10.3 Markov Matrices, Population, and Economics

This section is about ***positive matrices***: every $a_{ij} > 0$. The key fact is quick to state: ***The largest eigenvalue is real and positive and so is its eigenvector.*** In economics and ecology and population dynamics and random walks, that fact leads a long way:

Markov $\lambda_{\text{max}} = 1$ **Population** $\lambda_{\text{max}} > 1$ **Consumption** $\lambda_{\text{max}} < 1$

λ_{max} controls the powers of A. We will see this first for $\lambda_{\text{max}} = 1$.

Markov Matrices

Multiply a positive vector $\boldsymbol{u}_0$ again and again by this matrix A:

Markov matrix $$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad \boldsymbol{u}_1 = A\boldsymbol{u}_0 \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$$

After k steps we have $A^k\boldsymbol{u}_0$. The vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots$ will approach a "*steady state*" $\boldsymbol{u}_\infty = (.6, .4)$. This final outcome does not depend on the starting vector $\boldsymbol{u}_0$. ***For every*** $\boldsymbol{u}_0 = (a, 1-a)$ ***we converge to the same*** $\boldsymbol{u}_{\infty(.6,.4)}$. The question is why.

The steady state equation $A\boldsymbol{u}_\infty = \boldsymbol{u}_\infty$ makes $\boldsymbol{u}_\infty$ ***an eigenvector with eigenvalue*** $\mathbf{1}$:

Steady state $$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{u}_\infty.$$

Multiplying by A does not change $\boldsymbol{u}_\infty$. But this does not explain why so many vectors $\boldsymbol{u}_0$ lead to $\boldsymbol{u}_\infty$. Other examples might have a steady state, but it is not necessarily attractive:

Not Markov $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has the unattractive steady state $B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In this case, the starting vector $\boldsymbol{u}_0 = (0, 1)$ will give $\boldsymbol{u}_1 = (0, 2)$ and $\boldsymbol{u}_2 = (0, 4)$. The second components are doubled. In the language of eigenvalues, B has $\lambda = 1$ but also $\lambda = 2$— this produces instability. The component of $\boldsymbol{u}$ along that unstable eigenvector is multiplied by λ, and $|\lambda| > 1$ means blowup.

This section is about two special properties of A that guarantee a *stable steady state*. These properties define a positive ***Markov matrix***, and A above is one particular example:

Markov matrix	**1.** ***Every entry of*** A ***is positive***: $a_{ij} > 0$. **2.** ***Every column of*** A ***adds to*** $\mathbf{1}$.

Column 2 of B adds to 2, not 1. When A is a Markov matrix, two facts are immediate:
Because of **1**: Multiplying $\boldsymbol{u}_0 \geq 0$ by A produces a nonnegative $\boldsymbol{u}_1 = A\boldsymbol{u}_0 \geq 0$.
Because of **2**: If the components of $\boldsymbol{u}_0$ add to 1, so do the components of $\boldsymbol{u}_1 = A\boldsymbol{u}_0$.

Reason: The components of $\boldsymbol{u}_0$ add to 1 when $\begin{bmatrix}1 & \dots & 1\end{bmatrix}\boldsymbol{u}_0 = 1$. This is true for each column of A by Property 2. Then by matrix multiplication $[1 \ \dots \ 1]A = [1 \ \dots \ 1]$:

$$\textbf{Components of } \boldsymbol{Au_0} \textbf{ add to 1} \qquad [1 \ \dots \ 1]Au_0 = [1 \ \dots \ 1]u_0 = 1.$$

The same facts apply to $\boldsymbol{u}_2 = A\boldsymbol{u}_1$ and $\boldsymbol{u}_3 = A\boldsymbol{u}_2$. ***Every vector*** $A^k\boldsymbol{u}_0$ ***is nonnegative with components adding to*** 1. These are "***probability vectors***." The limit $\boldsymbol{u}_\infty$ is also a probability vector—but we have to prove that there is a limit. We will show that $\lambda_{\max} = 1$ for a positive Markov matrix.

Example 1 The fraction of rental cars in Denver starts at $\frac{1}{50} = .02$. The fraction outside Denver is .98. Every month, 80% of the Denver cars stay in Denver (and 20% leave). Also 5% of the outside cars come in (95% stay outside). This means that the fractions $\boldsymbol{u}_0 = (.02, .98)$ are multiplied by A:

$$\textbf{First month} \qquad A = \begin{bmatrix} \mathbf{.80} & \mathbf{.05} \\ \mathbf{.20} & \mathbf{.95} \end{bmatrix} \quad \text{leads to} \quad \boldsymbol{u}_1 = A\boldsymbol{u}_0 = A\begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .065 \\ .935 \end{bmatrix}.$$

Notice that $.065 + .935 = 1$. All cars are accounted for. Each step multiplies by A:

$$\textbf{Next month} \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = (.09875, \ .90125). \ \text{This is } A^2\boldsymbol{u}_0.$$

All these vectors are positive because A is positive. Each vector $\boldsymbol{u}_k$ will have its components adding to 1. The first component has grown from .02 and cars are moving toward Denver. What happens in the long run?

This section involves powers of matrices. The understanding of A^k was our first and best application of diagonalization. Where A^k can be complicated, the diagonal matrix Λ^k is simple. The eigenvector matrix X connects them: A^k equals $X\Lambda^k X^{-1}$. The new application to Markov matrices uses the eigenvalues (in Λ) and the eigenvectors (in X). We will show that $\boldsymbol{u}_\infty$ **is an eigenvector of** $\boldsymbol{A}$ **corresponding to** $\lambda = 1$.

Since every column of A adds to 1, nothing is lost or gained. We are moving rental cars or populations, and no cars or people suddenly appear (or disappear). The fractions add to 1 and the matrix A keeps them that way. The question is how they are distributed after k time periods—which leads us to A^k.

Solution $A^k\boldsymbol{u}_0$ gives the fractions in and out of Denver after k steps. We diagonalize A to understand A^k. The eigenvalues are $\lambda = \mathbf{1}$ and $\mathbf{.75}$ (the trace is 1.75).

$$Ax = \lambda x \qquad A\begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} \qquad \textbf{and} \qquad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = .75\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The starting vector $\boldsymbol{u}_0$ combines $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, in this case with coefficients 1 and .18:

$$\textbf{Combination of eigenvectors} \qquad \boldsymbol{u}_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now multiply by A to find $\boldsymbol{u}_1$. The eigenvectors are multiplied by $\lambda_1 = 1$ and $\lambda_2 = .75$:

$$\textbf{Each } \boldsymbol{x} \textbf{ is multiplied by } \lambda \qquad \boldsymbol{u}_1 = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Every month, another $\lambda = .75$ multiplies the vector $\boldsymbol{x}_2$. The eigenvector $\boldsymbol{x}_1$ is unchanged:

After k steps $$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = 1^k\begin{bmatrix}.2\\.8\end{bmatrix} + (.75)^k(.18)\begin{bmatrix}-1\\1\end{bmatrix}.$$

This equation reveals what happens. ***The eigenvector $\boldsymbol{x}_1$ with $\lambda = 1$ is the steady state***. The other eigenvector $\boldsymbol{x}_2$ disappears because $|\lambda| < 1$. The more steps we take, the closer we come to $\boldsymbol{u}_\infty = (.2, .8)$. In the limit, $\frac{2}{10}$ of the cars are in Denver and $\frac{8}{10}$ are outside. This is the pattern for Markov chains, even starting from $\boldsymbol{u}_0 = (0, 1)$:

> If A is a *positive* Markov matrix (entries $a_{ij} > 0$, each column adds to $\mathbf{1}$), then $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvector $\boldsymbol{x}_1$ is the ***steady state***:
>
> $$\boldsymbol{u_k = x_1 + c_2(\lambda_2)^k x_2 + \cdots + c_n(\lambda_n)^k x_n} \quad \textbf{\textit{always approaches}} \quad \boldsymbol{u_\infty = x_1}.$$

The first point is to see that $\lambda = 1$ is an eigenvalue of A. *Reason*: Every column of $A - I$ adds to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. Those rows are linearly dependent, so $A - I$ is singular. Its determinant is zero and $\lambda = 1$ is an eigenvalue.

The second point is that no eigenvalue can have $|\lambda| > 1$. With such an eigenvalue, the powers A^k would grow. But A^k is also a Markov matrix! A^k has positive entries still adding to 1—and that leaves no room to get large.

A lot of attention is paid to the possibility that another eigenvalue has $|\lambda| = 1$.

Example 2 $A = \begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{bmatrix}$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside, and vice versa. The powers A^k alternate between A and I. The second eigenvector $\boldsymbol{x}_2 = (-1, 1)$ will be multiplied by $\lambda_2 = -1$ at every step—and does not become smaller: No steady state.

Suppose the entries of A or any power of A are all *positive*—zero is not allowed. In this "regular" or "primitive" case, $\lambda = 1$ is strictly larger than any other eigenvalue. The powers A^k approach the rank one matrix that has the steady state in every column.

Example 3 ("**Everybody moves**") Start with three groups. At each time step, half of group 1 goes to group 2 and the other half goes to group 3. The other groups also *split in half and move*. Take one step from the starting populations p_1, p_2, p_3:

New populations $$\boldsymbol{u}_1 = A\boldsymbol{u}_0 = \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}p_2 + \frac{1}{2}p_3\\ \frac{1}{2}p_1 + \frac{1}{2}p_3\\ \frac{1}{2}p_1 + \frac{1}{2}p_2\end{bmatrix}.$$

A is a Markov matrix. Nobody is born or lost. A contains zeros, which gave trouble in Example 2. But after two steps in this new example, the zeros disappear from A^2:

Two-step matrix $$\boldsymbol{u}_2 = A^2\boldsymbol{u}_0 = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ (because A is Markov) and $\lambda_2 = \lambda_3 = -\frac{1}{2}$. For $\lambda = 1$, ***the eigenvector*** $\boldsymbol{x}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ***will be the steady state***. When three equal populations split in half and move, the populations are again equal. Starting from $\boldsymbol{u}_0 = (8, 16, 32)$, the Markov chain approaches its steady state:

$$\boldsymbol{u}_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \qquad \boldsymbol{u}_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 16 \\ 18 \\ 22 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}.$$

The step to $\boldsymbol{u}_4$ will split some people in half. This cannot be helped. The total population is $8 + 16 + 32 = 56$ at every step. The steady state is 56 times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. You can see the three populations approaching, but never reaching, their final limits $56/3$.

Challenge Problem 6.7.16 created a Markov matrix A from the number of links between websites. The steady state $\boldsymbol{u}$ will give the Google rankings. ***Google finds*** $\boldsymbol{u}_\infty$ ***by a random walk that follows links*** (*random surfing*). That eigenvector comes from counting the fraction of visits to each website—a quick way to compute the steady state.

The size $|\lambda_2|$ of the second eigenvalue controls the speed of convergence to steady state.

Perron-Frobenius Theorem

One matrix theorem dominates this subject. The Perron-Frobenius Theorem applies when all $a_{ij} \geq 0$. There is no requirement that columns add to 1. We prove the neatest form, when all $a_{ij} > 0$: any positive matrix A (not necessarily positive definite!).

Perron-Frobenius for $\boldsymbol{A > 0}$	***All numbers in*** $A\boldsymbol{x} = \lambda_{\text{max}}\boldsymbol{x}$ ***are strictly positive.***

Proof The key idea is to look at all numbers t such that $A\boldsymbol{x} \geq t\boldsymbol{x}$ for some nonnegative vector $\boldsymbol{x}$ (other than $\boldsymbol{x} = \boldsymbol{0}$). We are allowing inequality in $A\boldsymbol{x} \geq t\boldsymbol{x}$ in order to have many small positive candidates t. For the largest value t_{max} (which is attained), we will show that ***equality holds***: $A\boldsymbol{x} = t_{\text{max}}\boldsymbol{x}$.

Otherwise, if $A\boldsymbol{x} \geq t_{\text{max}}\boldsymbol{x}$ is not an equality, multiply by A. Because A is positive that produces a strict inequality $A^2\boldsymbol{x} > t_{\text{max}}A\boldsymbol{x}$. Therefore the positive vector $\boldsymbol{y} = A\boldsymbol{x}$ satisfies $A\boldsymbol{y} > t_{\text{max}}\boldsymbol{y}$, and t_{max} could be increased. This contradiction forces the equality $A\boldsymbol{x} = t_{\text{max}}\boldsymbol{x}$, and *we have an eigenvalue*. Its eigenvector $\boldsymbol{x}$ is positive because on the left side of that equality, $A\boldsymbol{x}$ is sure to be positive.

To see that no eigenvalue can be larger than t_{max}, suppose $A\boldsymbol{z} = \lambda\boldsymbol{z}$. Since λ and $\boldsymbol{z}$ may involve negative or complex numbers, we take absolute values: $|\lambda||\boldsymbol{z}| = |A\boldsymbol{z}| \leq A|\boldsymbol{z}|$ by the "triangle inequality." This $|\boldsymbol{z}|$ is a nonnegative vector, so this $|\lambda|$ is one of the possible candidates t. Therefore $|\lambda|$ cannot exceed t_{max}—which must be λ_{max}.

Population Growth

Divide the population into three age groups: age < 20, age 20 to 39, and age 40 to 59. At year T the sizes of those groups are n_1, n_2, n_3. Twenty years later, the sizes have changed for three reasons: births, deaths, and getting older.

1. **Reproduction** $n_1^{\textbf{new}} = F_1\,n_1 + F_2\,n_2 + F_3\,n_3$ gives a new generation
2. **Survival** $n_2^{\textbf{new}} = P_1 n_1$ and $n_3^{\textbf{new}} = P_2 n_2$ gives the older generations

The fertility rates are F_1, F_2, F_3 (F_2 largest). The *Leslie matrix* A might look like this:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{\textbf{new}} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} .04 & \mathbf{1.1} & .01 \\ .98 & 0 & 0 \\ 0 & \mathbf{.92} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ \boldsymbol{n_2} \\ n_3 \end{bmatrix}.$$

This is population projection in its simplest form, the same matrix A at every step. In a realistic model, A will change with time (from the environment or internal factors). Professors may want to include a fourth group, age ≥ 60, but we don't allow it.

The matrix has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem still applies because $A^3 > 0$. The largest eigenvalue is $\lambda_{\text{max}} \approx 1.06$. You can watch the generations move, starting from $n_2 = 1$ in the middle generation:

$$\textbf{eig}(A) = \begin{matrix} \mathbf{1.06} \\ -1.01 \\ -0.01 \end{matrix} \qquad A^2 = \begin{bmatrix} 1.08 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{1.08} & .01 \\ 0.90 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0.10 & \mathbf{1.19} & .01 \\ 0.06 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{0.99} & .01 \end{bmatrix}.$$

A fast start would come from $\boldsymbol{u}_0 = (0, 1, 0)$. That middle group will reproduce 1.1 and also survive $.92$. The newest and oldest generations are in $\boldsymbol{u}_1 = (1.1, 0, .92) =$ column 2 of A. Then $\boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$ is the second column of A^2. The early numbers (transients) depend a lot on $\boldsymbol{u}_0$, but ***the asymptotic growth rate*** λ_{max} ***is the same from every start***. Its eigenvector $\boldsymbol{x} = (.63, .58, .51)$ shows all three groups growing steadily together.

Caswell's book on *Matrix Population Models* emphasizes sensitivity analysis. The model is never exactly right. If the F's or P's in the matrix change by 10%, does λ_{max} go below 1 (which means extinction)? Problem 19 will show that a matrix change ΔA produces an eigenvalue change $\Delta\lambda = \boldsymbol{y}^{\mathrm{T}}(\Delta A)\boldsymbol{x}$. Here $\boldsymbol{x}$ and $\boldsymbol{y}^{\mathrm{T}}$ are the right and left eigenvectors of A, with $A\boldsymbol{x} = d\boldsymbol{x}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$.

Linear Algebra in Economics: The Consumption Matrix

A long essay about linear algebra in economics would be out of place here. A short note about one matrix seems reasonable. The ***consumption matrix*** tells how much of each input goes into a unit of output. This describes the manufacturing side of the economy.

Consumption matrix We have n industries like chemicals, food, and oil. To produce a unit of chemicals may require .2 units of chemicals, .3 units of food, and .4 units of oil. Those numbers go into row 1 of the consumption matrix A:

$$\begin{bmatrix} \text{chemical output} \\ \text{food output} \\ \text{oil output} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{chemical input} \\ \text{food input} \\ \text{oil input} \end{bmatrix}.$$

Row 2 shows the inputs to produce food—a heavy use of chemicals and food, not so much oil. Row 3 of A shows the inputs consumed to refine a unit of oil. The real consumption matrix for the United States in 1958 contained 83 industries. The models in the 1990's are much larger and more precise. We chose a consumption matrix that has a convenient eigenvector.

Now comes the question: Can this economy meet demands y_1, y_2, y_3 for chemicals, food, and oil? To do that, the inputs p_1, p_2, p_3 will have to be higher—because part of $\boldsymbol{p}$ is consumed in producing $\boldsymbol{y}$. The input is $\boldsymbol{p}$ and the consumption is $A\boldsymbol{p}$, which leaves the output $\boldsymbol{p} - A\boldsymbol{p}$. This net production is what meets the demand $\boldsymbol{y}$:

Problem Find a vector $\boldsymbol{p}$ such that $\boxed{\boldsymbol{p} - A\boldsymbol{p} = \boldsymbol{y}}$ or $\boxed{\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}.}$

Apparently the linear algebra question is whether $I - A$ is invertible. But there is more to the problem. The vector $\boldsymbol{y}$ of required outputs is nonnegative, and so is A. *The production levels in $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$ must also be nonnegative*. The real question is:

When is $(I - A)^{-1}$ a nonnegative matrix?

This is the test on $(I - A)^{-1}$ for a productive economy, which can meet any demand. If A is small compared to I, then $A\boldsymbol{p}$ is small compared to $\boldsymbol{p}$. There is plenty of output. If A is too large, then production consumes too much and the demand $\boldsymbol{y}$ cannot be met.

"Small" or "large" is decided by the largest eigenvalue λ_1 of A (which is positive):

If $\lambda_1 > 1$ then $(I - A)^{-1}$ has negative entries
If $\lambda_1 = 1$ then $(I - A)^{-1}$ fails to exist
If $\lambda_1 < 1$ then $(I - A)^{-1}$ is nonnegative as desired.

The main point is that last one. The reasoning uses a nice formula for $(I - A)^{-1}$, which we give now. The most important infinite series in mathematics is the ***geometric series*** $1 + x + x^2 + \cdots$. This series adds up to $1/(1 - x)$ provided x lies between -1 and 1. When $x = 1$ the series is $1 + 1 + 1 + \cdots = \infty$. When $|x| \geq 1$ the terms x^n don't go to zero and the series has no chance to converge.

The nice formula for $(I - A)^{-1}$ is the ***geometric series of matrices***:

Geometric series $\boxed{(I - A)^{-1} = I + A + A^2 + A^3 + \cdots .}$

If you multiply the series $S = I + A + A^2 + \cdots$ by A, you get the same series except for I. Therefore $S - AS = I$, which is $(I - A)S = I$. The series adds to $S = (I - A)^{-1}$ if it converges. ***And it converges if all eigenvalues of*** A ***have*** $|\lambda| < 1$.

In our case $A \geq 0$. All terms of the series are nonnegative. Its sum is $(I - A)^{-1} \geq 0$.

Example 4 $A = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix}$ has $\lambda_{\text{max}} = \mathbf{.9}$ and $(I - A)^{-1} = \frac{1}{93}\begin{bmatrix} 41 & 25 & 27 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$.

This economy is productive. A is small compared to I, because $\lambda_{\max}$ is .9. To meet the demand $\boldsymbol{y}$, start from $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$. Then $A\boldsymbol{p}$ is consumed in production, leaving $\boldsymbol{p} - A\boldsymbol{p}$. This is $(I - A)\boldsymbol{p} = \boldsymbol{y}$, and the demand is met.

Example 5 $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has $\lambda_{\text{max}} = \mathbf{2}$ and $(I - A)^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

This consumption matrix A is too large. Demands can't be met, because production consumes more than it yields. The series $I + A + A^2 + \ldots$ does not converge to $(I - A)^{-1}$ because $\lambda_{\text{max}} > 1$. The series is growing while $(I - A)^{-1}$ is actually negative.

In the same way $1 + 2 + 4 + \cdots$ is not really $1/(1 - 2) = -1$. But not entirely false !

Problem Set 10.3

Questions 1–12 are about Markov matrices and their eigenvalues and powers.

1 Find the eigenvalues of this Markov matrix (their sum is the trace):

$$A = \begin{bmatrix} .90 & .15 \\ .10 & .85 \end{bmatrix}.$$

What is the steady state eigenvector for the eigenvalue $\lambda_1 = 1$?

2 Diagonalize the Markov matrix in Problem 1 to $A = X\Lambda X^{-1}$ by finding its other eigenvector:

$$A = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & \\ & .75 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}.$$

What is the limit of $A^k = X\Lambda^k X^{-1}$ when $\Lambda^k = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{.75^k} \end{bmatrix}$ approaches $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$?

3 What are the eigenvalues and steady state eigenvectors for these Markov matrices?

$$A = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \quad A = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

4 For every 4 by 4 Markov matrix, what eigenvector of A^{T} corresponds to the (known) eigenvalue $\lambda = 1$?

5 Every year 2% of young people become old and 3% of old people become dead. (No births.) Find the steady state for

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} .98 & .00 & 0 \\ .02 & .97 & 0 \\ .00 & .03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k}.$$

6 For a Markov matrix, the sum of the components of $\boldsymbol{x}$ equals the sum of the components of $A\boldsymbol{x}$. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\lambda \neq 1$, prove that the components of this non-steady eigenvector $\boldsymbol{x}$ add to zero.

7 Find the eigenvalues and eigenvectors of A. Explain why A^k approaches A^∞:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Challenge problem: Which Markov matrices produce that steady state $(.6, .4)$?

8 The steady state eigenvector of a permutation matrix is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This is *not* approached when $\boldsymbol{u}_0 = (0, 0, 0, 1)$. What are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$? What are the four eigenvalues of P, which solve $\lambda^4 = 1$?

Permutation matrix = Markov matrix $$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

9 Prove that the square of a Markov matrix is also a Markov matrix.

10 If $A = \left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ is a Markov matrix, its eigenvalues are 1 and _____. The steady state eigenvector is $\boldsymbol{x}_1 =$ _____.

11 Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\boldsymbol{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ - & - & - \end{bmatrix}.$$

12 A Markov differential equation is not $d\boldsymbol{u}/dt = A\boldsymbol{u}$ but $d\boldsymbol{u}/dt = (A - I)\boldsymbol{u}$. The diagonal is negative, the rest of $A - I$ is positive. The columns add to zero, not 1.

Find λ_1 and λ_2 for $B = A - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$. Why does $A - I$ have $\lambda_1 = 0$?

When $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ multiply $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, what is the steady state as $t \to \infty$?

Questions 13–15 are about linear algebra in economics.

13 Each row of the consumption matrix in Example 4 adds to .9. Why does that make $\lambda = .9$ an eigenvalue, and what is the eigenvector?

14 Multiply $I + A + A^2 + A^3 + \cdots$ by $I - A$ to get I. The series adds to $(I - A)^{-1}$. For $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, find A^2 and A^3 and use the pattern to add up the series.

15 For which of these matrices does $I + A + A^2 + \cdots$ yield a nonnegative matrix $(I - A)^{-1}$? Then the economy can meet any demand:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}.$$

If the demands are $\boldsymbol{y} = (2, 6)$, what are the vectors $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$?

16 (Markov again) This matrix has zero determinant. What are its eigenvalues?

$$A = \begin{bmatrix} .4 & .2 & .3 \\ .2 & .4 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

Find the limits of $A^k\boldsymbol{u}_0$ starting from $\boldsymbol{u}_0 = (1, 0, 0)$ and then $\boldsymbol{u}_0 = (100, 0, 0)$.

17 If A is a Markov matrix, why doesn't $I + A + A^2 + \cdots$ add up to $(I - A)^{-1}$?

18 For the Leslie matrix show that $\det(A - \lambda I) = 0$ gives $F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2 = \lambda^3$. The right side λ^3 is larger as $\lambda \longrightarrow \infty$. The left side is larger at $\lambda = 1$ if $F_1 + F_2P_1 + F_3P_1P_2 > 1$. In that case the two sides are equal at an eigenvalue $\lambda_{\max} > 1$: *growth.*

19 **Sensitivity of eigenvalues**: A matrix change ΔA produces eigenvalue changes $\Delta\Lambda$. *Those changes* $\Delta\lambda_1, \ldots, \Delta\lambda_n$ *are on the diagonal of* $(X^{-1}\,\Delta A\,X)$. **Challenge**: Start from $AX = X\Lambda$. The eigenvectors and eigenvalues change by ΔX and $\Delta\Lambda$:

$$(A{+}\Delta A)(X{+}\Delta X) = (X{+}\Delta X)(\Lambda{+}\Delta\Lambda) \text{ becomes } A(\Delta X){+}(\Delta A)X = X(\Delta\Lambda){+}(\Delta X)\Lambda.$$

Small terms $(\Delta A)(\Delta X)$ and $(\Delta X)(\Delta\Lambda)$ are ignored. *Multiply the last equation by* X^{-1}. From the inner terms, the diagonal part of $X^{-1}(\Delta A)X$ gives $\Delta\Lambda$ as we want. *Why do the outer terms* $X^{-1}\,A\,\Delta X$ *and* $X^{-1}\,\Delta X\,\Lambda$ *cancel on the diagonal*?

$$\text{Explain } X^{-1}A = \Lambda X^{-1} \text{ and then } \quad \textbf{diag}(\Lambda\,X^{-1}\,\Delta X) = \textbf{diag}(X^{-1}\,\Delta X\,\Lambda).$$

20 Suppose $B > A > 0$, meaning that each $b_{ij} > a_{ij} > 0$. How does the Perron-Frobenius discussion show that $\lambda_{\max}(B) > \lambda_{\max}(A)$?