INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 8.1, page 407

1 With $\boldsymbol{w} = \mathbf{0}$ linearity gives $T(\boldsymbol{v} + \mathbf{0}) = T(\boldsymbol{v}) + T(\mathbf{0})$. Thus $T(\mathbf{0}) = \mathbf{0}$. With $c = -1$ linearity gives $T(-\mathbf{0}) = -T(\mathbf{0})$. This is a second proof that $T(\mathbf{0}) = \mathbf{0}$.

2 Combining $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ and $T(d\boldsymbol{w}) = dT(\boldsymbol{w})$ with addition gives $T(c\boldsymbol{v} + d\boldsymbol{w}) = cT(\boldsymbol{v}) + dT(\boldsymbol{w})$. Then one more addition gives $cT(\boldsymbol{v}) + dT(\boldsymbol{w}) + eT(\boldsymbol{u})$.

3 (d) $T(\boldsymbol{v}) = (0, 1) =$ constant and $(f)\, T(\boldsymbol{v}) = v_1 v_2$ are not linear.

4 (a) $S(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (-1, 0)$. Then $T(\boldsymbol{v}) + T(\boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w})$ but $T(\boldsymbol{v} + \boldsymbol{w}) = (0, 0)$.

6 (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ does not satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ or $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ (b) and (c) are linear (d) satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$.

7 (a) $T(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(T(\boldsymbol{v})) = \boldsymbol{v} + (2, 2)$ (c) $T(T(\boldsymbol{v})) = -\boldsymbol{v}$ (d) $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$.

8 (a) The range of $T(v_1, v_2) = (v_1 - v_2, 0)$ is the line of vectors $(c, 0)$. The nullspace is the line of vectors (c, c). (b) $T(v_1, v_2, v_3) = (v_1, v_2)$ has Range $\mathbf{R}^2$, kernel $\{(0, 0, \boldsymbol{v}_3)\}$ (c) $T(\boldsymbol{v}) = \mathbf{0}$ has Range $\{\mathbf{0}\}$, kernel $\mathbf{R}^2$ (d) $T(v_1, v_2) = (v_1, v_1)$ has Range = multiples of $(1, 1)$, kernel = multiples of $(1, -1)$.

9 If $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$ then $T(T(\boldsymbol{v})) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2)$; $T^3(\boldsymbol{v}) = \boldsymbol{v}$; $T^{100}(\boldsymbol{v}) = T(\boldsymbol{v})$.

10 (a) $T(1, 0) = \mathbf{0}$ (b) $(0, 0, 1)$ is not in the range (c) $T(0, 1) = \mathbf{0}$.

11 For multiplication $T(\boldsymbol{v}) = A\boldsymbol{v}$: $\boldsymbol{V} = \mathbf{R}^n$, $\boldsymbol{W} = \mathbf{R}^m$; the outputs fill the column space; $\boldsymbol{v}$ is in the kernel if $A\boldsymbol{v} = \mathbf{0}$.

12 $T(\boldsymbol{v}) = (4, 4)$; $(2, 2)$; $(2, 2)$; if $\boldsymbol{v} = (a, b) = b(1, 1) + \frac{a-b}{2}(2, 0)$ then $T(\boldsymbol{v}) = b(2, 2) + (0, 0)$.

13 The *distributive law* (page 69) gives $A(M_1 + M_2) = AM_1 + AM_2$. The *distributive law* over c's gives $A(cM) = c(AM)$.

14 This A is invertible. Multiply $AM = 0$ and $AM = B$ by A^{-1} to get $M = 0$ and $M = A^{-1}B$. The kernel contains only the zero matrix $M = 0$.

15 This A is *not* invertible. $AM = I$ is impossible. $A\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The range contains only matrices AM whose columns are multiples of $(1, 3)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 For $T(M) = MT$ (a) $T^2 = I$ is True (b) True (c) True (d) False.

18 $T(I) = 0$ but $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$; these M's fill the range. Every $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is in the kernel. Notice that $\dim$ (range) $+ \dim$ (kernel) $= 3 + 1 = \dim$ (input space of 2 by 2 M's).

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1, 0) = (a_{11}, 0)$.

21 $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles the width of the house. $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$ *projects* the house (since $A^2 = A$ from trace $= 1$ and $\lambda = 0, 1$). The projection is onto the column space of $A =$ line through $(.7, .3)$. $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ will *shear* the house horizontally: The point at (x, y) moves over to $(x + y, y)$.

22 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ leaves the house AH sitting straight up (b) $A = 3I$ expands the house by 3 (c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the house.

23 $T(\boldsymbol{v}) = -\boldsymbol{v}$ rotates the house by 180° around the origin. Then the affine transformation $T(\boldsymbol{v}) = -\boldsymbol{v} + (1, 0)$ shifts the rotated house one unit to the right.

24 A code to add a chimney will be gratefully received!

25 This code needs a correction: add spaces between $-10\ 10\ -10\ 10$

26 $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ compresses vertical distances by 10 to 1. $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects onto the $45°$ line. $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ rotates by $45°$ clockwise and contracts by a factor of $\sqrt{2}$ (the columns have length $1/\sqrt{2}$). $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant -1 so the house is "flipped and sheared." One way to see this is to factor the matrix as LDL^{T}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \text{(shear) (flip left-right) (shear)}.$$

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

28 A code that adds two eyes and a smile will be included here with public credit given!

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0, 0)$.)

30 Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed $\boldsymbol{v}$) go to two parallel edges (edges differing by $T(\boldsymbol{v})$). So the output is a parallelogram.

Problem Set 8.2, page 418

1 For $S\boldsymbol{v} = d^2\boldsymbol{v}/dx^2$

Basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 = 1, x, x^2, x^3$

$S\boldsymbol{v}_1 = S\boldsymbol{v}_2 = \mathbf{0}$, $S\boldsymbol{v}_3 = 2\boldsymbol{v}_1$, $S\boldsymbol{v}_4 = 6\boldsymbol{v}_2$;

The matrix for S is $B = \begin{bmatrix} 0 & 0 & \mathbf{2} & 0 \\ 0 & 0 & 0 & \mathbf{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 $Sv = d^2v/dx^2 = 0$ for linear functions $\boldsymbol{v}(\boldsymbol{x}) = a + bx$. All $(a, b, 0, 0)$ are in the nullspace of the second derivative matrix B.

3 (Matrix $A)^2 = B$ when transformation $T(T(\boldsymbol{v})) = S(\boldsymbol{v})$ and output basis = input basis.

4 The third derivative matrix has **6** in the $(1,4)$ position; since the third derivative of x^3 is 6. This matrix also comes from AB. The fourth derivative of a cubic is zero, and B^2 is the zero matrix.

5 $T(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 2\boldsymbol{w}_1 + \boldsymbol{w}_2 + 2\boldsymbol{w}_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $\boldsymbol{v} = c(\boldsymbol{v}_2 - \boldsymbol{v}_3)$ gives $T(\boldsymbol{v}) = \mathbf{0}$; nullspace is $(0, c, -c)$; solutions $(1,0,0) + (0,c,-c)$.

7 $(1,0,0)$ is not in the column space of the matrix A, and $\boldsymbol{w}_1$ is not in the range of the linear transformation T. Key point: *Column space* of matrix matches *range* of transformation. Nullspace matches normal.

8 We don't know $T(\boldsymbol{w})$ unless the $\boldsymbol{w}$'s are the same as the $\boldsymbol{v}$'s. In that case the matrix is A^2.

9 Rank of $A = 2 =$ dimension of the *range* of T. The outputs $A\boldsymbol{v}$ (column space) match the outputs $T(\boldsymbol{v})$ (the range of T). The "output space" $\boldsymbol{W}$ is like $\mathbf{R}^m$: it contains all outputs but may not be filled up by the column space.

10 The matrix for T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. For the output $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose input $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This means: For the output $\boldsymbol{w}_1$ choose the input $\boldsymbol{v}_1 - \boldsymbol{v}_2$.

11 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ so $T^{-1}(\boldsymbol{w}_1) = \boldsymbol{v}_1 - \boldsymbol{v}_2, T^{-1}(\boldsymbol{w}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_3, T^{-1}(\boldsymbol{w}_3) = \boldsymbol{v}_3$. The columns of A^{-1} describe T^{-1} from $\boldsymbol{W}$ back to $\boldsymbol{V}$. The only solution to $T(\boldsymbol{v}) = 0$ is $\boldsymbol{v} = 0$.

12 (c) $T^{-1}(T(\boldsymbol{w}_1)) = \boldsymbol{w}_1$ is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

13 (a) $T(\boldsymbol{v}_1) = \boldsymbol{v}_2, T(\boldsymbol{v}_2) = \boldsymbol{v}_1$ is its own inverse (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1, T(\boldsymbol{v}_2) = 0$ has $T^2 = T$ (c) If $T^2 = I$ for part (a) and $T^2 = T$ for part (b), then T must be I.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

15 (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} r \\ t \end{bmatrix}$ and $\begin{bmatrix} s \\ u \end{bmatrix}$; this is the "easy" direction. (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\mathbf{-1}}$ transforms in the inverse direction, back to the standard basis vectors. (c) $ad = bc$ will make the forward matrix singular and the inverse impossible.

16 $MW = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

17 Reordering basis vectors is done by a *permutation matrix*. Changing lengths is done by a *positive diagonal matrix*.

18 $(a, b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

19 $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$; $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ = first column of M^{-1} = coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ because $5 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

20 $\boldsymbol{w}_2(x) = 1 - x^2;\ \boldsymbol{w}_3(x) = \frac{1}{2}(x^2 - x);\ \boldsymbol{y} = 4\boldsymbol{w}_1 + 5\boldsymbol{w}_2 + 6\boldsymbol{w}_3$.

21 $\boldsymbol{w}$'s to $\boldsymbol{v}$'s: $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$. $\boldsymbol{v}$'s to $\boldsymbol{w}$'s: inverse matrix $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. *The key idea*: The matrix multiplies the coordinates in the $\boldsymbol{v}$ basis to give the coordinates in the $\boldsymbol{w}$ basis.

22 The 3 equations to match $4, 5, 6$ at $x = a, b, c$ are $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. This Vandermonde determinant equals $(b - a)(c - a)(c - b)$. So a, b, c must be distinct to have $\det \neq 0$ and one solution A, B, C.

23 The matrix M with these nine entries must be invertible.

24 Start from $A = QR$. Column 2 is $\boldsymbol{a}_2 = r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2$. This gives $\boldsymbol{a}_2$ as a combination of the $\boldsymbol{q}$'s. So the change of basis matrix is $\boldsymbol{R}$.

25 Start from $A = LU$. Row 2 of A is ℓ_{21}(row 1 of U) $+ \ell_{22}$ (row 2 of U). The change of basis matrix is always *invertible*, because basis goes to basis.

26 The matrix for $T(\boldsymbol{v}_i) = \lambda_i\boldsymbol{v}_i$ is $\Lambda = \mathsf{diag}(\lambda_1, \lambda_2, \lambda_3)$.

27 If T is not invertible, $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$ is not a basis. We couldn't choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$.

28 (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \mathbf{0}$ and $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_1$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_1$ (which combine into $T(\boldsymbol{v}_2) = \mathbf{0}$ by *linearity*).

29 $T(x, y) = (x, -y)$ is reflection across the x-axis. Then reflect across the y-axis to get $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

30 S takes (x, y) to $(-x, y)$. $S(T(\boldsymbol{v})) = (\mathbf{-1}, \mathbf{2})$. $S(\boldsymbol{v}) = (-2, 1)$ and $T(S(\boldsymbol{v})) = (\mathbf{1}, \mathbf{-2})$.

31 Multiply the two reflections to get $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ which is *rotation by* $2(\theta - \alpha)$. In words: $(1, 0)$ is reflected to have angle 2α, and that is reflected again to angle $2\theta - 2\alpha$.

32 The matrix for T in this basis is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

33 Multiplying by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\boldsymbol{v}_1 + c\boldsymbol{v}_3$. Similarly $T(\boldsymbol{v}_2) = a\boldsymbol{v}_2 + c\boldsymbol{v}_4$ and $T(\boldsymbol{v}_3) = b\boldsymbol{v}_1 + d\boldsymbol{v}_3$ and $T(\boldsymbol{v}_4) = b\boldsymbol{v}_2 + d\boldsymbol{v}_4$. The matrix for T in this basis is $\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$

34 False: We will not know $T(\boldsymbol{v})$ for *every* $\boldsymbol{v}$ unless the n $\boldsymbol{v}$'s are linearly independent.

Problem Set 8.3, page 429

1 For this matrix J, the rank of $J - 3I$ is **3** so the dimension of the nullspace is only 1. There is only 1 independent eigenvector even though $\lambda = 3$ is a *double root* of $\det(J - \lambda I) = 0$: a repeated eigenvalue.

$$J = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}.$$

2 $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to all other 2 by 2 matrices A that have **2** zero eigenvalues but only **1** independent eigenvector. Then $J = B_1^{-1} A_1 B_1$ is the same as $B_1 J = A_1 B_1$:

$$B_1 J = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = A_1 B_1$$

$$B_2 J = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} = A_2 B_2$$

3 Every matrix is similar to its transpose (same eigenvalues, same multiplicity, more than that the same Jordan form). In this example

$$BJ = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = J^{\mathrm{T}} B.$$

4 Here J and K are *different* Jordan forms (block sizes $2, 2$ versus block sizes $3, 1$). Even though J and K have the same λ's (all zero) and same rank, J and K are *not similar*. If $BK = JB$ then B is *not invertible*:

$$BK = B \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ 0 & b_{31} & b_{32} & 0 \\ 0 & b_{41} & b_{42} & 0 \end{bmatrix}$$

$$JB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Those right hand sides agree only if $b_{21} = 0, b_{41} = 0, b_{24} = 0, b_{44} = 0, b_{22} = 0$, $b_{42} = 0$. But then also $b_{11} = b_{22} = 0$ and $b_{31} = b_{42} = 0$. So the first column has $b_{11} = b_{21} = b_{31} = b_{41} = 0$ and B is not invertible.

5 If A^3 is the zero matrix then every eigenvalue of A is $\lambda = 0$ (because $A\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $\boldsymbol{0} = A^3\boldsymbol{x} = \lambda^3\boldsymbol{x}$). The Jordan form J will also have $J^3 = 0$ because $J = B^{-1}AB$ has $J^3 = B^{-1}A^3B = 0$. The blocks of J must become zero blocks in J^3. So those blocks of J can be

$$\begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{but not} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left(\begin{array}{c} \text{third power} \\ \text{is not zero} \end{array} \right)$$

The rank of J (and A) is largest if every block is 3 by 3 of rank 2. Then rank $\leq \dfrac{2}{3}n$.

If $A^n =$ zero matrix then A is *not invertible* and rank $(A) < n$.

6 This question substitutes $u_1 = te^{\lambda t}$ and $u_2 = e^{\lambda t}$ to show that u_1, u_2 solve the system $\boldsymbol{u}' = J\boldsymbol{u}$:

$$\begin{aligned} u_1' &= \lambda u_1 + u_2 & \qquad e^{\lambda t} + t\lambda e^{\lambda t} &= \lambda(te^{\lambda t}) + (e^{\lambda t}) \\ u_2' &= \qquad \lambda u_2 & \lambda e^{\lambda t} &= \qquad \lambda(e^{\lambda t}) \,. \end{aligned}$$

Certainly $u_1 = 0$ and $u_2 = 1$ at $t = 0$, so we have the solution and it involves $te^{\lambda t}$ (the factor t appears because λ is a double eigenvalue of J).

7 The equation $u_{k+2} - 2\lambda u_{k+1} + \lambda^2 u_k$ is certainly solved by $u_k = \lambda^k$. But this is a second order equation and there must be another solution. In analogy with $te^{\lambda t}$ for the differential equation in 8.3.6, that second solution is $u_k = k\lambda^k$. Check:

$$(k+2)\lambda^{k+2} - 2\lambda(k+1)\lambda^{k+1} + \lambda^2(k)\lambda^k = \big[k + 2 - 2(k+1) + k\big]\lambda^{k+2} = 0.$$

8 $\lambda^3 = 1$ has 3 roots $\lambda = 1$ and $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Those are $\mathbf{1}, \boldsymbol{\lambda}, \boldsymbol{\lambda^2}$ if we take $\lambda = e^{2\pi i/3}$. The Fourier matrix is

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda^2 \\ 1 & \lambda^2 & \lambda^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{8\pi i/3} \end{bmatrix}.$$

9 A 3 by 3 circulant matrix has the form on page 425:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix} \text{ with } C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (c_0 + c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = (c_0 + c_1\lambda + c_2\lambda^2) \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \qquad C \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix} = (c_0 + c_1\lambda^2 + c_2\lambda^4) \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix}.$$

Those 3 eigenvalues of C are exactly the 3 components of $F\boldsymbol{c} = F \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$,

10 The Fourier cosine coefficient c_3 is in formula (7) with integrals from $-\pi$ to π. Because f drops to zero at $x = L$, the integral stops at L:

$$a_3 = \frac{\int f(x) \cos 3x \, dx}{\int (\cos 3x)^2 \, dx} = \frac{1}{\pi} \int_{-L}^{L} (1)(\cos 3x) \, dx = \frac{1}{3\pi} \Big[\sin 3x \Big]_{x=-L}^{x=L} = \frac{2 \sin 3L}{3\pi}.$$

Note that we should have defined $f(x) = 0$ for $L < |x| < \pi$ (not 2π !).