Chapter 10

Applications

10.1 Graphs and Networks

Over the years I have seen one model so often, and I found it so basic and useful, that I always put it first. The model consists of ***nodes connected by edges***. This is called a ***graph***.

Graphs of the usual kind display functions $f(x)$. Graphs of this node-edge kind lead to matrices. This section is about the ***incidence matrix*** of a graph—which tells how the n nodes are connected by the m edges. Normally $m > n$, there are more edges than nodes.

For any m by n matrix there are two fundamental subspaces in $\mathbf{R}^n$ and two in $\mathbf{R}^m$. They are the row spaces and nullspaces of A and A^{T}. Their *dimensions* $r, n-r$ and $r, m-r$ come from the most important theorem in linear algebra. The second part of that theorem is the *orthogonality* of the row space and nullspace. Our goal is to show how examples from graphs illuminate this Fundamental Theorem of Linear Algebra.

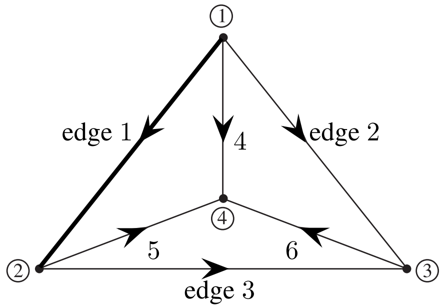
When I construct a ***graph*** and its ***incidence matrix***, the subspace dimensions will be easy to discover. But we want the subspaces themselves—and orthogonality helps. It is essential to connect the subspaces to the graph they come from. By specializing to incidence matrices, **the laws of linear algebra become Kirchhoff's laws**. Please don't be put off by the words "current" and "voltage." These rectangular matrices are the best.

Every entry of an incidence matrix is 0 or 1 or -1. This continues to hold during elimination. All pivots and multipliers are ± 1. Therefore both factors in $A = LU$ also contain $0, 1, -1$. So do the nullspace matrices! All four subspaces have basis vectors with these exceptionally simple components. The matrices are not concocted for a textbook, they come from a model that is absolutely essential in pure and applied mathematics.

The Incidence Matrix

Figure 10.1 displays a graph with $m = 6$ edges and $n = 4$ nodes. The 6 by 4 matrix A tells which nodes are connected by which edges. The first row $-1, 1, 0, 0$ shows that the first edge goes *from node* 1 *to node* 2 (-1 for node 1 because the arrow goes out, $+1$ for node 2 with arrow in).

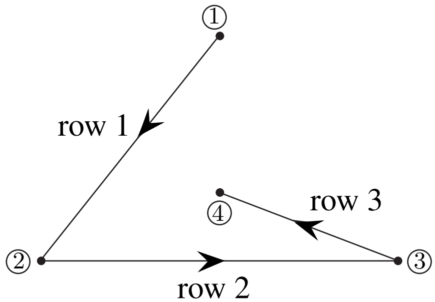
Row numbers in A are edge numbers, column numbers $1, 2, 3, 4$ are node numbers!



$$\begin{array}{cc} & \textbf{node} \\ & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ A = & \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \textbf{edge} \end{array}$$

Figure 10.1: Complete graph with $m=6$ edges and $n=4$ nodes: 6 by 4 incidence matrix A.

You can write down the matrix by looking at the graph. The second graph has the same four nodes but only three edges. Its incidence matrix B is 3 by 4.



$$\begin{array}{cc} & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ B = & \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \text{edge} \end{array}$$

Figure 10.1*: Tree with 3 edges and 4 nodes and no loops. Then B has independent rows.

The first graph is *complete*—every pair of nodes is connected by an edge. The second graph is a ***tree***—the graph has ***no closed loops***. Those are the two extremes. The maximum number of edges is $\frac{1}{2}n(n-1) = 6$ and the minimum to stay connected is $n-1 = 3$.

Elimination reduces every graph to a tree. Loops produce dependent rows in A and zero rows in the echelon forms U and R. Look at the large loop from edges 1, 2, 3 in the first graph, which leads to a zero row in U:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Those steps are typical. When edges 1 and 2 share node 1, elimination produces the "shortcut edge" without node 1. If the graph already has this shortcut edge making a loop, then elimination gives a row of zeros. When the dust clears we have a tree.

An idea suggests itself: ***Rows are dependent when edges form a loop***. Independent rows come from trees. This is the key to the row space. We are assuming that the graph is connected, and the arrows could go either way. On each edge, *flow with the arrow is "positive."* Flow in the opposite direction counts as negative. The flow might be a current or a signal or a force—or even oil or gas or water.

When x_1, x_2, x_3, x_4 are voltages at the nodes, Ax **gives voltage differences**:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}. \qquad (1)$$

Let me say that again. The incidence matrix A is a difference matrix. The input vector x gives voltages, the output vector Ax gives voltage differences (along edges 1 to 6). If the voltages are equal, the differences are zero. This tells us the nullspace of A.

1 The ***nullspace*** contains the solutions to $Ax = \mathbf{0}$. All six voltage differences are zero. This means: *All four voltages are equal.* Every x in the nullspace is a **constant vector**: $x = (c, c, c, c)$. The nullspace of A is a line in $\mathbf{R}^n$—its dimension is $n - r = 1$.

The second incidence matrix B has the same nullspace. It contains $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$:

1−dimensional nullspace: same for the tree

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can raise or lower all voltages by the same amount c, without changing the differences. There is an "arbitrary constant" in the voltages. Compare this with the same statement for functions. We can raise or lower a function by C, without changing its derivative.

Calculus adds "$+C$" to indefinite integrals. Graph theory adds (c, c, c, c) to the vector x. Linear algebra adds any vector x_n in the nullspace to one particular solution of $Ax = b$.

The "$+C$" disappears in calculus when a definite integral starts at a known point. Similarly the nullspace disappears when we fix $x_4 = \mathbf{0}$. The unknown x_4 is removed and so are the fourth columns of A and B (those columns multiplied x_4). Electrical engineers would say that node 4 has been "grounded."

2 The ***row space*** contains all combinations of the six rows. Its dimension is certainly not 6. The equation $r + (n - r) = n$ must be $3 + 1 = 4$. The rank is $r = 3$, as we saw from elimination. After 3 edges, we start forming loops! The new rows are not independent.

How can we tell if $v = (v_1, v_2, v_3, v_4)$ is in the row space? The slow way is to combine rows. The quick way is by orthogonality:

v is in the row space if and only if it is perpendicular to $(1, 1, 1, 1)$ in the nullspace.

The vector $v = (0, 1, 2, 3)$ fails this test—its components add to 6. The vector $(-6, 1, 2, 3)$ is in the row space: $-6+1+2+3 = 0$. That vector equals $6(\text{row } 1)+5(\text{row } 3)+3(\text{row } 6)$.

Each row of A adds to zero. This must be true for every vector in the row space.

3 The ***column space*** contains all combinations of the four columns. We expect three independent columns, since there were three independent rows. The first three columns of A are independent (so are any three). But the four columns add to the zero vector, which says again that $(1, 1, 1, 1)$ is in the nullspace. ***How can we tell if a particular vector b is in the column space of an incidence matrix***?

First answer Try to solve $A\boldsymbol{x} = \boldsymbol{b}$. That misses all the insight. As before, orthogonality gives a better answer. We are now coming to Kirchhoff's two famous laws of circuit theory—the voltage law and current law (**KVL** and **KCL**). Those are natural expressions of "laws" of linear algebra. It is especially pleasant to see the key role of the left nullspace.

Second answer $A\boldsymbol{x}$ is the vector of voltage differences $x_i - x_j$. If we add differences around a closed loop in the graph, they cancel to leave zero. Around the big triangle formed by edges $1, 3, -2$ (*the arrow goes backward on edge* 2) the differences cancel:

$$\textbf{Sum of differences is 0} \qquad (x_2 - x_1) + (x_3 - x_2) - (x_3 - x_1) = \mathbf{0}.$$

Kirchhoff's Voltage Law: The components of $\boldsymbol{Ax} = \boldsymbol{b}$ ***add to zero around every loop***.

Around the big triangle: $\quad b_1 + b_3 - b_2 = 0.$

By testing each loop, the Voltage Law decides whether $\boldsymbol{b}$ is in the column space. $A\boldsymbol{x} = \boldsymbol{b}$ can be solved exactly when the components of $\boldsymbol{b}$ satisfy all the same dependencies as the rows of A. Then elimination leads to $0 = 0$, and $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

4 The ***left nullspace*** contains the solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Its dimension is $m - r = 6 - 3$:

$$\textbf{Current Law} \qquad A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (2)$$

The true number of equations is $r = 3$ and not $n = 4$. Reason: The four equations add to $0 = 0$. The fourth equation follows automatically from the first three.

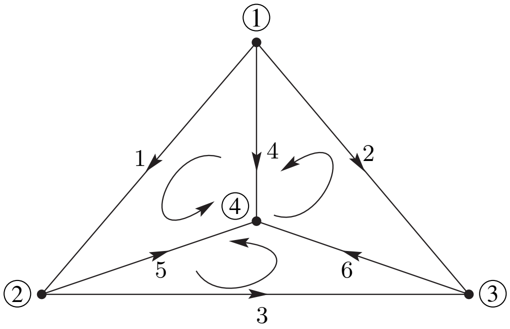
What do the equations mean? The first equation says that $-y_1 - y_2 - y_4 = 0$. ***The net flow into node* 1 *is zero***. The fourth equation says that $y_4 + y_5 + y_6 = 0$. ***Flow into node 4 minus flow out is zero***. The equations $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are famous and fundamental:

Kirchhoff's Current Law: $A^{\mathrm{T}}y = 0$ $\qquad$ ***Flow in equals flow out at each node.***

This law deserves first place among the equations of applied mathematics. It expresses "*conservation*" and "*continuity*" and "*balance*." Nothing is lost, nothing is gained. When currents or forces are balanced, the equation to solve is $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Notice the beautiful fact that the matrix in this balance equation is the transpose of the incidence matrix A.

What are the actual solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? The currents must balance themselves. The easiest way is to **flow around a loop**. If a unit of current goes around the big triangle (forward on edge 1 and 3, backward on 2), the six currents are $\boldsymbol{y} = (1, -1, 1, 0, 0, 0)$. This satisfies $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. *Every loop current is a solution to the Current Law.* Flow in equals flow out at every node. A smaller loop goes forward on edge 1, forward on 5, back on 4. Then $\boldsymbol{y} = (\mathbf{1}, 0, 0, -\mathbf{1}, \mathbf{1}, 0)$ is also in the left nullspace.

We expect three independent $\boldsymbol{y}$'s: $m-r=6-3=3$. The three small loops in the graph are independent. The big triangle seems to give a fourth $\boldsymbol{y}$, but that flow is the sum of flows around the small loops. *Flows around the* 3 *small loops are a basis for the left nullspace.*



$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\text{3 small loops}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{big loop}}$$

The incidence matrix A comes from a connected graph with n nodes and m edges. The row space and column space have dimensions $r = n - 1$. The nullspaces of A and A^{T} have dimensions 1 and $m - n + 1$:

$\boldsymbol{N}(A)$ The constant vectors $(c, c, \ldots, c)$ make up the nullspace of $A : \dim = 1$.

$\boldsymbol{C}(A^{\mathrm{T}})$ The edges of any tree give r independent rows of $A : r = n - 1$.

$\boldsymbol{C}(A)$ ***Voltage Law***: The components of $A\boldsymbol{x}$ add to zero around all loops: $\dim = n - 1$.

$\boldsymbol{N}(A^{\mathrm{T}})$ ***Current Law***: $A^{\mathrm{T}}\boldsymbol{y} = (\textbf{flow in}) - (\textbf{flow out}) = \mathbf{0}$ is solved by loop currents.

There are $\boldsymbol{m - r = m - n + 1}$ independent small loops in the graph.

For every graph in a plane, linear algebra yields ***Euler's formula***: Theorem 1 in topology!

$$(\textit{number of nodes}) - (\textit{number of edges}) + (\textit{number of small loops}) = \mathbf{1}.$$

This is $\boldsymbol{(n) - (m) + (m - n + 1) = 1}$. The graph in our example has $4 - 6 + 3 = 1$.

A single triangle has (3 nodes) $-$ (3 edges) $+$ (1 loop). On a 10-node tree with 9 edges and no loops, Euler's count is $10 - 9 + 0$. All planar graphs lead to the answer 1.

The next figure shows a network with a current source. Kirchhoff's Current Law changes from $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$, to balance the source $\boldsymbol{f}$ from outside. *Flow into each node still equals flow out.* The six edges would have conductances $c_1, \ldots, c_6$, and the current source goes into node 1. The source comes out from node 4 to keep the overall balance (**in** $=$ **out**). The problem is: ***Find the currents $\boldsymbol{y_1, \ldots, y_6}$ on the six edges.***
Flows in networks now lead us from the incidence matrix A to the Laplacian matrix $A^{\mathrm{T}}A$.

Voltages and Currents and $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$

We started with voltages $\boldsymbol{x} = (x_1, \ldots, x_n)$ at the nodes. So far we have $A\boldsymbol{x}$ to find voltage differences $x_i - x_j$ along edges. And we have the Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ to find edge currents $\boldsymbol{y} = (y_1, \ldots y_m)$. If all resistances in the network are 1, Ohm's Law will match $\boldsymbol{y} = A\boldsymbol{x}$. Then $A^{\mathrm{T}}\boldsymbol{y} = A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$. We are close but not quite there.

Without any sources, the solution to $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ will just be no flow: $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{y} = \boldsymbol{0}$. I can see three ways to produce $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$.

1 Assign fixed voltages x_i to one or more nodes.

2 Add batteries (voltage sources) in one or more edges.

3 Add current sources going into one or more nodes. See Figure 10.2

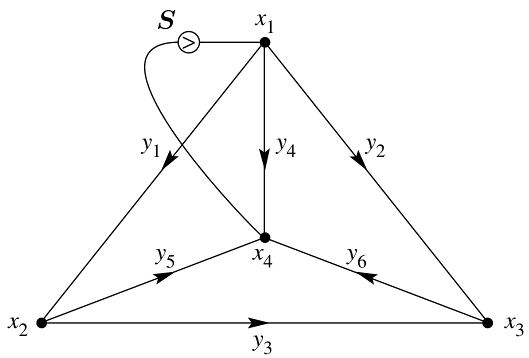


Figure 10.2: The currents y_1 to y_6 in a network with a source S from node 4 to node 1.

Example Figure 10.2 includes a current source S from node 4 to node 1. That current will trickle back through the network to node 4. Some current y_4 will go directly on edge 4. Other current will go the long way from node 1 to 2 to 4, or 1 to 3 to 4. By symmetry I expect no current ($y_3 = 0$) from node 2 to node 3. Solving the network equations will confirm this. **The matrix in those equations is $A^{\mathrm{T}}A$, *the graph Laplacian matrix***:

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}_{\boldsymbol{A^{\mathrm{T}}A}}$$

That Laplacian matrix is not invertible! We cannot solve for all four potentials because $(1,1,1,1)$ is in the nullspace of A and $A^{\mathrm{T}}A$. *One node has to be grounded*. Setting $x_4 = 0$ removes the fourth row and column, and this leaves a 3 by 3 invertible matrix. Now we solve $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$ for the unknown potentials x_1, x_2, x_3, with source S into node 1:

Voltages $A^{\mathrm{T}}Ax = f$

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

Currents $y = -Ax$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} S/2 \\ S/4 \\ S/4 \\ 0 \end{bmatrix} = \begin{bmatrix} S/4 \\ S/4 \\ 0 \\ S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

Half the current goes directly on edge 4. That is $y_4 = S/2$. No current crosses from node 2 to node 3. Symmetry indicated $y_3 = 0$ and now the solution proves it.

Admission of error I remembered that current flows from high voltage to low voltage. That produces the minus sign in $\boldsymbol{y} = -A\boldsymbol{x}$. And the correct form of Ohm's Law will be $R\boldsymbol{y} = -A\boldsymbol{x}$ when the resistances on the edges are not all 1. *Conductances* are neater than resistances: $C = R^{-1} =$ diagonal matrix. **We now present Ohm's Law $\boldsymbol{y} = -CA\boldsymbol{x}$.**

Networks and $A^{\mathrm{T}}CA$

In a real network, the current $\boldsymbol{y}$ along an edge is the product of two numbers. One number is the difference between the potentials $\boldsymbol{x}$ at the ends of the edge. This voltage difference is $A\boldsymbol{x}$ and it drives the flow. The other number c is the "***conductance***"—which measures how easily flow gets through.

In physics and engineering, c is decided by the material. For electrical currents, c is high for metal and low for plastics. For a superconductor, c is nearly infinite. If we consider elastic stretching, c might be low for metal and higher for plastics. In economics, c measures the capacity of an edge or its cost.

To summarize, the graph is known from its incidence matrix A. This tells the node-edge connections. A ***network*** goes further, and assigns a conductance c to each edge. *These numbers $c_1, \ldots, c_m$ go into the "conductance matrix" C—which is diagonal.*

For a network of resistors, the conductance is $c = 1/(\text{resistance})$. In addition to Kirchhoff's Laws for the whole system of currents, we have Ohm's Law for each current. Ohm's Law connects the current y_1 on edge 1 to the voltage difference $x_2 - x_1$:

Ohm's Law: Current along edge = conductance times voltage difference.

Ohm's Law for all m currents is $\boldsymbol{y} = -CA\boldsymbol{x}$. The vector $A\boldsymbol{x}$ gives the potential differences, and C multiplies by the conductances. Combining Ohm's Law with Kirchhoff's

Current Law $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, we get $A^{\mathrm{T}}CA\boldsymbol{x} = \mathbf{0}$. This is *almost* the central equation for network flows. The only thing wrong is the zero on the right side! The network needs power from outside—a voltage source or a current source—to make something happen.

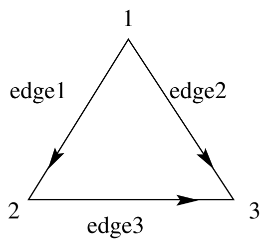
Note about signs In circuit theory we change from $A\boldsymbol{x}$ to $-A\boldsymbol{x}$. The flow is from higher potential to lower potential. There is (positive) current from node 1 to node 2 when $x_1 - x_2$ is positive—whereas $A\boldsymbol{x}$ was constructed to yield $x_2 - x_1$. The minus sign in physics and electrical engineering is a plus sign in mechanical engineering and economics. $A\boldsymbol{x}$ versus $-A\boldsymbol{x}$ is a general headache but unavoidable.

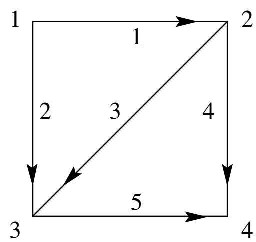
Note about applied mathematics Every new application has its own form of Ohm's Law. For springs it is Hooke's Law. The stress $\boldsymbol{y}$ is (elasticity C) times (stretching $A\boldsymbol{x}$). For heat conduction, $A\boldsymbol{x}$ is a temperature gradient. For oil flows it is a pressure gradient. For least squares regression in statistics (Chapter 12) C^{-1} is the covariance matrix.

My textbooks *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) are practically built on $A^{\mathrm{T}}CA$. This is the key to equilibrium in matrix equations and also in differential equations. Applied mathematics is more organized than it looks! *In new problems I have learned to watch for* $A^{\mathrm{T}}CA$.

Problem Set 10.1

Problems 1–7 and 8–14 are about the incidence matrices for these graphs.





1 Write down the 3 by 3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (x_1, x_2, x_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

2 Write down A^{T} for the triangle graph. Find a vector $\boldsymbol{y}$ in its nullspace. The components of $\boldsymbol{y}$ are currents on the edges—how much current is going around the triangle?

3 Eliminate x_1 and x_2 from the third equation to find the echelon matrix U. What tree corresponds to the two nonzero rows of U?

$$\begin{aligned} -x_1 + x_2 &= b_1 \\ -x_1 + x_3 &= b_2 \\ -x_2 + x_3 &= b_3. \end{aligned}$$

4 Choose a vector (b_1, b_2, b_3) for which $A\boldsymbol{x} = \boldsymbol{b}$ can be solved, and another vector $\boldsymbol{b}$ that allows no solution. How are those $\boldsymbol{b}$'s related to $\boldsymbol{y} = (1, -1, 1)$?

5 Choose a vector (f_1, f_2, f_3) for which $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ can be solved, and a vector $\boldsymbol{f}$ that allows no solution. How are those $\boldsymbol{f}$'s related to $\boldsymbol{x} = (1, 1, 1)$? The equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is Kirchhoff's _____ law.

6 Multiply matrices to find $A^{\mathrm{T}}A$. Choose a vector $\boldsymbol{f}$ for which $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{x}$. Put those potentials $\boldsymbol{x}$ and the currents $\boldsymbol{y} = -A\boldsymbol{x}$ and current sources $\boldsymbol{f}$ onto the triangle graph. Conductances are 1 because $C = I$.

7 With conductances $c_1 = 1$ and $c_2 = c_3 = 2$, multiply matrices to find $A^{\mathrm{T}}CA$. For $\boldsymbol{f} = (1, 0, -1)$ find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$. Write the potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the triangle graph, when the current source $\boldsymbol{f}$ goes into node 1 and out from node 3.

8 Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{x} = \boldsymbol{0}$ and two solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

9 Find two requirements on the b's for the five differences $x_2 - x_1, x_3 - x_1, x_3 - x_2, x_4 - x_2, x_4 - x_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's _____ law around the two _____ in the graph.

10 Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

11 Multiply matrices to find $A^{\mathrm{T}}A$ and guess how its entries come from the graph:

(a) The diagonal of $A^{\mathrm{T}}A$ tells how many _____ into each node.

(b) The off-diagonals -1 or 0 tell which pairs of nodes are _____.

12 Why is each statement true about $A^{\mathrm{T}}A$? *Answer for $A^{\mathrm{T}}A$ not A.*

(a) Its nullspace contains $(1, 1, 1, 1)$. Its rank is $n - 1$.

(b) It is positive semidefinite but not positive definite.

(c) Its four eigenvalues are real and their signs are _____.

13 With conductances $c_1 = c_2 = 2$ and $c_3 = c_4 = c_5 = 3$, multiply the matrices $A^{\mathrm{T}}CA$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f} = (1, 0, 0, -1)$. Write these potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the nodes and edges of the square graph.

14 The matrix $A^{\mathrm{T}}CA$ is not invertible. What vectors $\boldsymbol{x}$ are in its nullspace? Why does $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ have a solution if and only if $f_1 + f_2 + f_3 + f_4 = 0$?

15 A connected graph with 7 nodes and 7 edges has how many loops?

16 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$. If you connect it to two old nodes, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$.

17 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on $\boldsymbol{f}$ makes it possible to solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$?

(c) The diagonal entries of $A^{\mathrm{T}}A$ give the number of edges into each node. What is the sum of those diagonal entries?

18 Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A tree connecting 6 nodes has _____ edges.

Note The ***stoichiometric matrix*** in chemistry is an important "generalized" incidence matrix. Its entries show how much of each chemical species (each column) goes into each reaction (each row).