2.5   Inverse Matrices

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1 If the square matrix A has an inverse, then both $A^{-1}A = I$ and $AA^{-1} = I$.

2 The *algorithm* to test invertibility is elimination : A must have n (nonzero) pivots.

3 The *algebra* test for invertibility is the determinant of A : $\det A$ must not be zero.

4 The *equation* that tests for invertibility is $A\boldsymbol{x} = \boldsymbol{0}$: $\boldsymbol{x} = \boldsymbol{0}$ **must be the only solution.**

5 If A and B (same size) are invertible then so is AB : $\boxed{(AB)^{-1} = B^{-1}A^{-1}.}$

6 $AA^{-1} = I$ is n equations for n columns of A^{-1}. Gauss-Jordan eliminates $[A\ I]$ to $[I\ A^{-1}]$.

7 The last page of the book gives 14 equivalent conditions for a square A to be invertible.

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, such that A^{-1} *times* A *equals* I. Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}A\boldsymbol{x} = \boldsymbol{x}$. *But A^{-1} might not exist.*

What a matrix mostly does is to multiply a vector $\boldsymbol{x}$. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ by A^{-1} gives $A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$. ***This is*** $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero— matrices are more complicated and more interesting. The matrix A^{-1} is called "A inverse."

> **DEFINITION**   The matrix A is ***invertible*** if there exists a matrix A^{-1} that "inverts" A :
>
> **Two-sided inverse** $\qquad A^{-1}A = I \quad$ and $\quad AA^{-1} = I.$ $\qquad\qquad$ (1)

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible ? We don't mean that we immediately calculate A^{-1}. In most problems we never compute it ! Here are six "notes" about A^{-1}.

Note 1   ***The inverse exists if and only if elimination produces*** n ***pivots*** (row exchanges are allowed). Elimination solves $A\boldsymbol{x} = \boldsymbol{b}$ without explicitly using the matrix A^{-1}.

Note 2   The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses" :

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \qquad\qquad (2)$$

This shows that a *left-inverse B* (multiplying from the left) and a *right-inverse C* (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3   If A is invertible, the one and only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$:

> ***Multiply*** $\ A\boldsymbol{x} = \boldsymbol{b}\ $ ***by*** $\ A^{-1}.\ $ ***Then*** $\ \boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}.$

Note 4   (Important) ***Suppose there is a nonzero vector x such that $Ax = 0$. Then A cannot have an inverse.*** No matrix can bring $\mathbf{0}$ back to x.

If A is invertible, then $Ax = \mathbf{0}$ can only have the zero solution $x = A^{-1}\mathbf{0} = \mathbf{0}$.

Note 5   A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\text{2 by 2 \textbf{Inverse:}} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the *determinant* of A. A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 6   A diagonal matrix has an inverse provided no diagonal entries are zero :

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1   The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $Ax = \mathbf{0}$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B, the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product AB* has an inverse, if and only if the two factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in ***reverse order*** :

> If A and B are invertible then so is AB. The inverse of a product AB is
>
> $$(AB)^{-1} = B^{-1}A^{-1}. \tag{4}$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. Inside that is $BB^{-1} = I$:

$$\text{\textbf{Inverse of }} AB \qquad (AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I.

$B^{-1}A^{-1}$ illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____ . The same reverse order applies to three or more matrices:

Reverse order $$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \tag{5}$$

Example 2 *Inverse of an elimination matrix*. If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2 :

$$\begin{matrix} \textbf{\textit{E} subtracts} \\ \textbf{\textit{E}}^{-1} \textbf{ adds} \end{matrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ -\mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. If $AC = I$ then automatically $CA = I$.

For square matrices, an inverse on one side is automatically an inverse on the other side.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE. Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ \mathbf{20} & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ \mathbf{0} & 4 & 1 \end{bmatrix}. \tag{6}$$

The result is beautiful and correct. The product FE contains "20" but its inverse doesn't. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new* row 2 (changed by row 1) from row 3. *In this order FE, row 3 feels an effect from row 1.*

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn't change again. *In this order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.*

This is why the next section chooses $A = LU$, to go back from the triangular U to A. The multipliers fall into place perfectly in the lower triangular L.

> In elimination order F follows E. In reverse order E^{-1} follows F^{-1}.
> $E^{-1}F^{-1}$ *is quick. The multipliers 5, 4 fall into place below the diagonal of 1's.*

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is solved by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. But it is not necessary or efficient to compute A^{-1} and multiply it times $\boldsymbol{b}$. *Elimination goes directly to $\boldsymbol{x}$.* And elimination is also the way to calculate A^{-1}, as we now show. The Gauss-Jordan idea is to solve $AA^{-1} = I$, *finding each column of A^{-1}.*

A multiplies the first column of A^{-1} (call that $\boldsymbol{x}_1$) to give the first column of I (call that $\boldsymbol{e}_1$). This is our equation $A\boldsymbol{x}_1 = \boldsymbol{e}_1 = (1, 0, 0)$. There will be two more equations. *Each of the columns $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ of A^{-1} is multiplied by A to produce a column of I:*

3 columns of A^{-1} $\qquad AA^{-1} = A\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = I.$ $\qquad$ (7)

To invert a 3 by 3 matrix A, we have to solve three systems of equations: $A\boldsymbol{x}_1 = \boldsymbol{e}_1$ and $A\boldsymbol{x}_2 = \boldsymbol{e}_2 = (0, 1, 0)$ and $A\boldsymbol{x}_3 = \boldsymbol{e}_3 = (0, 0, 1)$. Gauss-Jordan finds A^{-1} this way.

The Gauss-Jordan method computes A^{-1} by solving all n equations together. Usually the "augmented matrix" $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ has one extra column $\boldsymbol{b}$. Now we have three right sides $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ (when A is 3 by 3). They are the columns of I, so the augmented matrix is really the block matrix $\begin{bmatrix} A & I \end{bmatrix}$. I take this chance to invert my favorite matrix K, with 2's on the main diagonal and -1's next to the 2's:

$$\begin{bmatrix} K & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \left[\begin{array}{rrr|rrr} \mathbf{2} & \mathbf{-1} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & 0 & 0 & 1 \end{array}\right] \quad \textbf{Start Gauss-Jordan on } K$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \qquad (\tfrac{1}{2}\,\textit{row } \mathbf{1} + \textit{row } \mathbf{2})$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \qquad (\tfrac{2}{3}\,\textit{row } \mathbf{2} + \textit{row } \mathbf{3})$$

We are halfway to K^{-1}. The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination*! He goes all the way to the ***reduced echelon form $R = I$***. Rows are added to rows above them, to produce ***zeros above the pivots***:

$$\left(\begin{array}{c} \text{Zero above} \\ \text{third pivot} \end{array}\right) \quad \rightarrow \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \qquad (\tfrac{3}{4}\,\textit{row } \mathbf{3} + \textit{row } \mathbf{2})$$

$$\left(\begin{array}{c} \text{Zero above} \\ \text{second pivot} \end{array}\right) \quad \rightarrow \left[\begin{array}{rrr|rrr} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \qquad (\tfrac{2}{3}\,\textit{row } \mathbf{2} + \textit{row } \mathbf{1})$$

The final Gauss-Jordan step is to divide each row by its pivot. The new pivots are all 1.

We have reached I in the first half of the matrix, because K is invertible. ***The three columns of K^{-1} are in the second half of*** $[\,I\ K^{-1}\,]$:

$$
\begin{array}{l}
\text{(divide by 2)}\\[2pt]
\text{(divide by } \tfrac{3}{2})\\[2pt]
\text{(divide by } \tfrac{4}{3})
\end{array}
\quad
\left[\begin{array}{ccc|ccc}
\mathbf{1} & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\[4pt]
0 & \mathbf{1} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\[4pt]
0 & 0 & \mathbf{1} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4}
\end{array}\right]
= \begin{bmatrix} I & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} I & \boldsymbol{K^{-1}} \end{bmatrix}.
$$

Starting from the 3 by 6 matrix $[\,K\ \ I\,]$, we ended with $[\,I\ \ K^{-1}\,]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix A:

Gauss-Jordan   ***Multiply*** $\begin{bmatrix} A & I \end{bmatrix}$ ***by*** A^{-1} ***to get*** $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

The elimination steps create the inverse matrix while changing A to I. For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about K^{-1}: an important example.

1. K is **symmetric** across its main diagonal. Then K^{-1} is also symmetric.

2. K is **tridiagonal** (only three nonzero diagonals). But K^{-1} is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.

3. The *product of pivots* is $2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right) = 4$. This number 4 is the **determinant** of K.

$$
\boldsymbol{K^{-1}} \text{ \textbf{\textit{involves division by the determinant of }} } \boldsymbol{K} \qquad K^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \qquad (8)
$$

This is why an invertible matrix cannot have a zero determinant: we need to divide.

Example 4   Find A^{-1} by Gauss-Jordan elimination starting from $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$.

$$
\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & 1 & 0 \\ \mathbf{4} & \mathbf{7} & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & -2 & 1 \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} U & L^{-1} \end{bmatrix} \right)
$$

$$
\rightarrow \begin{bmatrix} \mathbf{2} & \mathbf{0} & 7 & -\mathbf{3} \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & -\mathbf{2} & 1 \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} I & A^{-1} \end{bmatrix} \right).
$$

Example 5   ***If A is invertible and upper triangular, so is A^{-1}.*** Start with $\boldsymbol{AA^{-1} = I}$.

1 A times *column j of A^{-1}* equals *column j of I*, ending with $n - j$ zeros.

2   Back substitution keeps those $n - j$ zeros at the end of column j of A^{-1}.

3   Put those columns $[\ast \ldots \ast 0 \ldots 0]^{\mathrm{T}}$ into A^{-1} and that matrix is upper triangular!

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \mathbf{0} & 1 & -1 \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ \mathbf{0} & 1 & 1 \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \qquad \begin{array}{l} \text{Columns } j = 1 \text{ and } 2 \text{ end} \\ \text{with } 3 - j = 2 \text{ and } 1 \text{ zeros.} \end{array}$$

The code for $X = \mathbf{inv}(A)$ can use **rref**, the reduced row echelon form from Chapter 3:

$$\begin{array}{ll} I = \ \mathsf{eye}\ (n); & \text{\% Define the } n \text{ by } n \text{ identity matrix} \\ R = \ \mathsf{rref}\ ([A \quad I]); & \text{\% Eliminate on the augmented matrix } [A \quad I] \\ X = R(:,n+1:n+n) & \text{\% Pick } X = A^{-1} \text{ from the last } n \text{ columns of } R \end{array}$$

A must be invertible, or elimination cannot reduce it to I (in the left half of R).

Gauss-Jordan shows why A^{-1} is expensive. We solve n equations for its n columns. But all those equations involve the same matrix A on the left side (where most of the work is done). The total cost for A^{-1} is n^3 multiplications and subtractions. To solve a single $A\boldsymbol{x} = \boldsymbol{b}$ that cost (see the next section) is $n^3/3$.

To solve $Ax = b$ without A^{-1}, we deal with *one* column b to find one column x.

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: ***A^{-1} exists exactly when A has a full set of n pivots.*** (Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $A\boldsymbol{x}_i = \boldsymbol{e}_i$. The columns $\boldsymbol{x}_i$ go into A^{-1}. Then $AA^{-1} = I$ and A^{-1} is at least a ***right-inverse***.

2. Elimination is really a sequence of multiplications by E's and P's and D^{-1}:

Left-inverse C $\qquad\qquad CA = (D^{-1} \cdots E \cdots P \cdots E)A = I.$ $\qquad\qquad$ (9)

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. P will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a ***left-inverse of A***. With n pivots we have reached $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 at the start of in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that A ***must have n pivots if*** $AC = I$.

1. If A doesn't have n pivots, elimination will lead to a *zero row*.
2. Those elimination steps are taken by an invertible M. *So a row of MA is zero.*
3. If $AC = I$ had been possible, then $MAC = M$. The zero row of MA, times C, gives a zero row of M itself.
4. An invertible matrix M can't have a zero row! *A must* have n pivots if $AC = I$.

That argument took four steps, but the outcome is short and important. C is A^{-1}.

Elimination gives a complete test for invertibility of a square matrix. A^{-1} **exists (and Gauss-Jordan finds it) exactly when** A **has** n **pivots**. The argument above shows more:

$$\text{If} \quad AC = I \quad \text{then} \quad CA = I \quad \text{and} \quad C = A^{-1} \tag{10}$$

Example 6　If L is lower triangular with 1's on the diagonal, so is L^{-1}.

A triangular matrix is invertible if and only if no diagonal entries are zero.

Here L has 1's so L^{-1} also has 1's. Use the Gauss-Jordan method to construct L^{-1} from E_{32}, E_{31}, E_{21}. Notice how L^{-1} contains the strange entry 11, from 3 times 5 minus 4.

Gauss-Jordan on triangular L
$$\left[\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{1} & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} L & I \end{bmatrix}$$

$$\begin{array}{c} \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} (3 \text{ times row 1 from row 2}) \\ (4 \text{ times row 1 from row 3}) \\ (\text{then } 5 \text{ times row 2 from row 3}) \end{array}$$

The inverse is still triangular
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & -3 & 1 & \mathbf{0} \\ 0 & 0 & 1 & 11 & -5 & 1 \end{array}\right] = \begin{bmatrix} I & L^{-1} \end{bmatrix}.$$

Recognizing an Invertible Matrix

Normally, it takes work to decide if a matrix is invertible. The usual way is to find a full set of nonzero pivots in elimination. (Then the nonzero determinant comes from multiplying those pivots.) But for some matrices you can see quickly that they are invertible because every number a_{ii} on their main diagonal dominates the off-diagonal part of that row i.

Diagonally dominant matrices are invertible. Each a_{ii} on the diagonal is larger than the total sum along the rest of row i. On every row,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{means that} \quad |a_{ii}| > |a_{i1}| + \cdots (\text{skip } |a_{ii}|) \cdots + |a_{in}|. \tag{11}$$

Examples.　A is diagonally dominant $(3 > 2)$. B is not (but still invertible). C is singular.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Reasoning.   Take any nonzero vector $\boldsymbol{x}$. *Suppose its largest component is $|x_i|$.* Then $A\boldsymbol{x} = \boldsymbol{0}$ is impossible, because row i of $A\boldsymbol{x} = \boldsymbol{0}$ would need

$$a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = 0.$$

Those can't add to zero when A is diagonally dominant! The size of $a_{ii}x_i$ (that one particular term) is greater than all the other terms combined:

$$\textbf{All } |x_j| \le |x_i| \qquad \sum_{j \neq i} |a_{ij}x_j| \le \sum_{j \neq i} |a_{ij}|\,|x_i| < |a_{ii}|\,|x_i| \quad \textbf{because } a_{ii} \textbf{ dominates}$$

This shows that $A\boldsymbol{x} = \boldsymbol{0}$ is only possible when $\boldsymbol{x} = \boldsymbol{0}$. *So A is invertible.* The example B was also invertible but not quite diagonally dominant: 2 is not larger than $1 + 1$.

■ **REVIEW OF THE KEY IDEAS** ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.

2. A is invertible if and only if it has n pivots (row exchanges allowed).

3. *Important*. If $A\boldsymbol{x} = \boldsymbol{0}$ for a nonzero vector $\boldsymbol{x}$, then A has no inverse.

4. The inverse of AB is the reverse product $B^{-1}A^{-1}$. And $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1}. The augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ is row-reduced to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

6. Diagonally dominant matrices are invertible. Each $|a_{ii}|$ dominates its row.

■ **WORKED EXAMPLES** ■

2.5 A   The inverse of a triangular **difference matrix** A is a triangular **sum matrix** S:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] = \begin{bmatrix} I & A^{-1} \end{bmatrix} = \begin{bmatrix} I & sum\ matrix \end{bmatrix}.$$

If I change a_{13} to -1, then all rows of A add to zero. The equation $A\boldsymbol{x} = \boldsymbol{0}$ will now have the nonzero solution $\boldsymbol{x} = (1, 1, 1)$. A clear signal : ***This new A can't be inverted.***

2.5 B   Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $Ax = 0$) for the other three. The matrices are in the order A, B, C, D, S, E:

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36} \begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. E has two equal rows (and the second column minus the first column is zero). In other words $Ex = 0$ has the solution $x = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of A, D, E.

2.5 C   Apply the Gauss-Jordan method to invert this triangular "Pascal matrix" L. You see **Pascal's triangle**—adding each entry to the entry on its left gives the entry below. The entries of L are "binomial coefficients". The next row would be $1, 4, 6, 4, 1$.

$$\textbf{Triangular Pascal matrix} \quad L = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \end{bmatrix} = \text{abs(pascal (4,1))}$$

Solution   Gauss-Jordan starts with $[\,L \; I\,]$ and produces zeros by subtracting row 1:

$$[\boldsymbol{L} \; \boldsymbol{I}] = \left[\begin{array}{cccc|cccc} \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{0} & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ \mathbf{0} & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ \mathbf{0} & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \mathbf{0} & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & \mathbf{0} & 3 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 1 & 0 & \mathbf{1} & -\mathbf{2} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 & -\mathbf{1} & \mathbf{3} & -\mathbf{3} & \mathbf{1} \end{array}\right] = [\boldsymbol{I} \; \boldsymbol{L^{-1}}].$$

All the pivots were 1! So we didn't need to divide rows by pivots to get I. The inverse matrix L^{-1} looks like L itself, except odd-numbered diagonals have minus signs.

The same pattern continues to n by n Pascal matrices. L^{-1} has "alternating diagonals".

Problem Set 2.5

1   Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

2   For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3   Solve for the first column (x, y) and second column (t, z) of A^{-1}:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4   Show that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1}:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{pmatrix} \textit{For a different } A\textit{, could column 1 of } A^{-1} \\ \text{be possible to find but not column 2 ?} \end{pmatrix}$$

5   Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

6   (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

   (b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two different matrices such that $AB = AC$.

7   (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

   (a) Explain why $A\boldsymbol{x} = (0, 0, 1)$ cannot have a solution. Add eqn 1 + eqn 2.

   (b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{x} = \boldsymbol{b}$?

   (c) In elimination, what happens to equation 3 ?

8   If A has column 1 + column 2 = column 3, show that A is not invertible:

   (a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{0}$. The matrix is 3 by 3.

   (b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

9   Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

10   Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

11   (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

12   If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

13   If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

14   If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

Notice the order.   The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix}$ is   _____ .

15   Prove that a matrix with a column of zeros cannot have an inverse.

16   Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

17   (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18   If B is the inverse of A^2, show that AB is the inverse of A.

19   Find the numbers a and b that give the inverse of $5 * \text{eye}\,(4) - \text{ones}\,(4, 4)$:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of $6 * \text{eye}\,(5) - \text{ones}\,(5, 5)$?

20   Show that A $= 4 * \text{eye}\,(4) - \text{ones}\,(4, 4)$ is *not* invertible: Multiply A $* \text{ones}\,(4, 1)$.

21   There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22   Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

23   Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below
the pivots to reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

24   Use Gauss-Jordan elimination on $\begin{bmatrix} U & I \end{bmatrix}$ to find the upper triangular U^{-1}:

$$\boldsymbol{UU^{-1} = I} \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

25   Find A^{-1} and B^{-1} (*if they exist*) by elimination on $\begin{bmatrix} A & I \end{bmatrix}$ and $\begin{bmatrix} B & I \end{bmatrix}$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

26   What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix?
Multiply $D^{-1} E_{12} E_{21}$ to find A^{-1}.

27   Invert these matrices A by the Gauss-Jordan method starting with $\begin{bmatrix} A & I \end{bmatrix}$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

28   Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

29   True or false (with a counterexample if false and a reason if true):

   (a) A 4 by 4 matrix with a row of zeros is not invertible.

   (b) Every matrix with 1's down the main diagonal is invertible.

   (c) If A is invertible then A^{-1} and A^2 are invertible.

30   (Recommended) Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}). Then find three numbers c so that C is not invertible:

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \qquad C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

31   This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \ \ I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{x} = (1, 1, 1, 1).$$

32   Suppose the matrices P and Q have the same rows as I but in any order. They are "permutation matrices". Show that $P - Q$ is singular by solving $(P - Q)\,\boldsymbol{x} = \boldsymbol{0}$.

33   Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \qquad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \qquad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

34   Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

35   In the Worked Example **2.5 C**, the triangular Pascal matrix L has $L^{-1} = DLD$, where the diagonal matrix D has alternating entries $1, -1, 1, -1$. Then $LDLD = I$, so what is the inverse of $LD = \mathsf{pascal}\,(4, 1)$?

36   The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse invhilb (6). Then ask it to compute inv (hilb (6)). How can these be different, when the computer never makes mistakes?

37   (a) Use inv(P) to invert MATLAB's 4 by 4 symmetric matrix $P = \mathsf{pascal}\,(4)$.

(b) Create Pascal's lower triangular $L = \mathsf{abs}\,(\mathsf{pascal}\,(4, 1))$ and test $P = LL^{\mathrm{T}}$.

38   If $A = \mathsf{ones}\,(4)$ and $\boldsymbol{b} = \mathsf{rand}\,(4, 1)$, how does MATLAB tell you that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution? For the special $\boldsymbol{b} = \mathsf{ones}\,(4, 1)$, which solution to $A\boldsymbol{x} = \boldsymbol{b}$ is found by $A \backslash \boldsymbol{b}$?

Challenge Problems

39   (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

40   Suppose E_1, E_2, E_3 are 4 by 4 identity matrices, except E_1 has a, b, c in column 1 and E_2 has d, e in column 2 and E_3 has f in column 3 (below the 1's). Multiply $L = E_1 E_2 E_3$ to show that all these nonzeros are copied into L.

$E_1 E_2 E_3$ is in the *opposite* order from elimination (because E_3 is acting first). But $E_1 E_2 E_3 = L$ is in the *correct* order to invert elimination and recover A.

41   Second difference matrices have beautiful inverses if they start with $T_{11} = 1$ (instead of $K_{11} = 2$). Here is the 3 by 3 tridiagonal matrix T and its inverse:

$$T = \begin{bmatrix} \mathbf{1} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

One approach is Gauss-Jordan elimination on $\begin{bmatrix} T & I \end{bmatrix}$. I would rather write T as the product of first differences L times U. The inverses of L and U in Worked Example **2.5 A** are **sum matrices**, so here are $T = LU$ and $T^{-1} = U^{-1}L^{-1}$:

$$T = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}$$

$$\quad\;\; \textbf{difference} \quad\;\; \textbf{difference} \qquad\qquad\qquad \textbf{sum} \quad\;\; \textbf{sum}$$

Question.   (**4 by 4**) What are the pivots of T? What is its 4 by 4 inverse? The reverse order UL gives what matrix T^*? What is the inverse of T^*?

42   Here are two more difference matrices, both important. ***But are they invertible?***

$$\textbf{Cyclic } C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad \textbf{Free ends } F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

43   *Elimination for a block matrix*: When you multiply the first block row $\begin{bmatrix} A & B \end{bmatrix}$ by CA^{-1} and subtract from the second row $\begin{bmatrix} C & D \end{bmatrix}$, the "*Schur complement*" S appears:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \qquad \begin{matrix} A \text{ and } D \text{ are square} \\ S = D - CA^{-1}B. \end{matrix}$$

Multiply on the right to subtract $A^{-1}B$ times block column 1 from block column 2.

$$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = ? \quad \text{Find } S \text{ for} \quad \begin{bmatrix} A & B \\ C & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

The block pivots are A and S. If they are invertible, so is $\begin{bmatrix} A & B; & C & D \end{bmatrix}$.

44   How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.