

INTRODUCTION TO LINEAR ALGEBRA

Sixth Edition

SOLUTIONS TO PROBLEM SETS

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Problem Set 5.1, page 203

1 $\det(2A) = 2^4 \det A = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$. If $\det A = 0$ then $\det A/2 = \det(-A) = \det A^2 = 0$; no A^{-1} .

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (except when $n = 1$)

(b) *False*: $\det(4A)$ is $4^n \det A$

(c) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

4 Exchange rows 1 and 3 to show $\det J_3 = -1$. Exchange rows 1 and 4, then rows 2 and 3 to show $\det J_4 = 1$. Two exchanges = even permutation.

5 $|J_5| = 1$ by exchanging row 1 with 5 and row 2 with 4. $|J_6| = -1$, $|J_7| = -1$. Determinants 1, 1, -1 , -1 repeat in cycles of length 4 so the determinant of J_{101} is $+1$.

6 $\det A = 4$, $\det B = 0$, $\det C = 0$.

7 The 6 terms become $a(q+b)z - b(p+a)z + \dots$ (4 more). The approach in the display (using linearity to split up row 2) is better. Result: $\det$ does not change if row 2 is added to row 1.

$$\mathbf{8} \quad \det A^T = \begin{bmatrix} a & p & x \\ b & q & y \\ c & r & z \end{bmatrix} = \begin{matrix} aqz + cpy + brx \\ -ary - bpz - cqx \end{matrix} = \text{same six terms as } \det A$$

Key point: $\det P^T = \det P$ for every permutation, because the number of row exchanges is the same (just done in reverse order). Then P is even when P^T is even.

9 $\det A = 1$ from two row exchanges. $\det B = 2$ (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). $\det C = 0$ and $\det D = 0$ (equal rows).

10 If the entries in every row add to zero, then $(1, 1, \dots, 1)$ is in the nullspace: singular. A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

- 11** If P_1 needs n exchanges to reach I and P_2 needs N exchanges then $P_1 P_2$ reaches I after those $n+N$ exchanges. So $\det(P_1 P_2) = (-1)^{n+N} = (-1)^n (-1)^N = (\det P_1)(\det P_2)$.
- 12** We can pair off even permutations with odd permutations: odd = even followed by exchanging 1 and 2. Number of even permutations $= \frac{1}{2}n! =$ number of odd permutations.
- 13** A singular rank one matrix has determinant = 0. The skew-symmetric A also has $\det A = 0$. A skew-symmetric matrix A of odd order 3: Changing every sign will multiply $\det A$ by $(-1)^3$ but also keep the same $\det A = \det A^T$. So $\det A = 0$.
- 14** When the i, j entry is i times j , row 2 = 2 times row 1 so $\det A = 0$.
When the ij entry is $i + j$, row 3 - row 2 = row 2 - row 1 so A is singular: $\det A = 0$.
- 15** Fill a row (or column) by 4 zeros to guarantee $\det = 0$. Leave only the main diagonal (12 zeros) to allow $\det A \neq 0$.
- 16** The cofactor formula $\det A = a_{11}C_{11} + \dots + a_{1n}C_{1n}$ gives $\det = 0$ if all cofactors are zero. The 2 by 2 matrix of 1's has $\det = 0$ even though no cofactors are zero.
- 17** Two equal rows imply $\det = 0$. Proof for 3×3 if row 1 = row 2. Then $a = p, b = q, c = r$. Then $aqz + brx + cpy - ary - bpz - cqx = abz + bcx + cay - acy - baz - cbx = 0$.
- 18** If A has two equal rows then A^T has two equal columns (say columns j and k). Then the columns are not independent. So $\det A^T = 0$ and $\det A = 0$. Other proofs also reach this conclusion.
- 19** Start from $AC^T = (\det A)I$. Take determinants of both sides :

$$(\det A)(\det C) = (\det A)^n \quad \text{and} \quad \det C = (\det A)^{n-1}$$
 Note : If $\det A = 0$ (singular matrix) then A is the limit of invertible matrices $A_1, A_2, \dots$
 Apply $\det C_i = (\det A_i)^{n-1}$ and take the limit as $i \rightarrow \infty$. (How would you define C for a 1 by 1 matrix ??)
- 20** If you know C and if $\det A = 1$ then you know $A^{-1} = C^T / 1$. Then invert A^{-1} to find A .

Problem Set 5.2, page 209

- 1 If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$, $\det A^n = 2^n$, and $\det A^T = 2$.
- 2 $\det A = -2$, independent columns; $\det B = 0$, dependent columns; $\det C = -1$, independent columns but $\det D = 0$ because its submatrix B has dependent rows (and dependent columns).
- 3 The problem suggests 3 ways to see that $\det A = 0$: All cofactors of row 1 are zero. A has rank ≤ 2 . Each of the 6 terms in $\det A$ is zero. Notice also that column 2 has no pivot.
- 4 (a) $A = \begin{bmatrix} 0.9 & -0.9 \\ 0.9 & 0.9 \end{bmatrix}$ has $\det A = 1.62$ and $\det A^n = (1.62)^n \rightarrow \infty$.
 (b) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has $\det A = 0$ and $\det A^n = 0$ even if $A_{ij} = 2$.
- 5 (a) $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$, $|B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -6$, $|B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4$, $|B_1| = 3$, $|B_2| = -2$, $|B_3| = 1$.
 Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.
- 6 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$.
 That is because B_2 with $(1, 0, 0)$ in column 2 has $\det B_2 = fg - id$.
- 7 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = 0/0$: *undetermined*.
- 8 The determinant is linear in its first column so $|x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 \mathbf{a}_2 \mathbf{a}_3|$ splits into $x_1 |\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3| + x_2 |\mathbf{a}_2 \mathbf{a}_2 \mathbf{a}_3| + x_3 |\mathbf{a}_3 \mathbf{a}_2 \mathbf{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1 |\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3|$ which is $x_1 \det A$.
- 9 If the first column in A is also the right side b then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.
- 10 The pattern $\det = 1, 0, -1, -1, 0, 1$ repeats as in $E_{n+6} = E_n$. So $E_{100} = E_4$ after 16 repeats of length 6. And $E_4 = -1$.

11 If the entries are 1 to 9, the maximum determinant **may be** $412 = \begin{vmatrix} 9 & 3 & 5 \\ 4 & 8 & 1 \\ 2 & 6 & 7 \end{vmatrix}$.

12 **True.** We know that $\det AB = (\det A)(\det B)$. And we know $\det(ABC) = (\det AB)(\det C)$.
Put these together to prove $\det(ABC) = (\det A)(\det B)(\det C)$.

13 The combinations of $\mathbf{i} = (1, 0, 0)$ and $\mathbf{i} + \mathbf{j} = (1, 1, 0)$ fill the **xy plane** in xyz space.

14 (a) Sum = zero vector. (b) Sum = $-2:00$ vector = $8:00$ vector.

(c) $2:00$ is 30° from horizontal $= (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

15 Moving the origin to $6:00$ adds $\mathbf{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\mathbf{j} = (0, 12)$.

16 (a) $\det P = 1$ because columns 2, 1, 4, 3 have two exchanges from 1, 2, 3, 4.

(b) $\det P = -1$ because columns 3, 2, 1, 4 have only one exchange (of 3 and 1).

(c) $\det P = -1$ because columns 1, 2, 4, 3 have one exchange (3 and 4).

17 The sum is $(\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{u} - \mathbf{w}) = \mathbf{zero\ vector}$. Those three sides of a triangle are in the same plane !

18 All vectors in 3D are combinations of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as drawn (not in the same plane). Start by seeing that $c\mathbf{u} + d\mathbf{v}$ fills a plane, then adding all the vectors $e\mathbf{w}$ fills all of $\mathbf{R}^3$. Different answer when $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the same plane.

19 The only 4×4 column orders that start with 3, 2 are 3, 2, 1, 4 and 3, 2, 4, 1 (so 2 terms in $\det A$).

Problem Set 5.3, page 214

1 Edge matrix E = identity matrix I for a unit cube. With edges e_1, e_2, e_3 out from $(0, 0, 0)$, the other 7 corners are $e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3$.

2 Perpendicular e 's give an ordinary box with side lengths $\|e_1\|, \|e_2\|, \|e_3\|$.

3 The largest box with edge lengths 1, 2, 3 is the ordinary 90-degree box with volume 6. This is Hadamard's inequality (see Problem 4).

4 $|\det E| \leq (\|e_1\|)(\|e_2\|) \dots (\|e_n\|)$ is "Hadamard's inequality". One proof starts with $E = \begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} = QR$ from Section 4.4 (Gram-Schmidt producing orthogonal q 's from independent e 's). Each e_j is a combination of orthogonal q_i with $\|q_i\| = 1$:

$$a_j = \sum r_{ij} q_i \quad \|a_j\|^2 = \sum r_{ij}^2 \|q_i\|^2 \geq r_{jj}^2$$

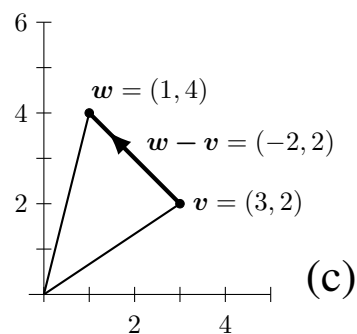
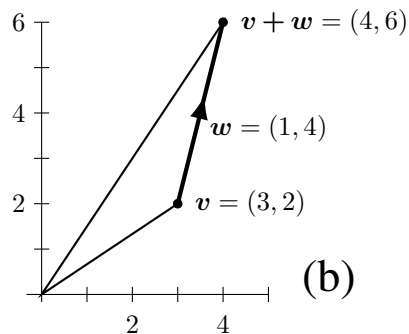
$$|\det E| = |\det Q| |\det R| = (r_{11}) \dots (r_{nn}) \leq \|e_1\| \dots \|e_n\|.$$

Wikipedia proves Hadamard's inequality from "geometric mean" $\leq$ "arithmetic mean".

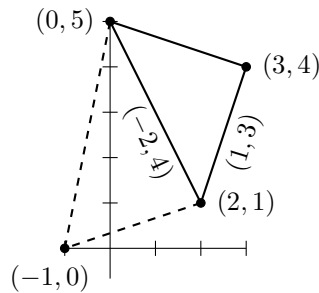
5 (a) The parallelogram area with edges $(3, 2)$ and $(1, 4)$ is the determinant of $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} =$

$$12 - 2 = 10. \quad \text{(b) Triangle area} = \frac{1}{2}(\text{parallelogram area}) = \frac{1}{2} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 5.$$

$$\text{(c) The triangle area is } \frac{1}{2}(\text{parallelogram area}) = \frac{1}{2} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 5.$$



- 6 (a) Two sides are $(3, 4) - (2, 1) = (1, 3)$ and $(0, 5) - (2, 1) = (-2, 4)$. The triangle area is half the parallelogram area $= \frac{1}{2} \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} = 5$.



- (b) has an additional triangle (dashed lines) of area $\frac{1}{2} \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 7$.
- 7 The hypercube in 4-dimensions has perpendicular sides of length $\sqrt{1+1+1+1} = 2$. So its volume is $2^4 = 16$. This must be $|\det H|$.
- 8 An n -dimensional cube has 2^n corners. This is the case $m = 0$ in Wikipedia's formula:
- An n -dimensional cube has $2^{n-m} \frac{n!}{m!(n-m)!}$ m -dimensional sides, edges, corners,...
- For edges ($m = 1$) this rule gives 2^{n-1} times n . 12 edges for a 3D cube ($n = 3$).
- For faces of dimension $m = n - 1$ this rule gives $2n$. Six faces for a 3D cube.
- The cube in $\mathbf{R}^n$ whose edges come from $2I$ has volume $\det(2I) = 2^n$.
- 9 The 3-dimensional "unit pyramid" in $\mathbf{R}^3$ has volume $\frac{1}{3!} = \frac{1}{6}$. I believe the 4-dimensional "unit pyramid" has volume $\frac{1}{4!} = \frac{1}{24}$.