INTRODUCTION TO LINEAR ALGEBRA

Sixth Edition

SOLUTIONS TO PROBLEM SETS

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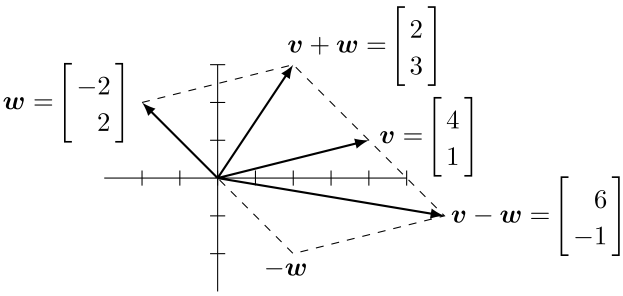
Problem Set 1.1, page 6

1 $c = ma$ and $d = mb$ lead to $ad = amb = bc$. With no zeros, $ad = bc$ is the equation for a 2×2 matrix to have rank 1.

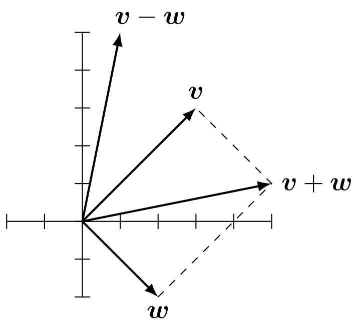
2 The three edges going around the triangle are $\boldsymbol{u} = (5, 0), \boldsymbol{v} = (-5, 12), \boldsymbol{w} = (0, -12)$. Their sum is $u + v + w = (0, 0)$. Their lengths are $||\boldsymbol{u}|| = 5, ||\boldsymbol{v}|| = 13, ||\boldsymbol{w}|| = 12$. This is a $5 - 12 - 13$ right triangle with $5^2 + 12^2 = 25 + 144 = 169 = 13^2$—the best numbers after the $3 - 4 - 5$ right triangle if we don't count $6 - 8 - 10$.

3 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $\boldsymbol{v} + \boldsymbol{w} = (2, 3)$ and $\boldsymbol{v} - \boldsymbol{w} = (6, -1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0, 0)$.



5 This problem gives the diagonals $\boldsymbol{v} + \boldsymbol{w} = (5, 1)$ and $\boldsymbol{v} - \boldsymbol{w} = (1, 5)$ of the parallelogram and asks for the sides $\boldsymbol{v}$ and $\boldsymbol{w}$: The opposite of Problem 4. In this example $\boldsymbol{v} = (3, 3)$ and $\boldsymbol{w} = (2, -2)$. Those come from $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$.

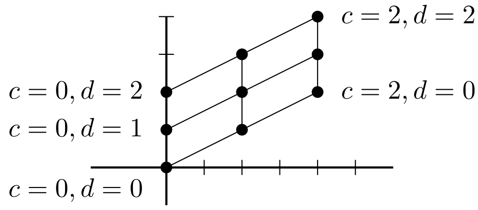


6 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

7 $\boldsymbol{u}+\boldsymbol{v} = (-2, 3, 1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = (0, 0, 0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w} =$ (add first answers) $= (-2, 3, 1)$. The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane because a combination $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ gives $(0, 0, 0)$. Stated another way : $\boldsymbol{u} = -\boldsymbol{v} - \boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

8 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v} = (1, -2, 1)$ and of $\boldsymbol{w} = (0, 1, -1)$ add to zero. $c = 3$ and $d = 9$ give $3\boldsymbol{v} + 9\boldsymbol{w} = (3, 3, -6)$. There is no solution to $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, 6)$ because $3 + 3 + 6$ is not zero.

9 The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = 0, 1, 2$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.



10 The question is whether (a, b, c) is a combination $x_1\boldsymbol{u} + x_2\boldsymbol{v}$. Can we solve

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} ?$$

Certainly x_1 has to be a. Certainly x_2 has to be c. So the middle components give the **requirement** $\boldsymbol{a + c = b}$.

11 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Draw 3 possible parallelograms !

12 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Centers of 6 faces : $\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 1\right)$ & $\left(0, \frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}, \frac{1}{2}\right)$ & $\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, 1, \frac{1}{2}\right)$. 12 edges.

13 The combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$ fill the $\boldsymbol{xy}$ **plane** in xyz space.

14 (a) Sum = zero vector. (b) Sum = −2:00 vector = 8:00 vector.

(c) 2:00 is 30° from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

15 Moving the origin to 6:00 adds $\boldsymbol{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from **0** to $12\boldsymbol{j} = (0, 12)$.

16 First part: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in the same direction.

Second part: Some combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ gives the zero vector but those 3 vectors are not on a line. Then their combinations fill a plane in 3D.

17 The two equations are $c + 3d = 14$ and $2c + d = 8$. The solution is $c = 2$ and $d = 4$.

18 The point $\dfrac{3}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \dfrac{1}{2}\boldsymbol{v} + \dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

19 The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $0 \le c \le 1$ and $0 \le d \le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$ then $c\boldsymbol{v} + d\boldsymbol{w}$ fills the unit square. In a special case like $\boldsymbol{v} = (a, 0)$ and $\boldsymbol{w} = (b, 0)$ these combinations only fill a segment of a line.

With $c \ge 0$ and $d \ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, then the cone is the whole first quadrant $x \ge 0$, $y \ge 0$. *Question*: What if $\boldsymbol{w} = -\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (-1, 0)$ only fill a line.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies halfway between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \ge 0$, $d \ge 0$, $e \ge 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors in 3D are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding all the vectors $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$. Different answer when $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane.

24 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges.

25 Fact: For any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, some combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ is the zero vector (beyond the obvious $c = d = e = 0$). So if there is one combination $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w}$ that produces $\boldsymbol{b}$, there will be many more—just add c, d, e or $2c, 2d, 2e$ to the particular solution C, D, E.

The example has $3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w} = 3(1,3) - 2(2,7) + 1(1,5) = (0,0)$. It also has $-2\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{b} = (0,1)$. Adding gives $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w} = (0,1)$. In this case c, d, e equal $3, -2, 1$ and $C, D, E = -2, 1, 0$.

Could another example have $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that could NOT combine to produce $\boldsymbol{b}$? Yes. The vectors $(1,1), (2,2), (3,3)$ are on a line and no combination produces $\boldsymbol{b}$. We can easily solve $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = 0$ but not $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w} = \boldsymbol{b}$.

26 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0), (0,1,0,0), (0,0,1,0)$, and $(0,0,0,1)$.

27 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rcl} \boldsymbol{2c} \;\; \boldsymbol{-d} \qquad\quad \boldsymbol{= 1} & \quad \text{So } d = 2e & \quad c = 3/4 \\ \boldsymbol{-c + 2d \;\; -e = 0} & \quad \text{then } c = 3e & \quad d = 2/4 \\ \boldsymbol{-d + 2e = 0} & \quad \text{then } 4e = 1 & \quad e = 1/4 \end{array}$$

Problem Set 1.2, page 15

1 $\boldsymbol{u}\cdot\boldsymbol{v} = -2.4 + 2.4 = \mathbf{0}$, $\boldsymbol{u}\cdot\boldsymbol{w} = -.6 + 1.6 = \mathbf{1}$, $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w} = \mathbf{0} + \mathbf{1}, \boldsymbol{w}\cdot\boldsymbol{v} = 4 + 6 = \mathbf{10} = \boldsymbol{v}\cdot\boldsymbol{w}$.

2 The lengths are $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = \sqrt{5}$. Then $|\boldsymbol{u}\cdot\boldsymbol{v}| = 0 < (1)(5)$ and $|\boldsymbol{v}\cdot\boldsymbol{w}| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{4}{5}, \frac{3}{5}) = (0.8, 0.6)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (1/\sqrt{5}, 2/\sqrt{5})$. The vectors $\boldsymbol{w}, (2,-1)$, and $-\boldsymbol{w}$ make $0^\circ, 90^\circ, 180^\circ$ angles with $\boldsymbol{w}$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = 10/5\sqrt{5} = 2/\sqrt{5}$.

4 For unit vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$: (a) $\boldsymbol{v}\cdot(-\boldsymbol{v}) = \mathbf{-1}$ (b) $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = \mathbf{0}$ so $\theta = 90^\circ$ (notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$) (c) $(\boldsymbol{v}-2\boldsymbol{w})\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{w}\cdot\boldsymbol{w} = 1 - 4 = \mathbf{-3}$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (1,3)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2,1,2)/3$. $\boldsymbol{U}_1 = (3,-1)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-3,1)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1,-2,0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v} = (2,-1)$. They lie on a line. All vectors (x,y,z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to both $(1,1,1)$ and $(1,2,3)$ lie on a *line* in 3-dimensional space.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60^\circ$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90^\circ$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60^\circ$ or $\pi/3$ (d) $\cos\theta = -5/\sqrt{10}\,\sqrt{5} = -1/\sqrt{2}$ so $\theta = 135^\circ$ or $3\pi/4$ radians.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular! The vectors $(1,4)$ and $(1,-\frac{1}{4})$ are perpendicular because $1 - 1 = 0$.

10 Slopes $2/1$ and $-1/2$ multiply to give -1. Then $\boldsymbol{v}\cdot\boldsymbol{w}=0$ and the two vectors (the arrow directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w}<0$ means angle $>90^\circ$; these $\boldsymbol{w}$'s fill half of 3-dimensional space. Draw a picture to show $\boldsymbol{v}$ and the $\boldsymbol{w}$'s.

12 $(1,1)$ is perpendicular to $(1,5)-c(1,1)$ if $(1,1)\cdot(1,5)-c(1,1)\cdot(1,1)=6-2c=0$ (then $\boldsymbol{c=3}$). $\boldsymbol{v}\cdot(\boldsymbol{w}-c\boldsymbol{v})=0$ if $c=\boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to constructing a perpendicular vector $\boldsymbol{w}-c\boldsymbol{v}$.

13 One possibility among many: $\boldsymbol{u}=(1,-1,0,0),\boldsymbol{v}=(0,0,1,-1),\boldsymbol{w}=(1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ in their 3D hyperplane and they will stay perpendicular."

14 $\frac{1}{2}(x+y)=(2+8)/2=\mathbf{5}$ and $\mathbf{5}>\mathbf{4}$; $\cos\theta=2\sqrt{16}/\sqrt{10}\sqrt{10}=\mathbf{8/10}$.

15 $\|\boldsymbol{v}\|^2=1+1+\cdots+1=9$ so $\|\boldsymbol{v}\|=\mathbf{3}$; $\boldsymbol{u}=\boldsymbol{v}/3=(\frac{1}{3},\ldots,\frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w}=(1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

16 $\cos\alpha=1/\sqrt{2}$, $\cos\beta=0$, $\cos\gamma=-1/\sqrt{2}$. For any vector $\boldsymbol{v}=(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$ the cosines with the 3 axes are $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=(v_1^2+v_2^2+v_3^2)/\|\boldsymbol{v}\|^2=1$.

17 $\|\boldsymbol{v}\|^2=4^2+2^2=20$ and $\|\boldsymbol{w}\|^2=(-1)^2+2^2=5$. Pythagoras is $\|(3,4)\|^2=25=20+5$ for the length of the hypotenuse $\boldsymbol{v}+\boldsymbol{w}=(3,4)$.

18 $||\boldsymbol{v}+\boldsymbol{w}||^2=(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})+\boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. This expands to $\boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}=||\boldsymbol{v}||^2+2||\boldsymbol{v}||\,||\boldsymbol{w}||\cos\theta+||\boldsymbol{w}||^2$.

19 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}-2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. Here θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When $\theta<90^\circ$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

Pythagoras changes from equality $a^2+b^2=c^2$ to *inequality* when $\theta<90^\circ$ or $\theta>90^\circ$.

20 $2\boldsymbol{v}\cdot\boldsymbol{w}\le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2=\boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}\le\|\boldsymbol{v}\|^2+2\|\boldsymbol{v}\|\|\boldsymbol{w}\|+\|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\|+\|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\|\le\|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.

21 $v_1^2w_1^2+2v_1w_1v_2w_2+v_2^2w_2^2\le v_1^2w_1^2+v_1^2w_2^2+v_2^2w_1^2+v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2+v_2^2w_1^2-2v_1w_1v_2w_2$ which is $(v_1w_2-v_2w_1)^2\ge 0$.

22 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True : $.96 < 1$.

23 The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than $1 : x^2/(x^2 + y^2) \le 1$.

24 These two lines add to $2||\boldsymbol{v}||^2 + 2||\boldsymbol{w}||^2$:

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$$
$$||\boldsymbol{v} - \boldsymbol{w}||^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$$

25 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

26 Three vectors in the plane could make angles greater than $90°$ with each other: for example $(1, 0), (-1, 4), (-1, -4)$. Four vectors could *not* do this ($360°$ total angle). How many can can be perpendicular to each other in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n+1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

27 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} \text{The columns have} \\ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1. \\ \text{Their dot products} \\ \text{are all zero.} \end{matrix} .$$

28 The commands $V = \textbf{randn}\,(3, 30); D = \textbf{sqrt}\,(\textbf{diag}\,(V' * V));\ U = V \backslash D$; will give 30 random unit vectors in the columns of U. Then $\boldsymbol{u}' * U$ is a row matrix of 30 dot products whose average absolute value should be close to $2/\pi$.

29 The four vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ must add to zero. Then the four corners of the quadrilateral could be 0 and $\boldsymbol{v}_1$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$. We are allowing the side vectors $\boldsymbol{v}$ to cross each other—can you answer if that is not allowed ?

Problem Set 1.3, page 24

1 The column space $\mathbf{C}(A_1)$ is a plane in $\mathbf{R}^3$: the two columns of A_1 are independent

The column space $\mathbf{C}(A_2)$ is all of $\mathbf{R}^3$

The column space $\mathbf{C}(A_3)$ is a line in $\mathbf{R}^3$

2 The combination $A\boldsymbol{x} =$ column $1 - 2$ (column 2) $+$ column 3 is zero for both matrices. This leaves 2 independent columns. So $\mathbf{C}(A)$ is a (2-dimensional) plane in $\mathbf{R}^3$.

3 B has 2 independent columns so its column space is a plane. The matrix C has the same 2 independent columns and the same column space as B.

4 $A\boldsymbol{x} = \begin{bmatrix} 14 \\ 28 \\ 2 \end{bmatrix}$ Typical dot product is $2(1) + 1(2) + 2(5) = 14$ $\quad B\boldsymbol{y} = \begin{bmatrix} 4 \\ 8 \\ 18 \end{bmatrix} \quad I\boldsymbol{z} = \boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

5 $$A\boldsymbol{x} = 1\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \\ 2 \end{bmatrix}$$

$$B\boldsymbol{y} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 18 \end{bmatrix}$$

$$I\boldsymbol{z} = z_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

6 A has **2** independent columns, B has **3**, and $A + B$ has **3**. These are the ranks of A and B and $A + B$. The rule is that rank$(A + B) \leq$ rank$(A) +$ rank(B).

7 (a) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad A + B = \begin{bmatrix} 4 & 4 \\ 6 & 6 \end{bmatrix} =$ rank 1

(b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix} \quad A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$ rank 0

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad A + B = I =$ rank 4

8 The column space of A is all of $\mathbf{R}^3$. The column space of B is a **line** in $\mathbf{R}^3$. The column space of C is a 2-dimensional plane in $\mathbf{R}^3$. If C had an additional row of zeros, its column space would be a 2-dimensional plane **in** $\mathbf{R}^4$.

9 $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ **Seven ones** is the maximum for rank 3. With eight ones, two columns will be equal

10 $A = \begin{bmatrix} 3 & \mathbf{9} \\ 5 & 15 \end{bmatrix}$ has rank $1:1$ independent column, 1 independent row

$B = \begin{bmatrix} 1 & 2 & -5 \\ 4 & 8 & \mathbf{-20} \end{bmatrix}$ has 1 independent column in $\mathbf{R}^2$, 1 independent row in $\mathbf{R}^3$

11 (a) If B has an extra zero column, A and B have the **same** column space. Different row spaces because of different row lengths !

(b) If column $3 =$ column $2 -$ column 1, A and B have the same column spaces.

(c) If the new column 3 in B is $(1,1,1)$, then the column space is not changed or changed depending whether $(1,1,1)$ was already in $\mathbf{C}(A)$.

12 If $\boldsymbol{b}$ is in the column space of A, then $\boldsymbol{b}$ is a combination of the columns of A and *the numbers in that combination* give a solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. The examples are solved by $(x_1, x_2) = (1,1)$ and $(1,-1)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

13 $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -2 \end{bmatrix}$ $\quad A + B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ -1 & -3 \end{bmatrix}$ has the same column space as A and B (other examples could have a smaller column space: for example if $B = -A$ in which case $A + B =$ zero matrix).

14 $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & \mathbf{10} \end{bmatrix}$ has column $3 = 2$ (column 1) $+ 3$ (column 2)

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \mathbf{9} \end{bmatrix}$ has column $3 = -1$ (column 1) $+ 2$ (column 2)

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & \boldsymbol{q} \end{bmatrix}$ has 2 independent columns if $\boldsymbol{q} \neq \mathbf{0}$

15 If $A\boldsymbol{x} = \boldsymbol{b}$ then the extra column $\boldsymbol{b}$ in $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ is a combination of the first columns, so the column space and the rank are not changed by including the $\boldsymbol{b}$ column.

16 (a) *False* : B could be $-A$, then $A + B$ has rank zero.

(b) *True* : If the n columns of A are independent, they could not be in a space $\mathbf{R}^m$ with $m < n$. Therefore $m \geq n$.

(c) *True* : If the entries are random and the matrix has $m = n$ (or $m \geq n$), then the columns are almost surely independent.

17 **rank 2** : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **rank 1** : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

rank 0 : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

18 $3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix} = S\boldsymbol{x} = \boldsymbol{b}$

$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and the 3 dot products in $S\boldsymbol{x}$ are $3, 7, 12$

19 Suppose $a = mc$ and $b = md$ (all nonzero). Then $amd = bmc$. Then $a/b = c/d$. If those ratios are M, then $(a, c) = M(b, d)$.

20 $S\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is solved by $\boldsymbol{y} = \begin{bmatrix} c_1 \\ c_2 - c_1 \\ c_3 - c_2 \end{bmatrix}$. This is $\boldsymbol{y} = S^{-1}\boldsymbol{c} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. S is square with independent columns. So S has an inverse with $SS^{-1} = S^{-1}S = I$.

21 To solve $A\boldsymbol{x} = \mathbf{0}$ we can simplify the 3 equations (this is the subject of Chapter 2).

Start from $A\boldsymbol{x} = \mathbf{0}$ $\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_1 + 5x_2 + 6x_3 &= 0 \\ 4x_1 + 7x_2 + 9x_3 &= 0 \end{aligned}$ $\quad$ Row 2 − 3(row 1), row 3 − 4(row 1) $\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_2 - 3x_3 &= 0 \\ -x_2 - 3x_3 &= 0 \end{aligned}$

If $x_3 = 1$ then $x_2 = -3$ and $x_1 = 3$. Any answer $\boldsymbol{x} = (3c, -3c, c)$ is correct.

22 $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c = \mathbf{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & c = \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{2} & 1 \\ 4 & \mathbf{2} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ have dependent columns

23 The equation $A\boldsymbol{x} = \mathbf{0}$ says that $\boldsymbol{x}$ is perpendicular to each row of A (three dot products are zero). So $\boldsymbol{x}$ is perpendicular to all combinations of those rows. In other words, $\boldsymbol{x}$ is perpendicular to the row space (here a plane).

An important fact for linear algebra : Every $\boldsymbol{x}$ in the nullspace of A (meaning $A\boldsymbol{x} = \mathbf{0}$) is perpendicular to every vector in the row space.

Problem Set 1.4, page 35

1 Here are the 4 ways to multiply AB and the operation counts. A is m by n, B is n by p.

Row i times column k	mp dot products, n multiplications each
Matrix A times column k	p columns, mn multiplications each
Row i times matrix B	m rows, np multiplications each
Column j of A times row j of B	n (columns) (rows), mp multiplications each

2 $A = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \end{bmatrix}$ factors into $CR = \begin{bmatrix} \boldsymbol{a} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

3 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

4 (a) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$$

5 A has 7 columns and 4 rows. Those columns are vectors in 4-dimensional space. We cannot have 5 independent column vectors because we cannot have 5 independent vectors in 4-dimensional space. (This is really just a restatement of the problem. The proof comes in Section 3.2 : Every m by n matrix C, with $m < n$ has a nonzero solution to $C\boldsymbol{x} = \mathbf{0}$. Here $m = 4$ and $n = 5$ and 5 columns of C cannot be independent.)

6 $A = \begin{bmatrix} 2 & -2 & 1 & 6 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 3 & -3 & 0 & 6 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

7 $CR = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$ in Problem 6.

8 $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = AI$ $\quad A = C$ and $R = I$

$B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 4 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = CR$

9 A random 4 by 4 matrix has independent columns ($C = A$ and $R = I$) with probability 1. (We could be choosing the 16 entries of A between 0 and 1 with uniform probability by $A = \textbf{rand}(4, 4)$. We could be choosing those 16 entries of A from a "bell-shaped" normal distribution by $A = \textbf{rand}(4, 4)$. If we were choosing those 16 entries from a finite list of numbers, then there is a nonzero probability that the columns of A are *dependent*. In fact a nonzero probability that all 16 numbers are the same.)

10 If A is a random 4 by 5 matrix, then (using **rand** or **randn** as above) with probability 1 the first 4 columns are independent and go into C. With probability zero (this does not mean it can't happen !) the first 4 columns will be dependent and C will be different (C will have r columns with $r \leq 4$).

11 $A = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} = CR$. Many other possibilities !

12 $A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$A_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 1.5 \end{bmatrix}$ $\quad A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

13 $C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 4 \end{bmatrix}$ have $CR = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$ and $RC = \begin{bmatrix} 14 \end{bmatrix}$

and $CRC = \begin{bmatrix} 14 \\ 42 \end{bmatrix}$ and $RCR = \begin{bmatrix} 28 & 56 \end{bmatrix}$.

Here is an interesting fact when A is m by n and B is n by m. The m numbers on the main diagonal of AB have the same total as the n numbers on the main diagonal of BA. Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \quad AB = \begin{bmatrix} 8 & 26 \\ 17 & 62 \end{bmatrix} \quad BA = \begin{bmatrix} 12 & 15 & 18 \\ 17 & 22 & 27 \\ 22 & 29 & 36 \end{bmatrix}$$

$8 + 62 = 12 + 22 + 36$

14 $\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$ rank one $\quad \begin{bmatrix} 6 & -7 \\ 7 & 6 \end{bmatrix}$ orthogonal columns $\quad \begin{bmatrix} 2 & 0 \\ 3 & 6 \end{bmatrix}$ rank 2 $\quad \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ $A^2 = I$

15
1. Column j of A equals the matrix C times column j of R.
   This is a combination of the **columns** of C.

2. Row i of A is row i of C times the matrix R.
   This is a combination of the **rows** of R.

3. (row i of C) $\cdot$ (column j of R) gives A_{ij}
   That dot product requires the number of columns of C to equal the number of rows of R.

4. C has r columns so R has r rows (to multiply CR). Those columns of C are independent (by construction). Those rows of R are independent (because R contains the r by r identity matrix).

16 (a) The vector $AB\boldsymbol{x}$ is the matrix A times the vector $B\boldsymbol{x}$. So it is a combination of the columns of A. Therefore $\mathbf{C}(AB) \subseteq \mathbf{C}(A)$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ give $AB =$ zero matrix and $\mathbf{C}(AB) =$ zero vectors.

17 (a) If A and B have rank 1, then AB has rank 1 or 0. $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ and $B = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ give $AB = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x})\boldsymbol{y}^{\mathrm{T}}$ so $AB =$ zero matrix if the dot product $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x}$ happens to be zero.

(b) If A and B are 3 by 3 matrices of rank 3, then it is **true** that AB has rank 3. *One approach*: If $AB\boldsymbol{x} = \mathbf{0}$ then $B\boldsymbol{x} = \mathbf{0}$ because A has 3 independent columns. But $B\boldsymbol{x} = \mathbf{0}$ only when $\boldsymbol{x} = \mathbf{0}$, because B has 3 independent columns.

(c) Suppose $AB = BA$ for all 2 by 2 matrices B. Choose $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $AB = \begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ e & f \end{bmatrix}$. This tells us that $\begin{bmatrix} c & 0 \\ e & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ and therefore $d = e = 0$. Now choose $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that $AB = \begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$. This tells us that $\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ and $c = f$ and $A = cI$.

18 (a) $AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $BC = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

(b) $(AB)C =$ column exchange of $AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$A(BC) =$ row exchange of $BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ $=$ same result ABC.

19 $$AB = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

20 $AB = (4 \times 3)\,(3 \times 2)$ needs $mnp = (4)\,(3)\,(2) = 24$ multiples.

Then $(AB)C = (4 \times 2)\,(2 \times 1)$ needs $(4)\,(2)\,(1) = 8$ more : TOTAL 32.

$BC = (3 \times 2)\,(2 \times 1)$ needs $mnp = (3)\,(2)\,(1) = 6$ multiplies.

Then $A(BC) = (4 \times 3)\,(3 \times 1)$ needs $(4)\,(3)\,(1) = 12$ more : TOTAL 18.

Best to start with $\boldsymbol{C}$ = vector. Multiply by B to get the vector BC, and then the vector $A(BC)$. Vectors need less computing time than matrices !