10.2 Matrices in Engineering

This section will show how engineering problems produce symmetric matrices K (often K is positive definite). The "linear algebra reason" for symmetry and positive definiteness is their form $K = A^{\mathrm{T}}A$ and $K = A^{\mathrm{T}}CA$. The "physical reason" is that the expression $\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}K\boldsymbol{u}$ represents *energy*—and energy is never negative. The matrix C, often diagonal, contains positive physical constants like conductance or stiffness or diffusivity.

Our best examples come from mechanical and civil and aeronautical engineering. K is the ***stiffness matrix***, and $K^{-1}\boldsymbol{f}$ is the structure's response to forces $\boldsymbol{f}$ from outside. Section 10.1 turned to electrical engineering—the matrices came from networks and circuits. The exercises involve chemical engineering and I could go on! Economics and management and engineering design come later in this chapter (the key is optimization).

Engineering leads to linear algebra in two ways, directly and indirectly:

Direct way The physical problem has only a finite number of pieces. The laws connecting their position or velocity are *linear* (movement is not too big or too fast). The laws are expressed by *matrix equations*.

Indirect way The physical system is "continuous". Instead of individual masses, the mass density and the forces and the velocities are functions of x or x, y or x, y, z. The laws are expressed by *differential equations*. ***To find accurate solutions we approximate by finite difference equations or finite element equations***.

Both ways produce matrix equations and linear algebra. I really believe that you cannot do modern engineering without matrices.

Here we present equilibrium equations $K\boldsymbol{u} = \boldsymbol{f}$. With motion, $Md^2\boldsymbol{u}/dt^2 + K\boldsymbol{u} = \boldsymbol{f}$ becomes dynamic. Then we would use eigenvalues from $K\boldsymbol{x} = \lambda M\boldsymbol{x}$, or finite differences.

Differential Equation to Matrix Equation

Differential equations are continuous. Our basic example will be $-d^2u/dx^2 = f(x)$. Matrix equations are discrete. Our basic example will be $K_0\boldsymbol{u} = \boldsymbol{f}$. By taking the step from second derivatives to second differences, you will see the big picture in a very short space. *Start with fixed boundary conditions at both ends* $x = 0$ *and* $x = 1$:

Fixed-fixed boundary value problem
$$-\frac{d^2u}{dx^2} = 1 \text{ with } u(0) = 0 \text{ and } u(1) = 0. \qquad (1)$$

That differential equation is linear. A particular solution is $u_p = -\frac{1}{2}x^2$ (then $d^2u/dx^2 = -1$). We can add any function "in the nullspace". Instead of solving $A\boldsymbol{x} = \boldsymbol{0}$ for a vector $\boldsymbol{x}$, we solve $-d^2u/dx^2 = 0$ for a function $u_n(x)$. (Main point: The right side is zero.)

The nullspace solutions are $u_n(x) = C + Dx$ (a 2-dimensional nullspace for a second order differential equation). The complete solution is $u_p + u_n$:

$$\textbf{Complete solution to} \quad -\frac{d^2u}{dx^2} = 1 \qquad \boxed{u(x) = -\frac{1}{2}x^2 + C + Dx.} \tag{2}$$

Now find C and D from the two boundary conditions: Set $x = 0$ and then $x = 1$. At $x = 0, u(0) = 0$ forces $\boldsymbol{C = 0}$. At $x = 1, u(1) = 0$ forces $-\frac{1}{2} + D = 0$. Then $\boldsymbol{D = \frac{1}{2}}$:

$$u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x = \boldsymbol{\frac{1}{2}(x - x^2)} \text{ solves the fixed-fixed boundary value problem.} \tag{3}$$

Differences Replace Derivatives

To get matrices instead of derivatives, we have three basic choices—*forward or backward or centered differences*. Start with first derivatives and first differences:

$$\frac{\boldsymbol{du}}{\boldsymbol{dx}} \approx \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad \text{or} \quad \frac{u(x) - u(x-\Delta x)}{\Delta x} \quad \text{or} \quad \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}.$$

Between $x = 0$ and $x = 1$, we divide the interval into $n + 1$ equal pieces. The pieces have width $\Delta x = 1/(n + 1)$. The values of u at the n breakpoints $\Delta x, 2\Delta x, \ldots$ will be the unknowns u_1 to u_n in our matrix equation $K\boldsymbol{u} = \boldsymbol{f}$:

Solution to compute: $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \approx (u(\Delta x), u(2\Delta x), \ldots, u(n\Delta x))$.
Zero values $u_0 = u_{n+1} = 0$ come from the boundary conditions $u(0) = u(1) = 0$.

Replace the derivatives in $-\dfrac{d}{dx}\left(\dfrac{du}{dx}\right) = 1$ *by forward and backward differences*:

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4}$$

This is our matrix equation when $n = 3$ and $\Delta x = \frac{1}{4}$. The two first differences are transposes of each other! The equation is $A^{\mathrm{T}}A\boldsymbol{u} = (\Delta x)^2\boldsymbol{f}$. When we multiply $A^{\mathrm{T}}A$, we get the positive definite second difference matrix K_0:

$$\begin{matrix}\boldsymbol{K_0\,u} = \\ \boldsymbol{(\Delta x)^2 f}\end{matrix} \qquad \boxed{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \quad \text{gives} \quad \boxed{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}}. \tag{5}$$

The wonderful fact in this example is that those numbers u_1, u_2, u_3 are exactly correct! They agree with the true solution $u = \frac{1}{2}(x - x^2)$ at the three meshpoints $x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$. Figure 10.3 shows the true solution (continuous curve) and the approximations u_1, u_2, u_3 (lying exactly on the curve). This curve is a parabola.

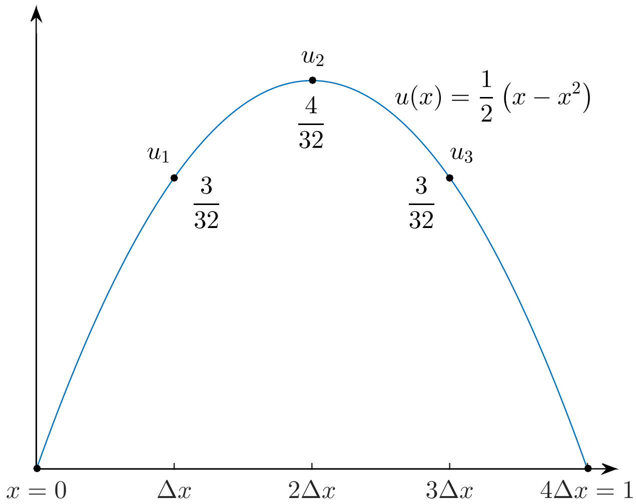

Figure 10.3: Solutions to $-\dfrac{d^2u}{dx^2} = 1$ and $K_0\boldsymbol{u} = \boldsymbol{(\Delta x)^2 f}$ with fixed-fixed boundaries.

How to explain this perfect answer, lying right on the graph of $u(x)$? In the matrix equation, $K_0 = A^{\mathrm{T}}A$ is a "second difference matrix." It gives a centered approximation to $-d^2u/dx^2$. I included the minus sign because the first derivative is *antisymmetric*. The second derivative by itself is *negative*:

$$\text{The "transpose" of } \frac{d}{dx} \text{ is } -\frac{d}{dx}. \text{ Then } \left(-\frac{d}{dx}\right)\left(\frac{d}{dx}\right) \text{ is positive definite.}$$

You can see that in the matrices A and A^{T}. The transpose of $A =$ *forward difference* is $A^{\mathrm{T}} = -$ *backward difference*. I don't want to choose a centered $u(x+\Delta x) - u(x-\Delta x)$. Centered is the best for a first difference, but then the second difference $A^{\mathrm{T}}A$ would stretch from $u(x + \mathbf{2}\Delta x)$ to $u(x - \mathbf{2}\Delta x)$: not good.

Now we can explain the perfect answers, exactly on the true curve $u(x) = \frac{1}{2}\left(x - x^2\right)$. Second differences $-1, 2, -1$ are exactly correct for straight lines $y = x$ and parabolas!

$$\begin{array}{llll} \boldsymbol{y = x} & -\dfrac{d^2y}{dx^2} = \mathbf{0} & -(x+\Delta x) \;\; +2x \;\; -(x-\Delta x) \;\; = & \mathbf{0}(\Delta x)^2 \\ \boldsymbol{y = x^2} & -\dfrac{d^2y}{dx^2} = \mathbf{-2} & -(x+\Delta x)^2 \; +2x^2 \; -(x-\Delta x)^2 = & \mathbf{-2}(\Delta x)^2 \end{array}$$

The miracle continues to $y = x^3$. The correct $-d^2y/dx^2 = -6x$ is produced by second differences. But for $y = x^4$ we return to earth. Second differences don't exactly match $-y'' = -12x^2$. The approximations u_1, u_2, u_3 won't fall on the graph of $u(x)$.

Fixed End and Free End and Variable Coefficient $c(x)$

To see two new possibilities, I will change the equation and also one boundary condition:

$$\boxed{-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right)=f(x)\ \text{ with }\ u(0)=0\ \text{ and }\ \boldsymbol{\frac{du}{dx}(1)=0}.} \tag{6}$$

The end $x=1$ is now **free**. There is no support at that end. "A hanging bar is fixed only at the top." There is no force at the free end $x=1$. That translates to $du/dx=0$ instead of the fixed condition $u=0$ at $x=1$.

The other change is in the coefficient $c(x)=1+x$. The stiffness of the bar is varying as you go from $x=0$ to $x=1$. Maybe its width is changing, or the material changes. This coefficient $1+x$ will bring a new matrix C into the difference equation.

Since u_4 is no longer fixed at 0, it becomes a new unknown. The backward difference A is 4 by 4. And the multiplication by $c(x)=1+x$ becomes a diagonal matrix C—which multiplies by $1+\Delta x,\ldots,1+4\Delta x$ at the meshpoints. Here are A^{T}, C, and A:

$$A^{\mathrm{T}}CA=\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}\begin{bmatrix}1.25&&&\\&1.5&&\\&&1.75&\\&&&2.0\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}. \tag{7}$$

This matrix $K=A^{\mathrm{T}}CA$ will be symmetric and positive definite! Symmetric because $(A^{\mathrm{T}}CA)^{\mathrm{T}}=A^{\mathrm{T}}C^{\mathrm{T}}A^{\mathrm{TT}}=A^{\mathrm{T}}CA$. Positive definite because it passes the energy test: A has independent columns, so $A\boldsymbol{x}\neq\mathbf{0}$ when $\boldsymbol{x}\neq\mathbf{0}$.

$$\text{Energy }=\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}=(A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})>0 \text{ for every } \boldsymbol{x}\neq\mathbf{0},\ \text{because } A\boldsymbol{x}\neq\mathbf{0}.$$

When you multiply the matrices $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}CA$ for this fixed-free combination, watch how 1 replaces 2 in the last corner of $A^{\mathrm{T}}A$. That fourth equation has u_4-u_3, a first (not second) difference coming from the free boundary condition $du/dx=0$.

Notice in $A^{\mathrm{T}}CA$ how c_1,c_2,c_3,c_4 come from $c(x)=1+x$ in equation (7). Previously the c's were simply $1,1,1,1$. Here are the **fixed-free** matrices:

$$A^{\mathrm{T}}A=\begin{bmatrix}2&-1&&\\-1&2&-1&\\&-1&2&-1\\&&-1&\mathbf{1}\end{bmatrix}\qquad A^{\mathrm{T}}CA=\begin{bmatrix}c_1+c_2&-c_2&&\\-c_2&c_2+c_3&-c_3&\\&-c_3&c_3+c_4&-c_4\\&&-c_4&\boldsymbol{c_4}\end{bmatrix}. \tag{8}$$

Free-free Boundary Conditions

Suppose both ends of the bar are free. Now $du/dx = 0$ at both $x = 0$ and $x = 1$. Nothing is holding the bar in place! Physically it is unstable—it can move with no force. Mathematically all constant functions like $u = 1$ satisfy these free conditions. **Algebraically our matrices $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}CA$ will not be invertible**:

Free-free examples
Unknown u_0, u_1, u_2
$\Delta x = 0.5$

$$A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{1} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \mathbf{1} \end{bmatrix} \qquad A^{\mathrm{T}}CA = \begin{bmatrix} \boldsymbol{c_0} & -c_0 & \\ -c_0 & c_0 + c_1 & -c_1 \\ & -c_1 & \boldsymbol{c_1} \end{bmatrix}.$$

The vector $(1, 1, 1)$ is in both nullspaces. This matches $u(x) = 1$ in the continuous problem. Free-free $A^{\mathrm{T}}A\boldsymbol{u} = \boldsymbol{f}$ and $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ are generally unsolvable.

Before explaining more physical examples, may I write down six of the matrices? The tridiagonal K_0 appears many times in this textbook. Now we are seeing its applications. These matrices are all symmetric, and the first four are positive definite:

$$K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} \mathbf{2} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{2} \end{bmatrix} \qquad A_0^{\mathrm{T}}C_0A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & \\ -c_2 & c_2 + c_3 & -c_3 \\ & -c_3 & c_3 + c_4 \end{bmatrix}$$

Fixed-fixed **Spring constants included**

$$K_1 = A_1^{\mathrm{T}}A_1 = \begin{bmatrix} \mathbf{2} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & \\ -c_2 & c_2 + c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix}$$

Fixed-free **Spring constants included**

$$K_{\textbf{singular}} = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad K_{\textbf{circular}} = \begin{bmatrix} 2 & -1 & \mathbf{-1} \\ -1 & 2 & -1 \\ \mathbf{-1} & -1 & 2 \end{bmatrix}$$

Free-free **Periodic $\boldsymbol{u(0) = u(1)}$**

The matrices $K_0, K_1, K_{\textbf{singular}}$, and $K_{\textbf{circular}}$ have $C = I$ for simplicity. This means that all the "spring constants" are $c_i = 1$. We included $A_0^{\mathrm{T}}C_0A_0$ and $A_1^{\mathrm{T}}C_1A_1$ to show how the spring constants enter the matrix (without changing its positive definiteness). Our next goal is to see these same stiffness matrices in other engineering problems.

A Line of Springs and Masses

Figure 10.4 shows three masses m_1, m_2, m_3 connected by a line of springs. The fixed-fixed case has four springs, with top and bottom fixed. That leads to K_0 and $A_0^{\mathrm{T}} C_0 A_0$. The fixed-free case has only three springs; the lowest mass hangs freely. That will lead to K_1 and $A_1^{\mathrm{T}} C_1 A_1$. A **free-free** problem produces $K_{\textbf{singular}}$.

We want equations for the mass movements $\boldsymbol{u}$ and the spring tensions $\boldsymbol{y}$:

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3) = \textbf{movements of the masses} \text{ (down is positive)} \\ \boldsymbol{y} &= (y_1, y_2, y_3, y_4) \text{ or } (y_1, y_2, y_3) = \textbf{tensions in the springs} \end{aligned}$$

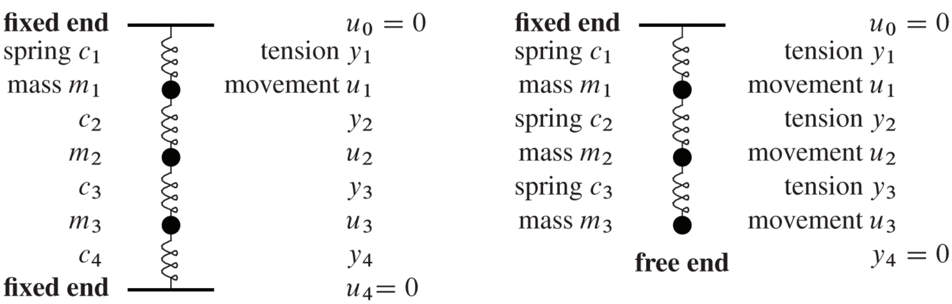

Figure 10.4: Lines of springs and masses: **fixed-fixed** and **fixed-free** ends.

When a mass moves downward, its displacement is positive ($u_j > 0$). For the springs, tension is positive and compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward. Each spring is controlled by its own Hooke's Law $y = c\,e$: (*stretching force* $\boldsymbol{y}$) = (*spring constant* $\boldsymbol{c}$) times (*stretching distance* $\boldsymbol{e}$).

Our job is to link these one-spring equations $y = c\,e$ into a vector equation $K\boldsymbol{u} = \boldsymbol{f}$ for the whole system. The force vector $\boldsymbol{f}$ comes from gravity. The gravitational constant g will multiply each mass to produce downward forces $\boldsymbol{f} = (m_1 g, m_2 g, m_3 g)$.

The real problem is to find the stiffness matrix (**fixed-fixed** and **fixed-free**). The best way to create K is in three steps, not one. Instead of connecting the movements $\boldsymbol{u}_j$ directly to the forces f_i, it is much better to connect each vector to the next in this list:

$$\begin{array}{lcll} u & = & \textit{Movements} \text{ of } n \text{ masses} & = (u_1, \ldots, u_n) \\ e & = & \textit{Elongations} \text{ of } m \text{ springs} & = (e_1, \ldots, e_m) \\ y & = & \textit{Internal forces} \text{ in } m \text{ springs} & = (y_1, \ldots, y_m) \\ f & = & \textit{External forces} \text{ on } n \text{ masses} & = (f_1, \ldots, f_n) \end{array}$$

A great framework for applied mathematics connects $\boldsymbol{u}$ to $\boldsymbol{e}$ to $\boldsymbol{y}$ to $\boldsymbol{f}$. Then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$:

$$\begin{array}{ccc} \boxed{u} & & \boxed{f} \\ A\downarrow & & \uparrow A^{\mathrm{T}} \\ \boxed{e} & \xrightarrow{C} & \boxed{y} \end{array} \qquad \begin{array}{ll} \boldsymbol{e} = A\boldsymbol{u} & A \text{ is } m \text{ by } n \\ \boldsymbol{y} = C\boldsymbol{e} & C \text{ is } m \text{ by } m \\ \boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y} & A^{\mathrm{T}} \text{ is } n \text{ by } m \end{array}$$

We will write down the matrices A and C and A^{T} for the two examples, first with fixed ends and then with the lower end free. Forgive the simplicity of these matrices, it is their form that is so important. Especially the appearance of A together with A^{T}.

The elongation $\boldsymbol{e}$ *is the stretching distance*—how far the springs are extended. Originally there is no stretching—the system is lying on a table. When it becomes vertical and upright, gravity acts. The masses move down by distances u_1, u_2, u_3. Each spring is stretched or compressed by $e_i = u_i - u_{i-1}$, *the difference in displacements of its ends*:

	First spring:	$\boldsymbol{e}_1 = \boldsymbol{u}_1$	(the top is fixed so $u_0 = 0$)
Stretching of	Second spring:	$\boldsymbol{e}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_1$	
each spring	Third spring:	$\boldsymbol{e}_3 = \boldsymbol{u}_3 - \boldsymbol{u}_2$	
	Fourth spring:	$\boldsymbol{e}_4 = \quad - \boldsymbol{u}_3$	(the bottom is fixed so $u_4 = 0$)

If both ends move the same distance, that spring is not stretched: $u_j = u_{j-1}$ and $e_j = 0$. The matrix in those four equations is a 4 by 3 *difference matrix* A, and $\boldsymbol{e} = A\boldsymbol{u}$:

Stretching distances (elongations)

$$\boldsymbol{e} = A\boldsymbol{u} \quad \text{is} \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \qquad (9)$$

The next equation $\boldsymbol{y} = C\boldsymbol{e}$ connects spring elongation $\boldsymbol{e}$ with spring tension $\boldsymbol{y}$. *This is Hooke's Law* $y_i = c_i e_i$ *for each separate spring*. It is the "constitutive law" that depends on the material in the spring. A soft spring has small c, so a moderate force y can produce a large stretching e. Hooke's linear law is nearly exact for real springs, before they are overstretched and the material becomes plastic.

Since each spring has its own law, the matrix in $\boldsymbol{y} = C\boldsymbol{e}$ is a diagonal matrix C:

Hooke's Law $\boldsymbol{y} = C\boldsymbol{e}$

$$\begin{aligned} y_1 &= \boldsymbol{c}_1\boldsymbol{e}_1 \\ y_2 &= \boldsymbol{c}_2\boldsymbol{e}_2 \\ y_3 &= \boldsymbol{c}_3\boldsymbol{e}_3 \\ y_4 &= \boldsymbol{c}_4\boldsymbol{e}_4 \end{aligned} \quad \text{is} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad (10)$$

Combining $\boldsymbol{e} = A\boldsymbol{u}$ with $\boldsymbol{y} = C\boldsymbol{e}$, the spring forces (tension forces) are $\boldsymbol{y} = CA\boldsymbol{u}$.

Finally comes the balance equation, the most fundamental law of applied mathematics. The internal forces from the springs balance the external forces on the masses. Each mass is pulled or pushed by the spring force y_j above it. From below it feels the spring force y_{j+1} plus f_j from gravity. Thus $y_j = y_{j+1} + f_j$ or $f_j = y_j - y_{j+1}$:

Force balance $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$

$$\begin{aligned} \boldsymbol{f}_1 &= \boldsymbol{y}_1 - \boldsymbol{y}_2 \\ \boldsymbol{f}_2 &= \boldsymbol{y}_2 - \boldsymbol{y}_3 \\ \boldsymbol{f}_3 &= \boldsymbol{y}_3 - \boldsymbol{y}_4 \end{aligned} \quad \text{is} \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (11)$$

That matrix is A^{T} ! *The equation for balance of forces is* $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$. Nature transposes the rows and columns of the $\boldsymbol{e}-\boldsymbol{u}$ matrix to produce the $\boldsymbol{f}-\boldsymbol{y}$ matrix. This is the beauty of the framework, that A^{T} appears along with A. The three equations combine into $K\boldsymbol{u} = \boldsymbol{f}$.

$$\left\{\begin{array}{rcl} \boldsymbol{e} & = & A\boldsymbol{u} \\ \boldsymbol{y} & = & C\boldsymbol{e} \\ \boldsymbol{f} & = & A^{\mathrm{T}}\boldsymbol{y} \end{array}\right\} \quad \begin{array}{l} \text{combine into} \quad A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f} \quad \text{or} \quad K\boldsymbol{u} = \boldsymbol{f} \\ K = A^{\mathrm{T}}CA \text{ is the } \textbf{stiffness matrix} \text{ (mechanics)} \\ K = A^{\mathrm{T}}CA \text{ is the } \textbf{conductance matrix} \text{ (networks)} \end{array}$$

Finite element programs spend major effort on assembling $K = A^{\mathrm{T}}CA$ from thousands of smaller pieces. We find K for four springs (**fixed-fixed**) by multiplying A^{T} times CA:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ -c_2 & c_2 & 0 \\ 0 & -c_3 & c_3 \\ 0 & 0 & -c_4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

If all springs are identical, with $c_1 = c_2 = c_3 = c_4 = 1$, then $C = I$. The stiffness matrix reduces to $A^{\mathrm{T}}A$. It becomes the special $-1, 2, -1$ matrix K_0.

Note the difference between $A^{\mathrm{T}}A$ from engineering and LU from linear algebra. The matrix A from four springs is 4 by 3. The triangular matrices from elimination are square. The stiffness matrix K is assembled from $A^{\mathrm{T}}A$, and then broken up into LU. One step is applied mathematics, the other is computational mathematics. Each K is built from rectangular matrices and factored into square matrices.

May I list some properties of $K = A^{\mathrm{T}}CA$? You know almost all of them:

1. K is **tridiagonal**, because mass 3 is not connected to mass 1.
2. K is **symmetric**, because C is symmetric and A^{T} comes with A.
3. K is **positive definite**, because $c_i > 0$ and A has **independent columns**.
4. K^{-1} is a **full matrix** (not sparse) with **all positive entries**.

Property 4 leads to an important fact about $\boldsymbol{u} = K^{-1}\boldsymbol{f}$: *If all forces act downwards* $(f_j > 0)$ *then all movements are downwards* $(u_j > 0)$. Notice that "positive" is different from "positive definite". K^{-1} is positive (K is not). Both are positive definite.

Example 1 Suppose all $c_i = c$ and $m_j = m$. Find the movements $\boldsymbol{u}$ and tensions $\boldsymbol{y}$.

All springs are the same and all masses are the same. But all movements and elongations and tensions will *not* be the same. K^{-1} includes $\frac{1}{c}$ because $A^{\mathrm{T}}CA$ includes c:

$$\textbf{Movements} \qquad \boldsymbol{u} = K^{-1}\boldsymbol{f} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{2} \\ \mathbf{3/2} \end{bmatrix}$$

The displacement u_2, for the mass in the middle, is greater than u_1 and u_3. The units are correct: the force mg divided by force per unit length c gives a length u. Then

$$\textbf{Elongations} \qquad \boldsymbol{e} = A\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{1/2} \\ \mathbf{-1/2} \\ \mathbf{-3/2} \end{bmatrix}.$$

Warning: *Normally you cannot write* $\quad K^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$.

The three matrices are mixed together by $A^{\mathrm{T}}CA$, and they cannot easily be untangled. In general, $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ has many solutions. And four equations $A\boldsymbol{u} = \boldsymbol{e}$ would usually have no solution with three unknowns. But $A^{\mathrm{T}}CA$ gives the correct solution to all three equations in the framework. Only when $m = n$ and the matrices are square can we go from $\boldsymbol{y} = (A^{\mathrm{T}})^{-1}\boldsymbol{f}$ to $\boldsymbol{e} = C^{-1}\boldsymbol{y}$ to $\boldsymbol{u} = A^{-1}\boldsymbol{e}$. We will see that now.

Fixed End and Free End

Remove the fourth spring. All matrices become 3 by 3. The pattern does not change! The matrix A loses its fourth row and (of course) A^{T} loses its fourth column. The new stiffness matrix K_1 becomes a product of square matrices:

$$A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The missing column of A^{T} and row of A multiplied the missing c_4. So the quickest way to find the new $A^{\mathrm{T}}CA$ is to set $c_4 = 0$ in the old one:

$$\textbf{FIXED} \atop \textbf{FREE} \qquad A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}. \tag{12}$$

Example 2 If $c_1 = c_2 = c_3 = 1$ and $C = I$, this is the $-1, 2, -1$ tridiagonal matrix K_1. The last entry of K_1 is 1 instead of 2 because the spring at the bottom is free. Suppose all $m_j = m$:

$$\textbf{Fixed-free} \qquad \boldsymbol{u} = K_1^{-1}\boldsymbol{f} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

Those movements are greater than the free-free case. The number 3 appears in u_1 because all three masses are pulling the first spring down. The next mass moves by that 3 plus an additional 2 from the masses below it. The third mass drops even more $(3 + 2 + 1 = 6)$. The elongations $\boldsymbol{e} = A\boldsymbol{u}$ in the springs display those numbers $3, 2, 1$:

$$\boldsymbol{e} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Two Free Ends: K is Singular

Freedom at *both ends* means trouble. The whole line can move. A is 2 by 3:

$$\textbf{FREE-FREE} \quad \boldsymbol{e} = A\boldsymbol{u} \qquad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{13}$$

Now there is a nonzero solution to $A\boldsymbol{u} = \mathbf{0}$. ***The masses can move with no stretching of the springs***. The whole line can shift by $\boldsymbol{u} = (1,1,1)$ and this leaves $\boldsymbol{e} = (0,0)$:

$$A\boldsymbol{u} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \textbf{no stretching}. \tag{14}$$

$A\boldsymbol{u} = \mathbf{0}$ certainly leads to $A^{\mathrm{T}}CA\boldsymbol{u} = \mathbf{0}$. Then $A^{\mathrm{T}}CA$ is only *positive semidefinite*, without c_1 and c_4. The pivots will be c_2 and c_3 and *no third pivot*. The rank is only 2:

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{15}$$

Two eigenvalues will be positive but $\boldsymbol{x} = (1,1,1)$ is an eigenvector for $\lambda = 0$. We can solve $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ only for special vectors $\boldsymbol{f}$. The forces have to add to $f_1 + f_2 + f_3 = 0$, or the whole line of springs (with both ends free) will take off like a rocket.

Circle of Springs

A third spring will complete the circle from mass 3 back to mass 1. This doesn't make K invertible—the stiffness matrix $K_{circular}$ matrix is still singular:

$$A^{\mathrm{T}}_{\textbf{circular}} A_{\textbf{circular}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{16}$$

The only pivots are 2 and $\frac{3}{2}$. The eigenvalues are 3 and 3 and 0. The determinant is zero. The nullspace still contains $\boldsymbol{x} = (1,1,1)$, when all the masses move together. This movement vector $(1,1,1)$ is in the nullspace of $A_{\textbf{circular}}$ and $K_{\textbf{circular}} = A^{\mathrm{T}}CA$.

May I summarize this section? I hope the example will help you connect calculus with linear algebra, replacing differential equations by difference equations. If your step Δx is small enough, you will have a totally satisfactory solution.

$$\textbf{The equation is} \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \ \ \textbf{with} \ \ u(0) = 0 \ \ \textbf{and} \ \ \left[u(1) \text{ or } \frac{du}{dx}(1)\right] = 0$$

Divide the bar into N pieces of length Δx. Replace du/dx by $A\boldsymbol{u}$ and $-dy/dx$ by $A^{\mathrm{T}}\boldsymbol{y}$. Now A and A^{T} include $1/\Delta x$. The end conditions are $u_0 = 0$ and $[u_N = 0 \text{ or } y_N = 0]$.

The three steps $-d/dx$ and $c(x)$ and d/dx correspond to A^{T} and C and A:

$$\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y} \text{ and } \boldsymbol{y} = C\boldsymbol{e} \text{ and } \boldsymbol{e} = A\boldsymbol{u} \text{ give } A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}.$$

This is a fundamental example in computational science and engineering.

1. Model the problem by a differential equation

2. Discretize the differential equation to a difference equation

3. Understand and solve the difference equation (and boundary conditions!)

4. Interpret the solution; visualize it; redesign if needed.

Numerical simulation has become a third branch of science, beside experiment and deduction. Computer design of the Boeing 777 was much less expensive than a wind tunnel.

The two texts *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) develop this whole subject further—see the course page **math.mit.edu/18085** with video lectures (The lectures are also on **ocw.mit.edu** and **YouTube**). I hope this book helps you to see the framework behind the computations.

Problem Set 10.2

1 Show that $\det A_0^{\mathrm{T}}C_0A_0 = c_1c_2c_3+c_1c_3c_4+c_1c_2c_4+c_2c_3c_4$. Find also $\det A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example.

2 Invert $A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example by multiplying $A_1^{-1}C_1^{-1}(A_1^{\mathrm{T}})^{-1}$.

3 In the free-free case when $A^{\mathrm{T}}CA$ in equation (15) is singular, add the three equations $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ to show that we need $f_1+f_2+f_3 = 0$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ when the forces $\boldsymbol{f} = (-1, 0, 1)$ balance themselves. Find all solutions!

4 Both end conditions for the free-free differential equation are $du/dx = 0$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \quad \text{with} \quad \frac{du}{dx} = 0 \text{ at both ends.}$$

Integrate both sides to show that the force $f(x)$ must balance itself, $\int f(x)\,dx = 0$, or there is no solution. The complete solution is one particular solution $u(x)$ plus any constant. The constant corresponds to $\boldsymbol{u} = (1, 1, 1)$ in the nullspace of $A^{\mathrm{T}}CA$.

5 In the fixed-free problem, the matrix A is square and invertible. We can solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ separately from $A\boldsymbol{u} = \boldsymbol{e}$. Do the same for the differential equation:

$$\text{Solve} \quad -\frac{dy}{dx} = f(x) \text{ with } y(1) = 0. \quad \text{Graph } y(x) \text{ if } f(x) = 1.$$

6 The 3 by 3 matrix $K_1 = A_1^{\mathrm{T}} C_1 A_1$ in equation (6) splits into three "element matrices" $c_1E_1 + c_2E_2 + c_3E_3$. Write down those pieces, one for each c. Show how they come from *column times row* multiplication of $A_1^{\mathrm{T}} C_1 A_1$. This is how finite element stiffness matrices are actually assembled.

7 For five springs and four masses with both ends fixed, what are the matrices A and C and K? With $C = I$ solve $K\boldsymbol{u} = \mathsf{ones}(4)$.

8 Compare the solution $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ in Problem 7 to the solution of the continuous problem $-u'' = 1$ with $u(0) = 0$ and $u(1) = 0$. The parabola $u(x)$ should correspond at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ to $\boldsymbol{u}$—is there a $(\Delta x)^2$ factor to account for?

9 Solve the fixed-free problem $-u'' = mg$ with $u(0) = 0$ and $u'(1) = 0$. Compare $u(x)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ with the vector $\boldsymbol{u} = (3mg, 5mg, 6mg)$ in Example 2.

10 Suppose $c_1 = c_2 = c_3 = c_4 = 1$, $m_1 = 2$ and $m_2 = m_3 = 1$. Solve $A^{\mathrm{T}} C A\, \boldsymbol{u} = (2, 1, 1)$ for this fixed-fixed line of springs. Which mass moves the most (largest u) ?

11 (MATLAB) Find the displacements $u(1), \ldots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of $\boldsymbol{u}$ with **fixed-fixed** and **fixed-free** ends. Note that $\mathsf{diag}(\mathsf{ones}(n, 1), d)$ is a matrix with n ones along diagonal d. This print command will graph a vector u:

```
plot(u,'+');    xlabel('mass number');    ylabel('movement');    print
```

12 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\Delta x = \frac{1}{8}$. Graph your three numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\,\frac{du}{dx} = 1 \;\text{ with }\; u(0) = u(1) = 0.$$

This ***convection-diffusion equation*** appears everywhere. It transforms to the Black-Scholes equation for option prices in mathematical finance.

Problem 12 is developed into the first MATLAB homework in my 18.085 course on Computational Science and Engineering at MIT. Videos on *ocw.mit.edu*.