10.4 Linear Programming

Linear programming is linear algebra plus two new ideas: ***inequalities*** and ***minimization***. The starting point is still a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. But the only acceptable solutions are *nonnegative*. We require $\boldsymbol{x} \geq 0$ (meaning that no component of $\boldsymbol{x}$ can be negative). The matrix has $n > m$, more unknowns than equations. If there are any solutions $\boldsymbol{x} \geq \mathbf{0}$ to $A\boldsymbol{x} = \boldsymbol{b}$, there are probably a lot. Linear programming picks the solution $\boldsymbol{x}^* \geq \mathbf{0}$ that minimizes the cost:

> ***The cost is*** $c_1x_1 + \cdots + c_nx_n$***. The winning vector*** x^* ***is the nonnegative solution of*** $Ax = b$ ***that has smallest cost.***

Thus a linear programming problem starts with a matrix A and two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$:

i) A has $n > m$: for example $A = [\,1 \ \ 1 \ \ 2\,]$ (one equation, three unknowns)

ii) $\boldsymbol{b}$ has m components for m equations $A\boldsymbol{x} = \boldsymbol{b}$: for example $\boldsymbol{b} = [\,4\,]$

iii) The ***cost vector c*** has n components: for example $\boldsymbol{c} = [\,5 \ \ 3 \ \ 8\,]$.

Then the problem is to minimize $\boldsymbol{c} \cdot \boldsymbol{x}$ subject to the requirements $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$:

Minimize $\;\mathbf{5x_1 + 3x_2 + 8x_3}\;$ ***subject to*** $\;\mathbf{x_1 + x_2 + 2x_3 = 4}\;$ ***and*** $\;\mathbf{x_1, x_2, x_3 \geq 0}$.

We jumped right into the problem, without explaining where it comes from. Linear programming is actually the most important application of mathematics to management. Development of the fastest algorithm and fastest code is highly competitive. You will see that finding $\boldsymbol{x}^*$ is harder than solving $A\boldsymbol{x} = \boldsymbol{b}$, because of the extra requirements: $\boldsymbol{x}^* \geq 0$ and minimum cost $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$. We will explain the background, and the famous *simplex method*, and *interior point methods*, after solving the example.

Look first at the "constraints": $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$. The equation $x_1 + x_2 + 2x_3 = 4$ gives a plane in three dimensions. The nonnegativity $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ chops the plane down to a triangle. The solution $\boldsymbol{x}^*$ must lie in the triangle PQR in Figure 8.6.

Inside that triangle, all components of $\boldsymbol{x}$ are positive. On the edges of PQR, one component is zero. At the corners P and Q and R, two components are zero. ***The optimal solution*** $\boldsymbol{x}^*$ ***will be one of those corners!*** We will now show why.

The triangle contains all vectors $\boldsymbol{x}$ that satisfy $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$. Those $\boldsymbol{x}$'s are called *feasible points*, and the triangle is the ***feasible set***. These points are the allowed candidates in the minimization of $\boldsymbol{c} \cdot \boldsymbol{x}$, which is the final step:

> ***Find*** x^* ***in the triangle*** PQR ***to minimize the cost*** $\mathbf{5x_1 + 3x_2 + 8x_3}$.

The vectors that have *zero* cost lie on the plane $5x_1 + 3x_2 + 8x_3 = 0$. That plane does not meet the triangle. We cannot achieve zero cost, while meeting the requirements on $\boldsymbol{x}$. So increase the cost C until the plane $5x_1 + 3x_2 + 8x_3 = C$ does meet the triangle. As C increases, we have *parallel planes moving toward the triangle*.

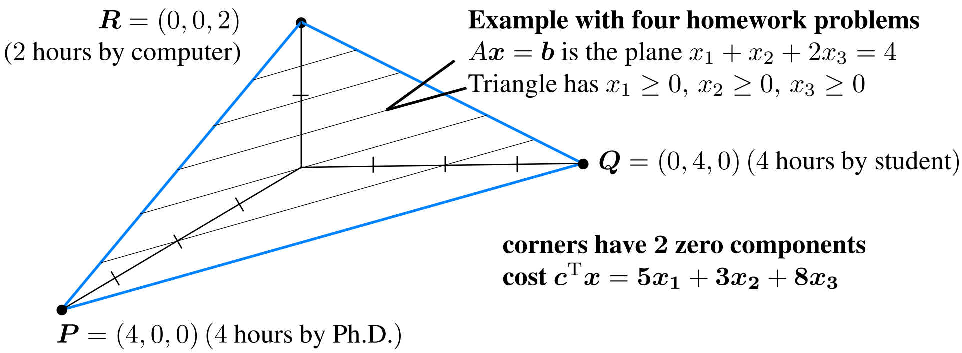

Figure 10.5: The triangle contains all nonnegative solutions: $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$. The lowest cost solution $\boldsymbol{x}^*$ is a corner $\boldsymbol{P}$, $\boldsymbol{Q}$, or $\boldsymbol{R}$ of this feasible set.

The first plane $5x_1 + 3x_2 + 8x_3 = C$ to touch the triangle has minimum cost C. ***The point where it touches is the solution*** $\boldsymbol{x}^*$**.** This touching point must be one of the corners $\boldsymbol{P}$ or $\boldsymbol{Q}$ or $\boldsymbol{R}$. A moving plane could not reach the inside of the triangle before it touches a corner! So check the cost $5x_1 + 3x_2 + 8x_3$ at each corner:

$\boldsymbol{P} = (4,0,0)$ costs 20	$\boldsymbol{Q} = (0,4,0)$ costs 12	$\boldsymbol{R} = (0,0,2)$ costs 16.

The winner is $\boldsymbol{Q}$. Then $\boldsymbol{x}^* = (0,4,0)$ solves the linear programming problem.

If the cost vector $\boldsymbol{c}$ is changed, the parallel planes are tilted. For small changes, $\boldsymbol{Q}$ is still the winner. For the cost $\boldsymbol{c} \cdot \boldsymbol{x} = 5x_1 + 4x_2 + 7x_3$, the optimum $\boldsymbol{x}^*$ moves to $\boldsymbol{R} = (0,0,2)$. The minimum cost is now $7 \cdot 2 = 14$.

Note 1 Some linear programs *maximize profit* instead of minimizing cost. The mathematics is almost the same. The parallel planes start with a large value of C, instead of a small value. They move toward the origin (instead of away), as C gets smaller. *The first touching point is still a corner.*

Note 2 The requirements $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$ could be impossible to satisfy. The equation $x_1 + x_2 + x_3 = -1$ cannot be solved with $\boldsymbol{x} \geq 0$. *That feasible set is empty.*

Note 3 It could also happen that the feasible set is *unbounded*. If the requirement is $x_1 + x_2 - 2x_3 = 4$, the large positive vector $(100, 100, 98)$ is now a candidate. So is the larger vector $(1000, 1000, 998)$. The plane $A\boldsymbol{x} = \boldsymbol{b}$ is no longer chopped off to a triangle. The two corners $\boldsymbol{P}$ and $\boldsymbol{Q}$ are still candidates for $\boldsymbol{x}^*$, but $\boldsymbol{R}$ moved to infinity.

Note 4 With an unbounded feasible set, the minimum cost could be $-\infty$ (*minus infinity*). Suppose the cost is $-x_1 - x_2 + x_3$. Then the vector $(100, 100, 98)$ costs $C = -102$. The vector $(1000, 1000, 998)$ costs $C = -1002$. We are being paid to include x_1 and x_2, instead of paying a cost. In realistic applications this will not happen. But it is theoretically possible that A, $\boldsymbol{b}$, and $\boldsymbol{c}$ can produce unexpected triangles and costs.

The Primal and Dual Problems

This first problem will fit $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ in that example. The unknowns x_1, x_2, x_3 represent hours of work by a Ph.D. and a student and a machine. The costs per hour are \$5, \$3, and \$8. (*I apologize for such low pay.*) The number of hours cannot be negative: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. The Ph.D. and the student get through one homework problem per hour. *The machine solves two problems in one hour.* In principle they can share out the homework, which has four problems to be solved: $x_1 + x_2 + 2x_3 = 4$.

The problem is to finish the four problems at minimum cost $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$.

If all three are working, the job takes one hour: $x_1 = x_2 = x_3 = 1$. The cost is $5 + 3 + 8 = 16$. But certainly the Ph.D. should be put out of work by the student (who is just as fast and costs less—this problem is getting realistic). When the student works two hours and the machine works one, the cost is $6 + 8$ and all four problems get solved. We are on the edge $\boldsymbol{QR}$ of the triangle because the Ph.D. is not working: $x_1 = 0$. But the best point is all work by student (at $\boldsymbol{Q}$) or all work by machine (at $\boldsymbol{R}$). In this example the student solves four problems in four hours for \$12—the minimum cost.

With only one equation in $A\boldsymbol{x} = \boldsymbol{b}$, the corner $(0, 4, 0)$ has only one nonzero component. ***When*** $A\boldsymbol{x} = \boldsymbol{b}$ ***has*** m ***equations, corners have*** m ***nonzeros***. We solve $A\boldsymbol{x} = \boldsymbol{b}$ for those m variables, with $n - m$ free variables set to zero. But unlike Chapter 3, ***we don't know which*** m ***variables to choose***.

The number of possible corners is the number of ways to choose m components out of n. This number "n choose m" is heavily involved in gambling and probability. With $n = 20$ unknowns and $m = 8$ equations (still small numbers), the "feasible set" can have $20!/8!12!$ corners. That number is $(20)(19)\cdots(13) = 5{,}079{,}110{,}400$.

Checking three corners for the minimum cost was fine. Checking five billion corners is not the way to go. The simplex method described below is much faster.

The Dual Problem In linear programming, problems come in pairs. There is a minimum problem and a maximum problem—the original and its "dual." The original problem was specified by a matrix A and two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. The dual problem transposes A and switches $\boldsymbol{b}$ and $\boldsymbol{c}$: ***Maximize*** $\boldsymbol{b} \cdot \boldsymbol{y}$. Here is the dual to our example:

> **A cheater offers to solve homework problems by selling the answers.** The charge is y dollars per problem, or $4y$ altogether. (Note how $\boldsymbol{b} = 4$ has gone into the cost.) The cheater must be as cheap as the Ph.D. or student or machine: $y \leq 5$ and $y \leq 3$ and $2y \leq 8$. (Note how $\boldsymbol{c} = (5, 3, 8)$ has gone into inequality constraints). The cheater maximizes the income $4y$.

Dual Problem ***Maximize*** $\boldsymbol{b} \cdot \boldsymbol{y}$ ***subject to*** $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$.

The maximum occurs when $y = 3$. The income is $4y = 12$. The maximum in the dual problem (\$12) equals the minimum in the original (\$12). *Max* = *min* is duality.

If either problem has a best vector (x^* or y^*) then so does the other.

Minimum cost $c \cdot x^*$ equals maximum income $b \cdot y^*$

This book started with a row picture and a column picture. The first "duality theorem" was about rank: The number of independent rows equals the number of independent columns. That theorem, like this one, was easy for small matrices. Minimum cost = maximum income is proved in our text *Linear Algebra and Its Applications*. One line will establish the easy half of the theorem: ***The cheater's income $b^{\mathrm{T}}y$ cannot exceed the honest cost***:

$$\textit{If}\;\; Ax = b, x \geq 0, A^{\mathrm{T}}y \leq c \;\;\textit{then}\;\; b^{\mathrm{T}}y = (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y) \leq x^{\mathrm{T}}c. \quad (1)$$

The full duality theorem says that when $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ reaches its maximum and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{c}$ reaches its minimum, they are equal: $\boldsymbol{b} \cdot \boldsymbol{y}^* = \boldsymbol{c} \cdot \boldsymbol{x}^*$. Look at the last step in (1), with $\leq$ sign:

The dot product of $\boldsymbol{x} \geq 0$ and $\boldsymbol{s} = \boldsymbol{c} - A^{\mathrm{T}}\boldsymbol{y} \geq 0$ gave $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s} \geq 0$. This is $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{c}$.

Equality needs $x^{\mathrm{T}}s = 0$ So the optimal solution has $x_j^* = 0$ or $s_j^* = 0$ for each j.

The Simplex Method

Elimination is the workhorse for linear equations. The simplex method is the workhorse for linear inequalities. We cannot give the simplex method as much space as elimination, but the idea can be clear. *The simplex method goes from one corner to a neighboring corner of lower cost*. Eventually (and quite soon in practice) it reaches the corner of minimum cost.

A ***corner*** is a vector $\boldsymbol{x} \geq 0$ that satisfies the m equations $A\boldsymbol{x} = \boldsymbol{b}$ with at most m positive components. *The other $n - m$ components are zero*. (Those are the free variables. Back substitution gives the m basic variables. All variables must be nonnegative or $\boldsymbol{x}$ is a false corner.) For a *neighboring corner*, one zero component of x becomes positive and one positive component becomes zero.

The simplex method must decide which component "enters" by becoming positive, and which component "leaves" by becoming zero. That exchange is chosen so as to lower the total cost. This is one step of the simplex method, moving toward x^*.

Here is the overall plan. Look at each zero component at the current corner. If it changes from 0 to 1, the other nonzeros have to adjust to keep $A\boldsymbol{x} = \boldsymbol{b}$. Find the new $\boldsymbol{x}$ by back substitution and compute the change in the total cost $\boldsymbol{c} \cdot \boldsymbol{x}$. This change is the "reduced cost" r of the new component. The ***entering variable*** is the one that gives the ***most negative*** r. This is the greatest cost reduction for a single unit of a new variable.

Example 1 Suppose the current corner is $\boldsymbol{P} = (4, 0, 0)$, with the Ph.D. doing all the work (the cost is \$20). If the student works one hour, the cost of $\boldsymbol{x} = (3, 1, 0)$ is down to \$18. The reduced cost is $r = -2$. If the machine works one hour, then $\boldsymbol{x} = (2, 0, 1)$ also costs \$18. The reduced cost is also $r = -2$. In this case the simplex method can choose either the student or the machine as the entering variable.

Even in this small example, the first step may not go immediately to the best $\boldsymbol{x}^*$. The method chooses the entering variable before it knows how much of that variable to include. We computed r when the entering variable changes from 0 to 1, but one unit may be too much or too little. The method now chooses the leaving variable (the Ph.D.). It moves to corner $\boldsymbol{Q}$ or $\boldsymbol{R}$ in the figure.

The more of the entering variable we include, the lower the cost. This has to stop when one of the positive components (which are adjusting to keep $A\boldsymbol{x} = \boldsymbol{b}$) hits zero. *The* ***leaving variable*** *is the first positive x_i to reach zero.* When that happens, a neighboring corner has been found. Then start again (from the new corner) to find the next variables to enter and leave.

When all reduced costs are positive, the current corner is the optimal $\boldsymbol{x}^*$. No zero component can become positive without increasing $\boldsymbol{c} \cdot \boldsymbol{x}$. No new variable should enter. The problem is solved (and we can show that $\boldsymbol{y}^*$ is found too).

Note Generally $\boldsymbol{x}^*$ is reached in αn steps, where α is not large. But examples have been invented which use an exponential number of simplex steps. Eventually a different approach was developed, which is guaranteed to reach $\boldsymbol{x}^*$ in fewer (but more difficult) steps. The new methods travel through the *interior* of the feasible set.

Example 2 Minimize the cost $\boldsymbol{c} \cdot \boldsymbol{x} = 3x_1 + x_2 + 9x_3 + x_4$. The constraints are $\boldsymbol{x} \geq 0$ and two equations $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 4 \qquad & m &= 2 \quad \text{equations} \\ x_2 + x_3 - x_4 &= 2 \qquad & n &= 4 \quad \text{unknowns.} \end{aligned}$$

A starting corner is $\boldsymbol{x} = (4, 2, 0, 0)$ which costs $\boldsymbol{c} \cdot \boldsymbol{x} = 14$. It has $m = 2$ nonzeros and $n - m = 2$ zeros. The zeros are x_3 and x_4. The question is whether x_3 or x_4 should enter (become nonzero). Try one unit of each of them:

If $x_3 = 1$ and $x_4 = 0$, then $\boldsymbol{x} = (2, 1, 1, 0)$ costs 16.

If $x_4 = 1$ and $x_3 = 0$, then $\boldsymbol{x} = (3, 3, 0, 1)$ costs 13.

Compare those costs with 14. The reduced cost of x_3 is $r = 2$, positive and useless. The reduced cost of x_4 is $r = -1$, negative and helpful. *The entering variable is x_4.*

How much of x_4 can enter? One unit of x_4 made x_1 drop from 4 to 3. Four units will make x_1 drop from 4 to zero (while x_2 increases all the way to 6). *The leaving variable is* x_1. The new corner is $\boldsymbol{x} = (0, 6, 0, 4)$, which costs only $\boldsymbol{c} \cdot \boldsymbol{x} = 10$. This is the optimal $\boldsymbol{x}^*$, but to know that we have to try another simplex step from $(0, 6, 0, 4)$. Suppose x_1 or x_3 tries to enter:

Start from the	If $x_1 = 1$ and $x_3 = 0$, then $\boldsymbol{x} = (1, 5, 0, 3)$ costs 11.
corner $(\mathbf{0}, \mathbf{6}, \mathbf{0}, \mathbf{4})$	If $x_3 = 1$ and $x_1 = 0$, then $\boldsymbol{x} = (0, 3, 1, 2)$ costs 14.

Those costs are higher than 10. Both r's are positive—it does not pay to move. The current corner $(0, 6, 0, 4)$ is the solution $\boldsymbol{x}^*$.

These calculations can be streamlined. Each simplex step solves three linear systems with the same matrix B. (This is the m by m matrix that keeps the m basic columns of A.) When a column enters and an old column leaves, there is a quick way to update B^{-1}. That is how most codes organize the simplex method.

Our text on *Computational Science and Engineering* includes a short code with comments. (The code is also on **math.mit.edu/cse**) The best $\boldsymbol{y}^*$ solves m equations $A^{\mathrm{T}}\boldsymbol{y}^* = \boldsymbol{c}$ in the m components that are nonzero in $\boldsymbol{x}^*$. Then we have optimality $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s} = 0$ and this is duality: *Either* $x_j^* = 0$ *or the "slack" in* $\boldsymbol{s}^* = \boldsymbol{c} - A^{\mathrm{T}}y^*$ *has* $s_j^* = 0$.

When $\boldsymbol{x}^* = (0, 4, 0)$ was the optimal corner $\boldsymbol{Q}$, the cheater's price was set by $y^* = 3$.

Interior Point Methods

The simplex method moves along the edges of the feasible set, eventually reaching the optimal corner $\boldsymbol{x}^*$. ***Interior point methods move inside the feasible set*** (where $\boldsymbol{x} > \boldsymbol{0}$). These methods hope to go more directly to $\boldsymbol{x}^*$. They work well.

One way to stay inside is to put a barrier at the boundary. Add extra cost as a *logarithm that blows up* when any variable x_j touches zero. The best vector has $\boldsymbol{x} > \boldsymbol{0}$. The number θ is a small parameter that we move toward zero.

Barrier problem **Minimize** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\,(\log x_1 + \cdots + \log x_n)$ **with** $A\boldsymbol{x} = \boldsymbol{b}$ (2)

This cost is nonlinear (but linear programming is already nonlinear from inequalities). The constraints $x_j \geq 0$ are not needed because $\log x_j$ becomes infinite at $x_j = 0$.

The barrier gives an *approximate problem* for each θ. The m constraints $A\boldsymbol{x} = \boldsymbol{b}$ have Lagrange multipliers $y_1, \ldots, y_m$. This is the good way to deal with constraints.

$\boldsymbol{y}$ from Lagrange $$L(x, y, \theta) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\,(\textstyle\sum \log x_i) - \boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{x} - \boldsymbol{b}) \quad (3)$$

$\partial L/\partial y = 0$ brings back $A\boldsymbol{x} = \boldsymbol{b}$. The derivatives $\partial L/\partial x_j$ are interesting !

Optimality in barrier pbm $$\frac{\partial L}{\partial x_j} = c_j - \frac{\theta}{x_j} - (A^{\mathrm{T}}y)_j = 0 \quad \text{which is} \quad \boldsymbol{x_j s_j = \theta}. \quad (4)$$

The true problem has $x_j s_j = 0$. The barrier problem has $x_j s_j = \theta$. The solutions $\boldsymbol{x}^*(\theta)$ lie on the ***central path*** to $\boldsymbol{x}^*(0)$. Those n optimality equations $x_j s_j = \theta$ are nonlinear, and we solve them iteratively by Newton's method.

The current $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}$ will satisfy $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ and $A^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$, *but not* $x_j s_j = \theta$. Newton's method takes a step $\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}$. By ignoring the second-order term $\Delta\boldsymbol{x}\Delta\boldsymbol{s}$

in $(\boldsymbol{x}+\Delta\boldsymbol{x})(\boldsymbol{s}+\Delta\boldsymbol{s})=\boldsymbol{\theta}$, the corrections in $\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}$ come from linear equations:

Newton step

$$\begin{aligned} A\,\Delta\boldsymbol{x} &= 0 \\ A^{\mathrm{T}}\Delta\boldsymbol{y}+\Delta\boldsymbol{s} &= 0 \\ s_j\Delta x_j + x_j\Delta s_j &= \theta - x_j s_j \end{aligned} \tag{5}$$

Newton iteration has quadratic convergence for each θ, and then θ approaches zero. The duality gap $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s}$ generally goes below 10^{-8} after 20 to 60 steps. The explanation in my *Computational Science and Engineering* textbook takes one Newton step in detail, for the example with four homework problems. I didn't intend that the student should end up doing all the work, but $\boldsymbol{x}^*$ turned out that way.

This interior point method is used almost "as is" in commercial software, for a large class of linear and nonlinear optimization problems.

Problem Set 10.4

1 Draw the region in the xy plane where $x+2y=6$ and $x\geq 0$ and $y\geq 0$. Which point in this "feasible set" minimizes the cost $c=x+3y$? Which point gives maximum cost? Those points are at corners.

2 Draw the region in the xy plane where $x+2y\leq 6,\ 2x+y\leq 6,\ x\geq 0,\ y\geq 0$. It has four corners. Which corner minimizes the cost $c=2x-y$?

3 What are the corners of the set $x_1+2x_2-x_3=4$ with x_1,x_2,x_3 all ≥ 0? Show that the cost x_1+2x_3 can be very negative in this feasible set. This is an example of unbounded cost: no minimum.

4 Start at $\boldsymbol{x}=(0,0,2)$ where the machine solves all four problems for \$16. Move to $\boldsymbol{x}=(0,1,\quad)$ to find the reduced cost r (the savings per hour) for work by the student. Find r for the Ph.D. by moving to $\boldsymbol{x}=(1,0,\quad)$ with 1 hour of Ph.D. work.

5 Start Example 1 from the Ph.D. corner $(4,0,0)$ with $\boldsymbol{c}$ changed to $[\,5\quad 3\quad 7\,]$. Show that r is better for the machine even when the total cost is lower for the student. The simplex method takes two steps, first to the machine and then to the student for $\boldsymbol{x}^*$.

6 Choose a different cost vector $\boldsymbol{c}$ so the Ph.D. gets the job. Rewrite the dual problem (maximum income to the cheater).

7 A six-problem homework on which the Ph.D. is fastest gives a second constraint $2x_1+x_2+x_3=6$. Then $\boldsymbol{x}=(2,2,0)$ shows two hours of work by Ph.D. and student on each homework. Does this $\boldsymbol{x}$ minimize the cost $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $\boldsymbol{c}=(5,3,8)$?

8 These two problems are also dual. Prove weak duality, that always $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}\leq\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$:

Primal problem Minimize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $A\boldsymbol{x}\geq\boldsymbol{b}$ and $\boldsymbol{x}\geq\mathbf{0}$.
Dual problem Maximize $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ with $A^{\mathrm{T}}\boldsymbol{y}\leq\boldsymbol{c}$ and $\boldsymbol{y}\geq\mathbf{0}$.