10.5 Fourier Series: Linear Algebra for Functions

This section goes from finite dimensions to *infinite* dimensions. I want to explain linear algebra in infinite-dimensional space, and to show that it still works. First step: look back. This book began with vectors and dot products and linear combinations. We begin by converting those basic ideas to the infinite case—then the rest will follow.

What does it mean for a vector to have infinitely many components? There are two different answers, both good:

1. The vector is infinitely long: $v = (v_1, v_2, v_3, \ldots)$. It could be $(1, \frac{1}{2}, \frac{1}{4}, \ldots)$.

2. The vector is a function $f(x)$. It could be $v = \sin x$.

We will go both ways. Then the idea of a Fourier series will connect them.

After vectors come *dot products*. The natural dot product of two infinite vectors $(v_1, v_2, \ldots)$ and $(w_1, w_2, \ldots)$ is an infinite series:

$$\textbf{Dot product} \qquad v \cdot w = v_1 w_1 + v_2 w_2 + \cdots. \qquad (1)$$

This brings a new question, which never occurred to us for vectors in $\mathbf{R}^n$. Does this infinite sum add up to a finite number? Does the series converge? Here is the first and biggest difference between finite and infinite.

When $v = w = (1, 1, 1, \ldots)$, the sum certainly does not converge. In that case $v \cdot w = 1 + 1 + 1 + \cdots$ is infinite. Since v equals w, we are really computing $v \cdot v = \|v\|^2$, the length squared. The vector $(1, 1, 1, \ldots)$ has infinite length. *We don't want that vector.* Since we are making the rules, we don't have to include it. The only vectors to be allowed are those with finite length:

DEFINITION The vector $v = (v_1, v_2, \ldots)$ and the function $f(x)$ are in our infinite-dimensional "***Hilbert spaces***" if and only if their lengths $\|v\|$ and $\|f\|$ are finite:

$$\|v\|^2 = v \cdot v = v_1^2 + v_2^2 + v_3^2 + \cdots \quad \text{must add to a finite number.}$$

$$\|f\|^2 = (f, f) = \int_0^{2\pi} |f(x)|^2 \, dx \qquad \text{must be a finite integral.}$$

Example 1 The vector $v = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$ is included in Hilbert space, because its length is $2/\sqrt{3}$. We have a geometric series that adds to $4/3$. The length of v is the square root:

$$\textbf{Length squared} \qquad v \cdot v = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Question If v and w have finite length, how large can their dot product be?

Answer The sum $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots$ also adds to a finite number. We can safely take dot products. The Schwarz inequality is still true:

$$\textbf{Schwarz inequality} \qquad |v \cdot w| \leq \|v\| \, \|w\|. \qquad (2)$$

The ratio of $v \cdot w$ to $\|v\| \, \|w\|$ is still the cosine of θ (the angle between v and w). Even in infinite-dimensional space, $|\cos \theta|$ is not greater than 1.

Now change over to functions. Those are the "vectors." The space of functions $f(x)$, $g(x)$, $h(x)$, ... defined for $0 \le x \le 2\pi$ must be somehow bigger than $\mathbf{R}^n$. *What is the dot product of $f(x)$ and $g(x)$? What is the length of $f(x)$?*

Key point in the continuous case: *Sums are replaced by integrals.* Instead of a sum of v_j times w_j, the dot product is an integral of $f(x)$ times $g(x)$. Change the "dot" to parentheses with a comma, and change the words "dot product" to *inner product*:

DEFINITION The ***inner product*** of $f(x)$ and $g(x)$, and the ***length squared*** of $f(x)$, are

$$(f, g) = \int_0^{2\pi} f(x)g(x)\, dx \qquad \text{and} \qquad \|f\|^2 = \int_0^{2\pi} \big(f(x)\big)^2 dx. \qquad (3)$$

The interval $[0, 2\pi]$ where the functions are defined could change to a different interval like $[0, 1]$ or $(-\infty, \infty)$. We chose 2π because our first examples are $\sin x$ and $\cos x$.

Example 2   The length of $f(x) = \sin x$ comes from its inner product with itself:

$$(f, f) = \int_0^{2\pi} (\sin\ x)^2\, dx = \pi. \quad \text{The length of } \sin x \text{ is } \sqrt{\pi}.$$

That is a standard integral in calculus—not part of linear algebra. By writing $\sin^2 x$ as $\frac{1}{2} - \frac{1}{2}\cos 2x$, we see it go above and below its average value $\frac{1}{2}$. Multiply that average by the interval length 2π to get the answer π.

More important: $\sin x$ ***and*** $\cos x$ ***are orthogonal in function space***: $(f, g) = 0$

Inner product $\displaystyle \int_0^{2\pi} \sin x \cos x\, dx = \int_0^{2\pi} \tfrac{1}{2}\sin 2x\, dx = \Big[-\tfrac{1}{4}\cos 2x\Big]_0^{2\pi} = 0. \quad (4)$
is zero

This zero is no accident. It is highly important to science. The orthogonality goes beyond the two functions $\sin x$ and $\cos x$, to an infinite list of sines and cosines. The list contains $\cos 0x$ (which is 1), $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, $\sin 3x$, $\cos 3x$, ...

Every function in that list is orthogonal to every other function in the list.

Fourier Series

The Fourier series of a function $f(x)$ is its expansion into sines and cosines:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots. \qquad (5)$$

We have an orthogonal basis! The vectors in "function space" are combinations of the sines and cosines. On the interval from $x = 2\pi$ to $x = 4\pi$, all our functions repeat what they did from 0 to 2π. They are "*periodic*." The distance between repetitions is the period 2π.

Remember: The list is infinite. The Fourier series is an infinite series. We avoided the vector $v = (1, 1, 1, \ldots)$ because its length is infinite, now we avoid a function like $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots$. (*Note*: This is π times the famous **delta function** $\delta(x)$. It is an infinite "spike" above a single point. At $x = 0$ its height $\frac{1}{2} + 1 + 1 + \cdots$ is infinite. At all points inside $0 < x < 2\pi$ the series adds in some average way to zero.) The integral of $\delta(x)$ is 1. But $\int \delta^2(x) = \infty$, so delta functions are not allowed into Hilbert space.

Compute the length of a typical sum $f(x)$:

$$(f, f) = \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots)^2 \, dx$$

$$= \int_0^{2\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \cdots) \, dx$$

$$\|f\|^2 = 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + \cdots). \tag{6}$$

The step from line 1 to line 2 used orthogonality. All products like $\cos x \cos 2x$ integrate to give zero. Line 2 contains what is left—the integrals of each sine and cosine squared. Line 3 evaluates those integrals. (The integral of 1^2 is 2π, when all other integrals give π.) If we divide by their lengths, our functions become *orthonormal*:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \text{ is an orthonormal basis for our function space.}$$

These are unit vectors. We could combine them with coefficients $A_0, A_1, B_1, A_2, \ldots$ to yield a function $F(x)$. Then the 2π and the π's drop out of the formula for length.

Function length = vector length $\qquad \|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \cdots. \tag{7}$

Here is the important point, for $f(x)$ as well as $F(x)$. *The function has finite length exactly when the vector of coefficients has finite length.* Fourier series gives us a perfect match between the Hilbert spaces for functions and for vectors. The function is in L^2, its Fourier coefficients are in ℓ^2.

> The function space contains $f(x)$ exactly when the Hilbert space contains the vector $v = (a_0, a_1, b_1, \ldots)$ of Fourier coefficients of $f(x)$. Both must have finite length.

Example 3 Suppose $f(x)$ is a "square wave," equal to 1 for $0 \le x < \pi$. Then $f(x)$ drops to -1 for $\pi \le x < 2\pi$. The $+1$ and -1 repeat forever. This $f(x)$ is an odd function like the sines, and all its cosine coefficients are zero. We will find its Fourier series, containing only sines:

$$\text{Square wave} \qquad f(x) = \frac{4}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right]. \tag{8}$$

The length of this function is $\sqrt{2\pi}$, because at every point $(f(x))^2$ is $(-1)^2$ or $(+1)^2$:

$$\|f\|^2 = \int_0^{2\pi} (f(x))^2 \, dx = \int_0^{2\pi} 1 \, dx = 2\pi.$$

At $x = 0$ the sines are zero and the Fourier series gives zero. This is half way up the jump from -1 to $+1$. The Fourier series is also interesting when $x = \frac{\pi}{2}$. At this point the square wave equals 1, and the sines in (8) alternate between $+1$ and -1:

$$\textbf{Formula for } \pi \qquad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \qquad (9)$$

Multiply by π to find a magical formula $4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$ for that famous number.

The Fourier Coefficients

How do we find the a's and b's which multiply the cosines and sines? For a given function $f(x)$, we are asking for its Fourier coefficients a_k and b_k:

$$\textbf{Fourier series} \qquad f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots .$$

Here is the way to find a_1. *Multiply both sides by* $\cos x$. *Then integrate from* 0 *to* 2π. The key is orthogonality! All integrals on the right side are zero, except for $\cos^2 x$:

$$\textbf{For coefficient } a_1 \qquad \int_0^{2\pi} f(x) \cos x \, dx = \int_0^{2\pi} a_1 \cos^2 x \, dx = \pi a_1. \qquad (10)$$

Divide by π and you have a_1. To find any other a_k, multiply the Fourier series by $\cos kx$. Integrate from 0 to 2π. Use orthogonality, so only the integral of $a_k \cos^2 kx$ is left. That integral is πa_k, and divide by π:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \qquad \text{and similarly} \qquad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx. \qquad (11)$$

The exception is a_0. This time we multiply by $\cos 0x = 1$. The integral of 1 is 2π:

$$\textbf{Constant term} \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 \, dx = \textbf{\textit{average value}} \text{ of } f(x). \qquad (12)$$

I used those formulas to find the Fourier coefficients for the square wave in equation (8). The integral of $f(x) \cos kx$ was zero. The integral of $f(x) \sin kx$ was $4/k$ for odd k.

Compare Linear Algebra in $\mathbf{R}^n$

Infinite-dimensional Hilbert space is very much like the n-dimensional space $\mathbf{R}^n$. Suppose the nonzero vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthogonal in $\mathbf{R}^n$. We want to write the vector $\boldsymbol{b}$ (instead of the function $f(x)$) as a combination of those $\boldsymbol{v}$'s:

$$\textbf{Finite orthogonal series} \quad \boldsymbol{b} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_n \boldsymbol{v}_n. \qquad (13)$$

Multiply both sides by $\boldsymbol{v}_1^{\mathrm{T}}$. Use orthogonality, so $\boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{v}_2 = 0$. Only the c_1 term is left:

$$\textbf{Coefficient } c_1 \quad \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{b} = c_1 \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{v}_1 + 0 + \cdots + 0. \quad \text{Therefore } c_1 = \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{b} / \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{v}_1. \qquad (14)$$

The denominator $\boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{v}_1$ is the length squared, like π in equation (11). The numerator $\boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{b}$ is the inner product like $\int f(x) \cos kx \, dx$. ***Coefficients are easy to find when the***

basis vectors are orthogonal. We are just doing one-dimensional projections, to find the components along each basis vector.

The formulas are even better when the vectors are orthonormal. Then we have unit vectors in Q. The denominators $v_k^T v_k$ are all 1. You know $c_k = v_k^T b$ in another form:

$$\textbf{Equation for } c\text{'s} \quad c_1 v_1 + \cdots + c_n v_n = b \quad \text{or} \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = b.$$

$$Qc = b \quad \text{yields} \quad c = Q^T b. \quad \text{Row by row this is } c_k = q_k^T b.$$

Fourier series is like having a matrix with infinitely many orthogonal columns. Those columns are the basis functions $1, \cos x, \sin x, \ldots$. After dividing by their lengths we have an "infinite orthogonal matrix." Its inverse is its transpose, Q^T. Orthogonality is what reduces a series of terms to one single term, when we integrate.

Problem Set 10.5

1 Integrate the trig identity $2 \cos jx \cos kx = \cos(j + k)x + \cos(j - k)x$ to show that $\cos jx$ is orthogonal to $\cos kx$, provided $j \neq k$. What is the result when $j = k$?

2 Show that $1, x$, and $x^2 - \frac{1}{3}$ are orthogonal, when the integration is from $x = -1$ to $x = 1$. Write $f(x) = 2x^2$ as a combination of those orthogonal functions.

3 Find a vector $(w_1, w_2, w_3, \ldots)$ that is orthogonal to $v = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$. Compute its length $\|w\|$.

4 The first three *Legendre polynomials* are $1, x$, and $x^2 - \frac{1}{3}$. Choose c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. All integrals go from -1 to 1.

5 For the square wave $f(x)$ in Example 3 jumping from 1 to -1, show that

$$\int_0^{2\pi} f(x) \cos x \, dx = 0 \quad \int_0^{2\pi} f(x) \sin x \, dx = 4 \quad \int_0^{2\pi} f(x) \sin 2x \, dx = 0.$$

Which three Fourier coefficients come from those integrals?

6 The square wave has $\|f\|^2 = 2\pi$. Then (6) gives what remarkable sum for π^2?

7 Graph the square wave. Then graph by hand the sum of two sine terms in its series, or graph by machine the sum of 2, 3, and 10 terms. The famous ***Gibbs phenomenon*** is the oscillation that overshoots the jump (this doesn't die down with more terms).

8 Find the lengths of these vectors in Hilbert space:

(a) $v = \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \ldots \right)$

(b) $\boldsymbol{v} = (1, a, a^2, \ldots)$

(c) $f(x) = 1 + \sin x$.

9 Compute the Fourier coefficients a_k and b_k for $f(x)$ defined from 0 to 2π:

(a) $f(x) = 1$ for $0 \le x \le \pi$, $f(x) = 0$ for $\pi < x < 2\pi$

(b) $f(x) = x$.

10 When $f(x)$ has period 2π, why is its integral from $-\pi$ to π the same as from 0 to 2π? If $f(x)$ is an *odd* function, $f(-x) = -f(x)$, show that $\int_{-\pi}^{\pi} f(x)\,dx$ is zero. Odd functions only have sine terms, even functions only have cosines.

11 Using trigonometric identities find the two terms in the Fourier series for $f(x)$:

(a) $f(x) = \cos^2 x$ (b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$ (c) $f(x) = \sin^3 x$

12 The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are a basis for Hilbert space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5 by 5 "differentiation matrix" for these functions?

13 Find the Fourier coefficients a_k and b_k of the square pulse $F(x)$ centered at $x = 0$: $F(x) = 1/h$ for $|x| \le h/2$ and $F(x) = 0$ for $h/2 < |x| \le \pi$.

As $h \to 0$, this $F(x)$ approaches a delta function. Find the limits of a_k and b_k.

Section 4.1 of *Computational Science and Engineering* explains the sine series, cosine series, complete series, and complex series $\Sigma c_k e^{ikx}$ on **math.mit.edu/cse**.

Section 9.3 of this book explains the *Discrete Fourier Transform*. This is "Fourier series for vectors" and it is computed by the **Fast Fourier Transform**. That fast algorithm comes quickly from special complex numbers $z = e^{i\theta} = \cos\theta + i\sin\theta$ when the angle is $\theta = 2\pi k/n$.