lambda_1 I$ is singular so its two columns are in the same direction. Also $(A - \lambda_1 I)\boldsymbol{x}_2 = (\lambda_2 - \lambda_1)\boldsymbol{x}_2$. So $\boldsymbol{x}_2$ is in the column space and both columns must be multiples of $\boldsymbol{x}_2$. Here is also a **second proof**: Columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$ because $M = (A - \lambda_2 I)(A - \lambda_1 I)$ is the zero matrix [this is the *Cayley-Hamilton Theorem* in Problem 6.2.30]. Notice that M has *zero eigenvalues* $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$. So those columns solve $(A - \lambda_2 I)\,\boldsymbol{x} = \mathbf{0}$, they are eigenvectors.

12 The projection matrix P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0)$, $(2, -1, 0)$, $(0, 0, 1)$. Add the first and last vectors: $(1, 2, 1)$ also has $\lambda = 1$. The whole column space of P contains eigenvectors with $\lambda = 1$! *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}$ times $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{u}$ times 1. So $\lambda = 1$.

(b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$.

(c) $\boldsymbol{x}_1 = (-1, 1, 0, 0)$, $\boldsymbol{x}_2 = (-3, 0, 1, 0)$, $\boldsymbol{x}_3 = (-5, 0, 0, 1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$.

14 $\det(Q - \lambda I) = \lambda^2 - 2\lambda\cos\theta + 1 = 0$ when $\lambda = \cos\theta \pm i\sin\theta = e^{i\theta}$ and $e^{-i\theta}$. Check $\lambda_1\lambda_2 = \cos^2\theta + \sin^2\theta = 1$ and $\lambda_1 + \lambda_2 = 2\cos\theta$. Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$ (or $c\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ with $cd \neq 0$).

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$. Those three eigenvalues add to $0 =$ trace of P. The three eigenvalues of the second P are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 Comparing $\lambda^2-(a+d)\lambda+(ad-bc)$ with $(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2$ shows:

$$a+d=\lambda_1+\lambda_2=\text{ trace} \qquad ad-bc=\lambda_1\lambda_2=\text{ determinant}$$

If $\lambda_1=3$ and $\lambda_2=4$ then $\det(A-\lambda I)=\lambda^2-7\lambda+12$.

18 Trace $=9$. Three possibilities are $A=\begin{bmatrix}4&0\\0&5\end{bmatrix}, \begin{bmatrix}10&-1\\30&-1\end{bmatrix}, \begin{bmatrix}4&6\\0&5\end{bmatrix}$.

19 (a) rank $=2$ (b) $\det(B^{\mathrm{T}}B)=0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 $A=\begin{bmatrix}0&1\\\mathbf{-28}&\mathbf{11}\end{bmatrix}$ has trace 11 and determinant 28, so $\lambda=4$ and 7. Moving to a 3 by 3 companion matrix, for eigenvalues $1,2,3$ we want $\det(C-\lambda I)=(1-\lambda)(2-\lambda)(3-\lambda)$. Multiply out to get $-\lambda^3+6\lambda^2-11\lambda+6$. To get those numbers $6,-11,6$ from a companion matrix you just put them into the last row:

$C=\begin{bmatrix}0&1&0\\0&0&1\\\mathbf{6}&\mathbf{-11}&\mathbf{6}\end{bmatrix}$ Notice the trace $6=1+2+3$ and determinant $6=(1)(2)(3)$.

21 $(A-\lambda I)$ has the same determinant as $(A-\lambda I)^{\mathrm{T}}$ because every square matrix has $\det M=\det M^{\mathrm{T}}$. Pick $M=A-\lambda I$.

$\begin{bmatrix}1&0\\1&0\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ have *different eigenvectors* $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$.

22 We can choose $M=\begin{bmatrix}.1&0&0\\.2&.4&0\\.7&.6&1\end{bmatrix}$. Its eigenvalues $\lambda=.1,\ .4,\ 1.0$ are on the diagonal. Clearly M^{T} has rows adding to 1 so M^{T} times the column $\boldsymbol{v}=\begin{bmatrix}1&1&1\end{bmatrix}^{\mathrm{T}}$ equals $\boldsymbol{v}$. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has $\boldsymbol{\lambda=0,\ 1,\ -\frac{1}{2}}$.

23 $\begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}-1&1\\-1&1\end{bmatrix}$. Always A^2 is the zero matrix if $\lambda=0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.30.

24 $\lambda=\mathbf{0,0,6}$ (*notice rank* 1 *and trace* 6). Two eigenvectors of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are perpendicular to $\boldsymbol{v}$ and the third eigenvector is $\boldsymbol{u}$: $\boldsymbol{x}_1=(0,-2,1)$, $\boldsymbol{x}_2=(1,-2,0)$, $\boldsymbol{x}_3=(1,2,1)$.

25 When A and B have the same n λ's and $\boldsymbol{x}$'s, look at any combination $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. Multiply by A and B: $A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **equals** $B\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **for all vectors** $\boldsymbol{v}$. So $A = B$.

26 A has eigenvalues 1 and 2 from block B (with eigenvectors ending in $0, 0$). A also has eigenvalues 5 and 7 from block D because A^{T} has eigenvalues $5, 7$ from block D^{T} (and transposing doesn't change eigenvalues).

27 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

28 The 4 by 4 matrix A of 1's has $\lambda = 0, 0, 0, 4$. Then $B = A - I$ has $\lambda = -1, -1, -1, 3$. And $C = I - A$ has $\lambda = 1, 1, 1, -3$.

29 A is triangular: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; C has rank one: $\lambda(C) = 0, 0, 6$.

30 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when $a + b = c + d$. Thus $\lambda_1 = a + b$.
Then $\lambda_2 = \text{trace} - \lambda_1 = (a + d) - (a + b) = d - b$.

31 If PA exchanges rows 1 and 2 of A, then AP^{T} exchanges columns 1 and 2. In fact

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P^{\mathrm{T}} = P^{-1} \quad \text{and} \quad B = PAP^{\mathrm{T}} = PAP^{-1}.$$

Then B is **similar** to A and they have the same eigenvalues. In this rank 1 and trace 11 example, the eigenvalues of A and B are $0, 0, 11$. From $A - 11I = \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix}$

the eigenvector for $\lambda = 11$ is $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace (we know $A\boldsymbol{u} = 0\boldsymbol{u}$); $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space (we know $A\boldsymbol{v}$ and $A\boldsymbol{w}$ are in the column space).

(b) $A(\boldsymbol{v}/3 + \boldsymbol{w}/5) = 3\boldsymbol{v}/3 + 5\boldsymbol{w}/5 = \boldsymbol{v} + \boldsymbol{w}$. So $\boldsymbol{x} = \boldsymbol{v}/3 + \boldsymbol{w}/5$ is a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Add any $c\boldsymbol{u}$ from the nullspace to find all solutions.

(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space : wrong dimension 3.

33 Always $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})\,\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})$ so $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ with $\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$. (Watch numbers $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$, vectors $\boldsymbol{u}$, matrices $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$!!) If $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$ then $A^2 = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{v}^{\mathrm{T}}$ is the zero matrix and $\lambda^2 = 0, 0$ and $\lambda = 0, 0$ and trace $(A) = 0$. This zero trace also comes from adding the diagonal entries of $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix} \quad \text{has trace } u_1v_1 + u_2v_2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$$

34 The vector $(1, 1, 1, 1)$ is not changed by P. It is the eigenvector for $\lambda = 1$. The other 3 eigenvectors (discussed in detail in Section 6.4) are

$$\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ (-i)^2 \\ (-i)^3 \end{bmatrix}.$$

35 The six 3 by 3 permutation matrices include $P = I$ and three single row exchange matrices P_{12}, P_{13}, P_{23} and two double exchange matrices like $P_{12}P_{13}$. Since $P^{\mathrm{T}}P = I$ gives $(\det P)^2 = 1$, the determinant of P is 1 or -1. The pivots are always 1 (but there may be row exchanges). The trace of P can be 3 (for $P = I$) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and -1 and $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

36 $AB - BA = I$ can happen only for infinite matrices. If $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = -B$ then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\,(AB - BA)\,\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\,(A^{\mathrm{T}}B + B^{\mathrm{T}}A)\,\boldsymbol{x} \le ||A\boldsymbol{x}||\,||B\boldsymbol{x}|| + ||B\boldsymbol{x}||\,||A\boldsymbol{x}||.$$

Therefore $||A\boldsymbol{x}||\,||B\boldsymbol{x}|| \ge \frac{1}{2}||\boldsymbol{x}||^2$ and $(||A\boldsymbol{x}||/||\boldsymbol{x}||)\,(||B\boldsymbol{x}||/||\boldsymbol{x}||) \ge \frac{1}{2}$.

37 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and $\det$. So does every $M^{-1}AM$!

38 (a) Since the columns of A add to 1, one eigenvalue is $\lambda = 1$ and the other is $c - 0.6$ (to give the correct trace $c + 0.4$).

(b) If $c = 1.6$ then both eigenvalues are 1, and all solutions to $(A - I)\,\boldsymbol{x} = \mathbf{0}$ are multiples of $\boldsymbol{x} = (1, -1)$. In this case A has rank 1.

(c) If $c = 0.8$, the eigenvectors for $\lambda = 1$ are multiples of (1, 3). Since all powers A^n also have column sums $= 1$, A^n will approach $\dfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ = rank-1 matrix A^∞ with eigenvalues $1, 0$ and correct eigenvectors. $(1, 3)$ and $(1, -1)$.

Problem Set 6.2, page 242

1 Eigenvectors in $\boldsymbol{X}$ and eigenvalues 1 and 3 in Λ. Then $A = X\Lambda X^{-1}$ is

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. The second matrix has $\lambda = 0$ (rank 1) and

$\lambda = 4$ (trace $= 4$). Then $A = X\Lambda X^{-1}$ is $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

$A^3 = X\Lambda^3 X^{-1}$ and $A^{-1} = X\Lambda^{-1}X^{-1}$.

2 Put the eigenvectors in X and eigenvalues $2, 5$ in Λ. $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 If $A = X\Lambda X^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still X. So $A + 2I = X(\Lambda + 2I)X^{-1} = X\Lambda X^{-1} + X(2I)X^{-1} = A + 2I$.

4 (a) False: We are not given the λ's (b) True (c) True since X has independent columns. (d) False: For this we would need the eigenvectors of X.

5 With $X = I, A = X\Lambda X^{-1} = \Lambda$ is a diagonal matrix. If X is triangular, then X^{-1} is triangular, so $X\Lambda X^{-1}$ is also triangular.

6 The columns of X are nonzero multiples of $(2,1)$ and $(0,1)$: either order. The same eigenvector matrices diagonalize A and A^{-1}.

7 Every matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ has the form

$$A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} /2 = \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}.$$

You could check trace $= \lambda_1 + \lambda_2$ and $\det = \frac{1}{4}\, 4\lambda_1\lambda_2 = \lambda_1\lambda_2$.

8 $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

$$X\Lambda^k X^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The second component is $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 (a) The equations are $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ with $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$. This matrix has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1,1),\ \boldsymbol{x}_2 = (1,-2)$

(b) $A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know if $\lambda = 0$ or not.

(b) True: an eigenvector is missing, which can only happen for a repeated eigenvalue.

(c) True: We know there is only one line of eigenvectors.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (the new A will have two different eigenvalues)

15 $A^k = X\Lambda^k X^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; A_1 is a Markov matrix so $\lambda_{\text{max}} = 1$ and $A_1^k \to A_1^\infty$, A_2 has $\lambda = .6 \pm .3$ so $A_2^k \to 0$.

16 $\begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} = X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $A_1^k = X\Lambda^k X^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 A_2 is $X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$ and $X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Then $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = X\Lambda X^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and

$A^k = X\Lambda^k X^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Multiply those last three matrices to get $A^k = \dfrac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 $B^k = X\Lambda^k X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 $\det A = (\det X)(\det \Lambda)(\det X^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof ($\det =$ product of λ's) works when A is *diagonalizable*. The formula is always true.

21 trace $XY = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace YX. Diagonalizable case: the trace of $X\Lambda X^{-1} =$ trace of $(\Lambda X^{-1})X =$ trace of $\Lambda = \Sigma\lambda_i$. $AB - BA = I$ is impossible since the left side has **trace = 0**.

22 If $A = X\Lambda X^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}\begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix}$. So B has the original λ's from A and the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$ from $2A$.

23 The A's form a subspace since cA and $A_1 + A_2$ all have the same X. When $X = I$ the A's with those eigenvectors give the subspace of **diagonal matrices**. The dimension of that matrix space is 4 since the matrices are 4 by 4.

24 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces $\mathbf{C}(A)$ and $\mathbf{N}(A)$ add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

25 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

26 $R = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\boldsymbol{R^2 = A}$.

$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, the trace (their sum) is not real so $\sqrt{B}$ cannot be real. Note that the square root of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has *two* imaginary eigenvalues $\sqrt{-1} = i$ and $-i$, real trace 0, real square root $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

27 The factorizations of A and B into $X\Lambda X^{-1}$ are the same. So $A = B$.

28 $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from

$X\Lambda_1 X^{-1} X\Lambda_2 X^{-1} = X\boldsymbol{\Lambda_1\Lambda_2}X^{-1} = X\boldsymbol{\Lambda_2\Lambda_1}X^{-1} = X\Lambda_2 X^{-1} X\Lambda_1 X^{-1} = BA.$

29 (a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\det(A - \lambda I) = \lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

30 When $A = X\Lambda X^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = X(\Lambda - \lambda_j I)X^{-1}$ will have 0 in the j,j diagonal entry of $\Lambda - \lambda_j I$. The product $p(A)$ becomes

$$p(A) = (A - \lambda_1 I) \cdots (A - \lambda_n I) = X(\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I)X^{-1}.$$

That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen the following Cayley-Hamilton proof but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.1 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)C^{\mathrm{T}}(\lambda) = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix A for λ. If other matrices B are substituted for λ, does the identity remain true? If $AB \neq BA$, even the order of multiplication seems unclear . . .

31 If $AB = BA$, then B has the same eigenvectors $(1,0)$ and $(0,1)$ as A. So B is also diagonal $b = c = 0$. The nullspace for the following equation is 2-dimensional:

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{b} \\ \boldsymbol{c} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Those 4 equations $0 = 0, -b = 0, c = 0, 0 = 0$ have a 4 by 4 coefficient matrix with rank $= 4 - 2 = 2$.

32 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1. Then $B^4 = I$ and $B^{1024} = I$.
C has $\lambda = (1 \pm \sqrt{3}i)/2$. This λ is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ which leads to $C^{1024} = (-I)^{341}C = -C$.

33 The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and determinant $\lambda_1\lambda_2 = 1$). Their eigenvectors are $(1,-i)$ and $(1,i)$:

$$A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}/2i$$
$$= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \cdots \\ (e^{in\theta} - e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

34 Columns of X times rows of ΛX^{-1} gives a sum of r rank-1 matrices ($r =$ rank of A). Those matrices are $\lambda_1 \boldsymbol{x}_1 \boldsymbol{y}_1^{\mathrm{T}}$ to $\lambda_r \boldsymbol{x}_r \boldsymbol{y}_r^{\mathrm{T}}$.

35 Multiply $\mathsf{ones}(n) * \mathsf{ones}(n) = n * \mathsf{ones}(n)$. Then

$$AA^{-1} = (\mathsf{eye}(n) + \mathsf{ones}(n)) * (\mathsf{eye}(n) + C * \mathsf{ones}(n))$$

$$= \mathsf{eye}(n) + (1 + C + Cn) * \mathsf{ones}(n) = \mathsf{eye}(n) \text{ for } C = \boldsymbol{-1/(n+1)}.$$

36 $B = A_1^{-1}$ leads to $A_2A_1 = B(A_1A_2)B^{-1}$. Then A_2A_1 is similar to A_1A_2 : they have the same eigenvectors (not zero because A_1 and A_2 are invertible).

37 Choose $B = A_1^{-1}$ to show that A_2A_1 is **similar** to A_1A_2. Assuming invertibility (no zero eigenvalues) this shows that A_2A_1 and A_1A_2 have the same eigenvalues.

38 This matrix has column $1 = 2$ (column 2) so $\boldsymbol{x}_1 = (1, -2, 0)$ is an eigenvector with $\lambda_1 = 0$. Also $A(1,1,1) = (1,1,1)$ and $\lambda_2 = 1$. Trace $=$ zero so $\lambda_3 = -1$. Then $1^{2020} = 1$ and $(-1)^{2020} = 1$ and $(0)^{2020} = 0$. So A^{2019} has the same eigenvalues and eigenvectors as A: $A^{2019} = A$ and $A^{2020} = A^2$. **TO COMPLETE FOR 2023**

Problem Set 6.3, page 238

1 (a) ASB stays symmetric like S when $B = A^{\mathrm{T}}$

(b) ASB is similar to S when $B = A^{-1}$

To have both (a) and (b) we need $B = A^{\mathrm{T}} = A^{-1}$ to be an **orthogonal matrix** Q. Then QSQ^{T} is similar to S and also symmetric like S.

2 $\lambda = 0, 4, -2$; unit vectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$. Those are for S. The eigenvalues of T are $\lambda = 0, \sqrt{5}, -\sqrt{5}$ in Λ (trace $= 0$). The eigenvectors of T are $\frac{1}{3}(2, 2, -1)$ and $(1+\sqrt{5}, 1-\sqrt{5}, 2)$ and $(1-\sqrt{5}, 1+\sqrt{5}, 4)$.

3 $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has $\lambda = 0$ and 25 so the columns of Q are the two eigenvectors: $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or we can exchange columns or reverse the signs of any column.

4 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots $1, 1 - b^2$ have the same signs as the λ's

(c) The trace is $\lambda_1 + \lambda_2 = 2$, so S can't have two negative eigenvalues.

5 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C will be 6 by 6 and the triple product $A^{\mathrm{T}}CA$ is 3 by 3.

6 $\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. Then $S = Q\Lambda Q^{\mathrm{T}}$.

If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is *symmetric* then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ requires $\Lambda = 0$. The only *symmetric* A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

7 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

8 $\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix}$ is an orthogonal matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} =$

$\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = QQ^{\mathrm{T}} = I$; also $P_1P_2 = \boldsymbol{x}_1(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2)\boldsymbol{x}_2^{\mathrm{T}} =$ zero matrix.

Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

9 $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ has $\lambda = ib$ and $-ib$. The block matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are also skew-symmetric with $\lambda = ib$ (twice) and $\lambda = -ib$ (twice).

10 M is skew-symmetric and **orthogonal**; every λ is imaginary with $|\lambda| = 1$. So λ's must be i, i, $-i$, $-i$ to have trace zero.

11 $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$. The good property for complex matrices is not $A^{\mathrm{T}} = A$ (symmetric) but $\overline{A}^{\mathrm{T}} = A$ (Hermitian with real eigenvalues and orthogonal eigenvectors).

12 S has $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular in Q
Not perpendicular in X
since $S^{\mathrm{T}} = S$ but $B^{\mathrm{T}} \neq B$

13 $S = \begin{bmatrix} 1 & 3+4i \\ 3-4i & 1 \end{bmatrix}$ is a *Hermitian matrix* $(\overline{S}^{\mathrm{T}} = S)$. Its eigenvalues 6 and -4 are *real*. Here is the proof that λ is always real when $\overline{S}^{\mathrm{T}} = S$:

$S\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $\overline{S}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. Transpose to $\overline{x}^{\mathrm{T}}S = \overline{x}^{\mathrm{T}}\overline{\lambda}$ using $\overline{S}^{\mathrm{T}} = S$.

Then $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x}$ and also $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda}\boldsymbol{x}$. So $\lambda = \overline{\lambda}$ is real.

14 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$
(c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

(e) True. If $\boldsymbol{x}$ is a column of the identity matrix, then the energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is a diagonal entry of S. Since S is positive definite in this problem, each diagonal entry is a positive number $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$.

15 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ is first for A^{T}.

16 A is invertible, orthogonal, permutation, diagonalizable; B is projection, diagonalizable. A allows $QR, X\Lambda X^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $X\Lambda X^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

17 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no X if $b = -1$; singular if $b = 0$.

18 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $S = \pm I$ or

$$\pm S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

19 Eigenvectors $(1, 0)$ and $(\mathbf{1}, \mathbf{1})$ give a $45°$ angle even with A^{T} very close to A.

20 a_{11} is $\begin{bmatrix} q_{11} \dots q_{1n} \end{bmatrix} \begin{bmatrix} \lambda_1 \overline{q}_{11} \dots \lambda_n \overline{q}_{1n} \end{bmatrix}^{\mathrm{T}} \le \lambda_{\max} \left(|q_{11}|^2 + \cdots + |q_{1n}|^2 \right) = \lambda_{\max}$.

21 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ so $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 0$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: because pairs of λ's $= ib, -ib$ multiply to give $+b^2$.

22 Since S is diagonalizable with eigenvalue matrix $\Lambda = 2I$, the matrix S itself has to be $X\Lambda X^{-1} = X(2I)X^{-1} = 2I$. The unsymmetric matrix $[2\ 1\ ;\ 0\ 2]$ also has $\lambda = 2, 2$ but this matrix can't be diagonalized.

23 (a) $S^{\mathrm{T}} = S$ and $S^{\mathrm{T}}S = I$ lead to $S^2 = I$.

(b) The only possible eigenvalues of S are 1 and -1.

(c) $\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ so $\boldsymbol{S} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \Lambda \begin{bmatrix} Q_1^{\mathrm{T}} \\ Q_2^{\mathrm{T}} \end{bmatrix} = \boldsymbol{Q_1 Q_1^{\mathrm{T}} - Q_2 Q_2^{\mathrm{T}}}$ with $Q_1^{\mathrm{T}} Q_2 = 0$.

24 Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$.

(i) The eigenvalues have the *same sign* because $\lambda_1 \lambda_2 = \det = ac - b^2 > 0$.

(ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

25 Only $S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 10^2$.

$\boldsymbol{x}^{\mathrm{T}} S_1 \boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has $\det = 0$.

26 Positive definite for $-3 < b < 3$ $\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}.$

Positive definite for $c > |b|$ $L = \begin{bmatrix}1 & 0\\ -b/c & 1\end{bmatrix}$ $D = \begin{bmatrix}c & 0\\ 0 & c-b^2/c\end{bmatrix}$ $S = LDL^{\mathrm{T}}.$

27 $x^2+4xy+3y^2 = (x+2y)^2-y^2 =$ *difference of squares* is negative at $x = 2$, $y = -1$, where the first square is zero.

28 $S = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ produces $f(x,y) = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \boldsymbol{2xy}$. S has $\lambda = 1$ and $\lambda = -1$. Then S is an *indefinite matrix* and $f(x,y) = 2xy$ has a *saddle point*.

29 $A^{\mathrm{T}}A = \begin{bmatrix}1 & 2\\ 2 & 13\end{bmatrix}$ and $A^{\mathrm{T}}A = \begin{bmatrix}6 & 5\\ 5 & 6\end{bmatrix}$ are positive definite; $A^{\mathrm{T}}A = \begin{bmatrix}2 & 3 & 3\\ 3 & 5 & 4\\ 3 & 4 & 5\end{bmatrix}$ is singular (and positive semidefinite). The first two A's have independent columns. The 2 by 3 A cannot have full column rank 3, with only 2 rows; third $A^{\mathrm{T}}A$ is singular.

30 $S = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $T = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $T\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

31 Corner determinants $|S_1| = 2$, $|S_2| = 6$, $|S_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

32 S is positive definite for $c > 1$; determinants c, c^2-1, and $(c-1)^2(c+2) > 0$. T is *never* positive definite (determinants $d-4$ and $-4d+12$ are never both positive).

33 $S = \begin{bmatrix}1 & 5\\ 5 & 10\end{bmatrix}$ is an example with $a + c > 2b$ but $ac < b^2$, so not positive definite.

34 The eigenvalues of S^{-1} are positive because they are $1/\lambda(S)$. Also the energy is $\boldsymbol{x}^{\mathrm{T}}S^{-1}\boldsymbol{x} = (S^{-1}\boldsymbol{x})^{\mathrm{T}}S(S^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$.

35 $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is zero when $(x_1,x_2,x_3) = (0,1,0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1,-10,0)$ because the second pivot is *negative*.

36 If a_{jj} were smaller than all λ's, $S - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $S - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 35.

37 (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues
(d) $S = -I$ has $\det = +1$ when n is even, but this S is *negative* definite.

38 S is positive definite when $s > 8$; T is positive definite when $t > 5$ by determinants.

39 $A = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $A = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

40 The ellipse $x^2 + xy + y^2 = 1$ comes from $S = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ with $\lambda = \dfrac{1}{2}$ and $\dfrac{3}{2}$.
The axes have half-lengths $\sqrt{2}$ and $\sqrt{2/3}$.

41 $\begin{matrix} S = C^{\mathrm{T}}C \\ \boldsymbol{S} \text{ not } \boldsymbol{A} \end{matrix} = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

42 The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = S$.

43 (a) $\det S = (1)(10)(1) = 10$; (b) $\lambda = 2$ and 5; (c) $\boldsymbol{x}_1 = (\cos\theta \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; (d) The λ's are positive, so S is positive definite.

44 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$) because the determinant $ac - b^2$ is *negative*.

45 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero along the line $2x + 3y = 0$.

46 A product ST of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $ST = M^{-1}K$. (Often we use **eig**(K, M) without actually inverting M.) All eigenvalues λ of ST are positive:

$$ST\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (T\boldsymbol{x})^{\mathrm{T}}ST\boldsymbol{x} = (T\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}T^{\mathrm{T}}ST\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0.$$

47 Put parentheses in $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \boldsymbol{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \boldsymbol{0}$ only happens when $\boldsymbol{x} = \boldsymbol{0}$. Thus $A^{\mathrm{T}}CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}}CA$ in a wide range of applications. I believe positive definiteness of $A^{\mathrm{T}}CA$ is a unifying concept from linear algebra.

48 (a) The eigenvalues of $\lambda_1 I - S$ are $\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$. Those are ≥ 0; $\lambda_1 I - S$ is semidefinite.

(b) Semidefinite matrices have energy $\boldsymbol{x}^{\mathrm{T}}\left(\lambda_1 I - S\right)\boldsymbol{x}_2 \geq 0$. Then $\lambda_1 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$.

(c) Part (b) says $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq \lambda_1$ for all $\boldsymbol{x}$. Equality at the eigenvector with $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$. So the maximum value of $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is λ_1.

49 Energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = a\,(x_1 + x_2 + x_3)^2 + c\,(x_2 - x_3)^2 \geq 0$ if $a \geq 0$ and $c \geq 0$: semidefinite. S has rank ≤ 2 and determinant $= 0$; cannot be positive definite for any a and c.

Problem Set 6.4, page 269

1 $z = 1 - i$ leads to $\overline{z} = 1 + i$ and $r = \sqrt{2}$ and $\dfrac{1}{z} = \dfrac{1+i}{(1-i)\,(1+i)} = \dfrac{1}{2}(1+i)$ and $\theta = -\dfrac{\pi}{4} = -45°$.

2 $\det \begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 - 2 = 0$ gives eigenvalues $\lambda = \mathbf{3}$ and $\mathbf{0}$.

3 If $Q\boldsymbol{x} = \lambda\boldsymbol{x}$ then $||Q\boldsymbol{x}|| = |\lambda|\ ||\boldsymbol{x}||$. Square both sides and use $\overline{Q}^{\mathrm{T}}Q = I$ to find $|\lambda|^2 = 1$. Therefore $|\lambda| = 1$ for unitary matrices Q.

4 $$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi 1/3} & e^{4\pi 1/3} \\ 1 & e^{4\pi 1/3} & e^{8\pi 1/3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2}\left(-1+\sqrt{3}\,i\right) & \frac{1}{2}\left(-1-\sqrt{3}\,i\right) \\ 1 & \frac{1}{2}\left(-1-\sqrt{3}\,i\right) & \frac{1}{2}\left(-1+\sqrt{3}\,i\right) \end{bmatrix}$$

5 $$F_6 = 6 \text{ by } 6 \text{ matrix} = \begin{bmatrix} I & B \\ I & -B \end{bmatrix} \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix} \begin{bmatrix} \text{columns} \\ 0,2,4,1,3,5 \\ \text{of } I \text{ (6 by 6)} \end{bmatrix}$$

The 3 by 3 matrix B is diagonal with entries $1, e^{2\pi i/6}, e^{4\pi i/6}$.

6 $$CD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}$$

$$\begin{array}{rccccc} & & & 1 & 2 & 1 \\ & & & 1 & 1 & 1 \\ \hline & & & 1 & 2 & 1 \\ & & 1 & 2 & 1 & \\ & 1 & 2 & 1 & & \\ \hline \text{convolution } c * d & 1 & 3 & 4 & 3 & 1 \end{array}$$

reduces to $4\ \ 4\ \ 4$ for cyclic convolution $c \circledast d$

7 Convolution Rule $F(\boldsymbol{c} \circledast \boldsymbol{d}) = (F\boldsymbol{c})\ .\!*\ (F\boldsymbol{d})$. This is $F\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = F\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\ .\!*\ F\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

with the 3 by 3 Fourier matrix $F = F_3$: **Multiply components for $.\!*$.**

$$F\begin{bmatrix}4\\4\\4\end{bmatrix}=\begin{bmatrix}1&1&1\\1&e^{2\pi i/3}&e^{4\pi i/3}\\1&e^{4\pi i/3}&e^{8\pi i/3}\end{bmatrix}\begin{bmatrix}4\\4\\4\end{bmatrix}=\begin{bmatrix}12\\0\\0\end{bmatrix}$$

$$F\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}3\\0\\0\end{bmatrix}\qquad F\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}4\\e^{2\pi i/3}\\e^{4\pi i/3}\end{bmatrix}\text{ and }\begin{bmatrix}3\\0\\0\end{bmatrix}.*\begin{bmatrix}4\\e^{2\pi i/3}\\e^{4\pi i/3}\end{bmatrix}\begin{bmatrix}12\\0\\0\end{bmatrix}$$

8 $\cos\theta+i\sin\theta=\left(1-\frac{1}{2}\theta^2+\cdots\right)+i\left(\theta-\frac{\theta^3}{6}+\cdots\right)=1+i\theta+\frac{1}{2}(i\theta)^2+\frac{1}{6}(i\theta)^3+\cdots$

9 $(e^{i\theta})\,(e^{i\theta})=e^{2i\theta}$ is $(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$.

The left side is $\cos^2\theta+2i\cos\theta\sin\theta+i^2\sin^2\theta$.

Matching the right side gives $\cos 2\theta=\cos^2\theta-\sin^2\theta$ and $\sin 2\theta=2\cos\theta\sin\theta$

10 The eigenvalues of a circulant matrix C are $F\boldsymbol{c}$ in equation (10).

If C is invertible then all its eigenvalues must be nonzero.

In that case C^{-1} is also a circulant because its entries (from the formula for C^{-1}) are also constant down each (cyclic) diagonal. There are other proofs too.

11 This problem is looking for a solution !

12 An n by n circulant matrix has $\overline{C}^{\mathrm{T}}=C$ (Hermitian) if its diagonal entries have c_0 real, $\overline{c}_1=c_{n-1},\overline{c}_2=c_{n-2},\ldots$ The circulant has $\overline{C}^{\mathrm{T}}C=I$ (unitary) if $|c_0+c_1x+\cdots+c_{n-1}x^{n-1}|^2=1$.

13 Columns 0 and 2 of the Fourier matrix F_4 in equation (7) add to $(2,0,2,0)$. Columns 1 and 3 add to $(2,0,-2,0)$.

14 $z=w^2=e^{2\pi i/32}$ would be a 32nd root of $1:\ z^{32}=1$.

$z=\sqrt{w}=e^{2\pi i/128}$ would be a 128th root of 1.

15 The 4 eigenvalues $0,2,4,2$ of C come from the eigenvalues $1,i,-1,-i$ of P_4.

$\lambda=2-1-1=0\qquad\lambda=2-i-i^3=2\qquad\lambda=2-(-1)-(-1)^3=4\qquad\lambda=2+i+i^3=2.$

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1 Eigenvalues 4 and 1 with eigenvectors $(1,0)$ and $(1,-1)$ give solutions $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{u}_2 = e^t\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = \begin{bmatrix}5\\-2\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix}$, then use those coefficients 3 and 2: $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^t\begin{bmatrix}1\\-1\end{bmatrix}$.

2 $z(t) = 2e^t$ solves $dz/dt = z$ with $z(0) = 2$. Then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 (a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix}-2 & 3\\2 & -3\end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3,2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1,-1)$. Then the usual 3 steps:

1. Write $u(0) = \begin{bmatrix}4\\1\end{bmatrix}$ as $\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2 =$ combination of eigenvectors
2. The solutions follow those eigenvectors: $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{u}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3,2)$ since $e^{-5t} \to 0$.

4 $d(v+w)/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\\lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

$$\begin{bmatrix}v(0)\\w(0)\end{bmatrix} = \begin{bmatrix}30\\10\end{bmatrix} = 20\begin{bmatrix}1\\1\end{bmatrix} + 10\begin{bmatrix}1\\-1\end{bmatrix} \text{ leads to } \begin{matrix}v(1) = 20 + 10e^{-2} & v(\infty) = 20\\w(1) = 20 - 10e^{-2} & w(\infty) = 20\end{matrix}$$

5 $\dfrac{d}{dt}\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ has $\lambda = 0$ and $\lambda = \boldsymbol{+2}$: $v(t) = \boldsymbol{20 + 10e^{2t}} \to -\infty$ as $t \to \infty$.

6 $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$. In this case the solutions of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ approach zero.

$B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$. In this case all solutions of $d\boldsymbol{v}/dt = B\boldsymbol{v}$ approach zero.

7 A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (c, c)$. Eigenvectors with $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (c, -c)$. For the solution to $d\boldsymbol{u}/dt = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; (somewhat against nature) e^{5t} dominates.

9 (a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ i \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $u(t) = 2e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

10 $\dfrac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$. This correctly gives $y' = y'$ and $y'' = 4y + 5y'$.

$A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

11 The series for e^{At} is $e^{At} = I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} +$ zeros $= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then

$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$. This $y(t) = y(0) + y'(0)t$ solves the equation—the factor t tells us that A had only one eigenvector: not diagonalizable.

12 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$. Substitute $y = te^{3t}$ to show that this gives the needed second solution ($y = e^{3t}$ is the first solution).

13 (a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with eigenvectors $x = \begin{bmatrix} 1 \\ 3i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$. Then $\boldsymbol{u}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} \mathbf{3\cos 3}\boldsymbol{t} \\ \mathbf{-9\sin 3}\boldsymbol{t} \end{bmatrix}$.

14 When A is skew-symmetric, the derivative of $||u(t)||^2$ is *zero*. Then $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ stays at $\|\boldsymbol{u}(0)\|$. So the matrix e^{At} is *orthogonal* when A is skew-symmetric ($A^{\mathrm{T}} = -A$).

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$. For the matrix equation, the particular solution $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix} 1 \\ t \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

16 $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \cdots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots)$. This is exactly Ae^{At}, the derivative we expect from e^{At}.

17 $e^{Bt} = I + Bt$ (short series with $B^2 = 0$) $= \begin{bmatrix} \mathbf{1} & \mathbf{-4}\boldsymbol{t} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. Derivative $= \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = Be^{Bt} = B$ in this example.

18 The solution at time $t + T$ is $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

19 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \frac{1}{6}\boldsymbol{At^3} + \cdots = I + (e^t - 1)A$.

20 $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$ from **21** and $e^B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ from **19**. By direct multiplication

$e^A e^B \neq e^B e^A \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$.

21 The matrix has $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$. *Then all* $A^n = A$. So $e^{At} =$

$I + (t + t^2/2! + \cdots)A = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t-1) \\ 0 & 0 \end{bmatrix}$ as in Problem 19.

22 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

To see $e^{At}\boldsymbol{x}$, write $(I + At + \frac{1}{2}A^2t^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$.

23 Invert $\begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}$ to produce $\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \boldsymbol{U}_n = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix} \boldsymbol{U}_n$.

At $\Delta t = 1$, $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$. Both eigenvalues have $\lambda^6 = 1$ so

$\boldsymbol{A^6 = I}$. Therefore $\boldsymbol{U}_6 = A^6 \boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

24 First A has $\lambda = \pm i$ and $A^4 = I$.

Second A has $\lambda = -1, -1$ and $A^n = (-1)^n \begin{bmatrix} 1-2n & -2n \\ 2n & 2n+1 \end{bmatrix}$ Linear growth.

25 With $a = \Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1} = \dfrac{1}{1+a^2} \begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} \boldsymbol{U}_n$.

That matrix has orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\| = \|\boldsymbol{U}_n\|$

26 For proof 2, square the start of the series to see $(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3)^2 = I + 2A + \frac{1}{2}(2A)^2 + \frac{1}{6}(2A)^3 + \cdots$. The diagonalizing proof is easiest when it works (but it needs a diagonalizable A).