18.06 Professor Edelman Final Exam December 15, 2010

Your PRINTED name is: ____________________

Please circle your recitation:

R01	T 9	2-132	S. Kleiman	2-278	3-4996	kleiman
R02	T 10	2-132	S. Kleiman	2-278	3-4996	kleiman
R03	T 11	2-132	S. Sam	2-487	3-7826	ssam
R04	T 12	2-132	Y. Zhang	2-487	3-7826	yanzhang
R05	T 1	2-132	V. Vertesi	2-233	3-2689	18.06
R06	T 2	2-131	V. Vertesi	2-233	3-2689	18.06

Grading

1

2

3

4

5

6

7

8

9

1 (16 pts.)

a. (8 pts) Give bases for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 0 & \pi & e \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the first two columns are independant and generate the column space.

The left null space is the orthogonal of the column space. A basis is given by the vector :

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A possible basis for the nullspace is :

$$\begin{bmatrix} \pi \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} e \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

The row space has dimension 2. The two first rows clearly generate it, therefore they form a basis.

b. (8 pts) Give bases for each of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(Each of the three matrices in the above product has orthogonal columns.)

The matrices on both side are non singular. Therefore the rank of the product is 3.

To compute the column space, we can restrict to the product of the first two matrices. The third is invertible and will not change the column space. Clearly the column space of the product of the first two matrices is spanned by the first three columns of the leftmost matrix. Since these vectors are independant they form a basis.

To compute the row space we can restrict to the last two matrices of the product for the same reason as above. We find that the row space has as basis the first three rows of the rightmost matrix.

Since the nullspace is the orthogonal of the row space, and we know that the rows of the rightmost matrix are orthogonal, the last row of the right most matrix is a basis for the nullspace.

Similarly, the last column of the leftmost matrix is a basis for the left nullspace.

Remark. Another way to see it is to notice that this is almost the SVD of A (we just need to normalize the columns of the leftmost and rightmost matrix). Our answer is exactly the usual way to find a basis of the four fundamental subspaces when we have found the SVD.

This page intentionally blank.

2 (14 pts.)

Let P_1 be the projection matrix onto the line through $(1, 1, 0)$ and P_2 is the projection onto the line through $(0, 1, 1)$.

(a) (4 pts) What are the eigenvalues of P_1?

The eigenvalue of a projection matrix are 1 and 0. The question remain of the multiplicity of each eigenvalue. The multiplicity of 0 is the dimension of the nullspace which is $3-$ the rank. Here P_1 is a projection on a line, therefore the columnspace is made of vector on that line and has dimension 1. Therefore 0 has multiplicity 2 and 1 has multiplicity 1.

(a2)(bonus) (This question is from an earlier version of the exam.)

Find an eigenvalue and an eigenvector of $P_1 + P_2$.

The problem asks only for one eigenvalue and one eigenvector, but since you're taking this exam for practice, you may as well find all three.

For simplicity of notation, let $\vec{a}_1 := (1, 1, 0)$ and $\vec{a}_2 := (0, 1, 1)$. Note that the vectors $\vec{a}_1$ and $\vec{a}_2$ have the same length, $\sqrt{2}$. Since $\vec{a}_1 \cdot \vec{a}_2 = 1 = 2\cos(60°)$, we know that the angle between $\vec{a}_1$ and $\vec{a}_2$ is 60°. It is possible to see the following facts geometrically:

- First, $\vec{v}_1 := \vec{a}_1 \times \vec{a}_2 = (1, -1, 1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_1 := 0$. Indeed, since v_1 is perpendicular to both $(1, 1, 0)$ and $(0, 1, 1)$, we know that v_1 lies in the nullspace of both P_1 and P_2, and hence in the nullspace of $P_1 + P_2$ as well.
- Second, $\vec{v}_2 := \vec{a}_1 + \vec{a}_2 = (1, 2, 1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_2 := 2\cos^2(60°/2) = 1 + \cos(60°) = 3/2$.
- Third, $\vec{v}_3 := \vec{a}_1 - \vec{a}_2 = (1, 0, -1)$ is an eigenvector of $P_1 + P_2$, with eigenvalue $\lambda_3 := 2\sin^2(60°/2) = 1 - \cos(60°) = 1/2$.

The second and third facts can be derived with a bit of trigonometry, but if you don't want to get into that, you can just do the usual linear-algebra calculation. Note that

$$P_1 = \frac{1}{1^2+1^2+0^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and similarly

$$P_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

so that

$$P_1 + P_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

You may solve for the roots $\lambda_1, \lambda_2, \lambda_3$ of $\det(P_1 + P_2 - \lambda I) = 0$ and find the nullspaces of $P_1 + P_2 - \lambda_i I$ (for $i = 1, 2, 3$) as usual, and get the results that we described above.

(b) (10 pts) Compute $P = P_2 P_1$. (Careful, the answer is not 0)

Let's just do it directly:

$$P = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

This page intentionally blank.

3 (10 pts.)

The nullspace of the matrix A is exactly the multiples of $(1, 1, 1, 1, 1)$.

(a) (2 pts.) How many columns are in A?

Five columns. Otherwise it doesn't make sense to multiply A with the given vector.

(b) (3 pts.) What is the rank of A?

We know that the sum of the rank of A and the dimension of the nullspace is the number of columns. Since the nullspace has dimension 1, the rank must be 4.

(c) (5 pts.) Construct a 5×5 matrix A with exactly this nullspace.

We need to construct a matrix whose sum of columns is 0 and with rank 4.

The following works :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This page intentionally blank.

4 (15 pts.)

Find the solution to

$$\frac{du}{dt} = -\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} u$$

starting with $u(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.
(Note the minus sign.)

The general formula for the solution to this differential equation is entirely analogous to the formula you learned in 18.01:

$$\vec{u}(t) = e^{-t\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \vec{u}(0).$$

Of course, you should actually rewrite this in simplest form.

To facilitate taking the matrix exponential, let's diagonalize

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues are easily computed to be 0 (because the matrix is obviously singular) and 3 (because the trace is 3), and the corresponding eigenvectors are $(2, -1)$ and $(1, 1)$, so

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}.$$

It follows that

$$e^{-t\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} e^{-t\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-0t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}.$$

Now

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix},$$

so

$$
\begin{aligned}
\vec{u}(t) &= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-0t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\vec{u}(0) \\
&= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ e^{-3t} \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t} + 2 \\ e^{-3t} - 1 \end{bmatrix}.
\end{aligned}
$$

This page intentionally blank.

5 (10 pts.)

The 3×3 matrix A satisfies $\det(tI - A) = (t-2)^3$.

(a) (2pts) What is the determinant of A?

If we take $t = 0$, we find $\det(-A) = -8$. Since A is 3×3, $\det(-A) = -\det(A)$. Therefore $\det(A) = 8$.

(b) (8pts) Describe all possible Jordan normal forms for A.

The eigenvalues of A are 2 with multiplicity 3. The possible Jordan form of A are :

$$\begin{bmatrix} 2 & 1 \text{ or } 0 & 0 \\ 0 & 2 & 1 \text{ or } 0 \\ 0 & 0 & 2 \end{bmatrix}$$

There are 4 possibilities.

This page intentionally blank.

6 (7 pts.)

The matrix $A = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}$

(a) (2 pts) What are the eigenvalues of A?

This is a triangular matrix, therefore the eigenvalues are the diagonal entries 1 and 1.

(b) (5 pts) Suppose σ_1 and σ_2 are the two singular values of A. What is $\sigma_1^2 + \sigma_2^2$?

σ_1^2 and σ_2^2 are the eigenvalues of $A^T A$. The question is asking for the trace of $A^T A$.

$$A^T A = \begin{bmatrix} 1 + C^2 & ? \\ ? & 1 \end{bmatrix}$$

(The question marks are entries that we don't care about)

The trace is $2 + C^2$.

This page intentionally blank.

7 (8 pts.)

For each transformation below, say whether it is linear or nonlinear, and briefly explain why.

(a) (2 pts) $T(v) = v/\|v\|$

Not linear. For a positive real number c, $T(cv) = T(v)$ and if T was linear we would have $T(cv) = cT(v)$.

(b) (2 pts) $T(v) = v_1 + v_2 + v_3$

Linear. We have $T(0) = 0$, $T(v + w) = T(v) + T(w)$ and $T(cv) = cT(v)$.

(c) (2 pts) $T(v) =$ smallest component of v

Not linear. Let's say that v is 2-dimensional. Take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $T(-v) = -1 \neq -T(v) = 0$.

(d) (2 pts) $T(v) = 0$

Linear. Clearly satisfies all the requirement.

This page intentionally blank.

8 (10 pts.)

V is the vector space of (at most) quadratic polynomials with basis $v_1 = 1, v_2 = (x-1), v_3 = (x-1)^2$. W is the same vector space, but we will use the basis $w_1, w_2, w_3 = 1, x, x^2$.

(a) (5 pts) Suppose $T(p(x)) = p(x+1)$. What is the 3×3 matrix for T from V to W in the indicated bases?

We have to compute $T(v_1)$, $T(v_2)$, $T(v_3)$ and write them in the basis w_1, w_2, w_3. We have $T(v_1) = 1 = w_1$, $T(v_2) = (x+1-1) = x = w_2$ and $T(v_3) = (x-1+1)^2 = w_3$. Therefore the matrix of T in these basis is the identity matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that since the two basis are different, this does not imply that T is the identity.

(b) (5 pts) Suppose $T(p(x)) = p(x)$. What is the 3×3 matrix for T from V to W in the indicated bases?

We do the same thing, $T(v_1) = w_1$, $T(v_2) = x - 1 = w_2 - w_1$, $T(v_3) = x^2 - 2x + 1 = w_3 - 2w_2 + w_1$. Therefore the matrix of T is :

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

This page intentionally blank.

9 (10 pts.)

In all of the following we are looking for a real 2×2 matrix or a simple and clear reason that one can not exist.

Please remember we are asking for a real 2×2 matrix.

(a) (2 pts) A with determinant -1 and singular values 1 and 1.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Clearly $|A| = -1$ and $A^T A = I$ therefore, the singular values are 1 and 1.

(b) (2 pts) A with eigenvalues 1 and 1 and singular values 1 and 0.

This is impossible. If the eigenvalues are 1 and 1, the matrix is non-singular, therefore it has two non-zero singular values (in general the number of non-zero singular values is the rank of the matrix).

(c) (2 pts) A with eigenvalues 0 and 0 and singular values 0 and 1

Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) (2 pts) A with rank $r = 1$ and determinant 1

Impossible. If the rank is 1, the matrix is singular and has determinant 0.

(e) (2 pts) A with complex eigenvalues and determinant 1

The determinant is the product of the eigenvalues, hence we need to find complex numbers whose product is 1. One possibility is i and $-i$. We need to find a real matrix with i and $-i$ as eigenvalues. The following matrix works :

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Bonus problem (From an earlier version of the exam)

A basis for the nullspace of the matrix A consists of the three vectors :

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the column space of the matrix A consists of the two vectors :

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(a) (2 pts.) How many rows and how many columns are in A?

If A is an $m \times n$ matrix, then the nullspace $N(A)$ is a subspace of $\mathbb{R}^n$, while the column space $C(A)$ is a subspace of $\mathbb{R}^m$. In our case, this tells us $n = 5$ and $m = 3$. So A is a 3×5 matrix; A has 3 rows and 5 columns.

(b) (6 pts.) Provide a basis for $N(A^T)$ and $C(A^T)$.

The left-nullspace of A, i.e., $N(A^T)$, is the orthogonal complement of $C(A)$ in $\mathbb{R}^3$. Let's take the cross product of the two vectors in the basis for $C(A)$:

$$(1, -1, 1) \times (2, -1, 0) = (1, 2, 1).$$

The vector $(1, 2, 1)$ is perpendicular to the plane $C(A)$, so it's a basis for the line that is orthogonal to this plane, i.e., the line $N(A^T)$. But suppose you didn't think of using the cross product; what could you have done instead? Well, $N(A^T)$ is the space of solutions (x_1, x_2, x_3) to the equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let's put the 2×3 matrix in reduced row-echelon form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Now x_3 is a free variable; set it to 1 and use back substitution to find x_1 and x_2. You get exactly $(x_1, x_2, x_3) = (1, 2, 1)$ as above; this special solution forms a basis for $N(A^T)$.

Similarly, the row space of A, i.e., $C(A^T)$, is the orthogonal complement of $N(A)$, so it consists of solutions $(x_1, \ldots, x_5)$ to the equation

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -2 & 0 & 1 & 0 \\ 6 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One could solve this by Gaussian elimination, but it's probably easiest to observe that, *if you reversed the order of the columns*, the 3×5 matrix would already be in reduced row-echelon form, so the special solutions can be found by setting (x_1, x_2) to $(1, 0)$ or $(0, 1)$, and solving for (x_3, x_4, x_5). In this way, we get the special solutions $(1, 0, -1, -3, -6)$ and $(0, 1, 2, 2, -5)$; these two vectors form a basis for $C(A^T)$.

(c) (6 pts.) Write down an example of such a matrix A.

Thanks to our answer to (b), we know that A has the same column space as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix}$$

(you could also just say

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$$

here, but our method makes the subsequent computations a bit easier). Also from our answer to (b), we know that A has the same row space as

$$\begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \end{bmatrix}.$$

So A is any matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} B \begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \end{bmatrix},$$

where B is an invertible 2×2 matrix (if you don't see why, please ask). For example, if we take B to be the identity, we get

$$A = \begin{bmatrix} 1 & 0 & -1 & -3 & -6 \\ 0 & 1 & 2 & 2 & -5 \\ -1 & -2 & -3 & -1 & 16 \end{bmatrix}.$$

18.06 Final Exam May 18, 2010 Professor Strang

1. **(12 points)** This question is about the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}.$$

(a) Find a lower triangular L and an upper triangular U so that $A = LU$.

Answer:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the reduced row echelon form $R = rref(A)$. How many independent columns in A?

Answer: 2

$$R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \text{ in this example.}$$

(c) Find a basis for the nullspace of A.

Answer:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(d) If the vector b is the sum of the four columns of A, write down the complete solution to $Ax = b$.

Answer:

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

2. **(11 points)** This problem finds the curve $y = C + D\,2^t$ which gives the best least squares fit to the points $(t, y) = (0, 6), (1, 4), (2, 0)$.

(a) Write down the 3 equations that would be satisfied *if* the curve went through all 3 points.

Answer:

$$\begin{aligned} \text{C} + \text{1D} &= 6 \\ \text{C} + \text{2D} &= 4 \\ \text{C} + \text{4D} &= 0 \end{aligned}$$

(b) Find the coefficients C and D of the best curve $y = C + D2^t$.

Answer:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Solve $A^T A\hat{x} = A^T b$:

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \text{gives} \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

(c) What values should y have at times $t = 0, 1, 2$ so that the best curve is $y = 0$?

Answer:

The projection is $p = (0, 0, 0)$ if $A^T b = 0$. In this case, $b =$ values of $y = c(2, -3, 1)$.

3. **(11 points)** Suppose $Av_i = b_i$ for the vectors $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ in R^n. Put the v's into the columns of V and put the b's into the columns of B.

(a) Write those equations $Av_i = b_i$ in matrix form. *What condition on which vectors* allows A to be determined uniquely? Assuming this condition, *find A from V and B.*

Answer:

$A\,[v_1 \cdots v_n] = [b_1 \cdots b_n]$ or $AV = B$. Then $A = BV^{-1}$ if the $v's$ are independent.

(b) Describe the column space of that matrix A in terms of the given vectors.

Answer:

The column space of A consists of all linear combinations of $b_1, \cdots, b_n$.

(c) What additional condition on which vectors makes A an *invertible* matrix? Assuming this, find A^{-1} from V and B.

Answer:

If the $b's$ are independent, then B is invertible and $A^{-1} = VB^{-1}$.

4. **(11 points)**

(a) Suppose x_k is the fraction of MIT students who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k+1$, $1/5$ of those who prefer calculus change their mind (possibly after taking 18.03). Also at year $k+1$, $1/10$ of those who prefer linear algebra change their mind (possibly because of this exam).

Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and find the limit of $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $k \to \infty$.

Answer:

$$A = \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}.$$

The eigenvector with $\lambda = 1$ is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

This is the steady state starting from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\frac{2}{3}$ of all students prefer linear algebra! I agree.

(b) Solve these differential equations, starting from $x(0) = 1, \quad y(0) = 0$:

$$\frac{dx}{dt} = 3x - 4y \qquad \frac{dy}{dt} = 2x - 3y\,.$$

Answer:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}.$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $x_1 = (2,1)$ and $x_2 = (1,1)$.
The initial vector $(x(0), y(0)) = (1,0)$ is $x_1 - x_2$.
So the solution is $(x(t), y(t)) = e^t(2,1) + e^{-t}(1,1)$.

(c) For what initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$ does the solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ to this differential equation lie on a single straight line in R^2 for all t?

Answer:

If the initial conditions are a multiple of either eigenvector $(2, 1)$ or $(1, 1)$, the solution is at all times a multiple of that eigenvector.

5. **(11 points)**

(a) Consider a $120°$ rotation around the axis $x = y = z$. Show that the vector $i = (1, 0, 0)$ is rotated to the vector $j = (0, 1, 0)$. (Similarly j is rotated to $k = (0, 0, 1)$ and k is rotated to i.) How is $j - i$ related to the vector $(1, 1, 1)$ along the axis?

Answer:

$$j - i = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

is orthogonal to the axis vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So are $k - j$ and $i - k$. By symmetry the rotation takes i to j, j to k, k to i.

(b) Find the matrix A that produces this rotation (so Av is the rotation of v). Explain why $A^3 = I$. What are the eigenvalues of A?

Answer:

$A^3 = I$ because this is three $120°$ rotations (so $360°$). The eigenvalues satisfy $\lambda^3 = 1$ so $\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3} = e^{4\pi i/3}$.

(c) If a 3 by 3 matrix P projects every vector onto the plane $x+2y+z = 0$, find three eigenvalues and three independent eigenvectors of P. No need to compute P.

Answer: The plane is perpendicular to the vector $(1, 2, 1)$. This is an eigenvector of P with $\lambda = 0$. The vectors $(-2, 1, 0)$ and $(1, -1, 1)$ are eigenvectors with $\lambda = 0$.

6. **(11 points)** This problem is about the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^T A$ and also of AA^T. For both matrices find a complete set of orthonormal eigenvectors.

Answer:

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$$

has $\lambda_1 = 70$ and $\lambda_2 = 0$ with eigenvectors $x_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$$\mathrm{AA}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{bmatrix}$$ has $\lambda_1 = 70,\ \lambda_2 = 0,\ \lambda_3 = 0$ with

$$\mathrm{x}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } x_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}.$$

(b) If you apply the Gram-Schmidt process (orthonormalization) to the columns of this matrix A, what is the resulting output?

Answer:

Gram-Schmidt will find the unit vector

$$q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

But the construction of q_2 fails because column 2 = 2 (column 1).

(c) If A is *any* m by n matrix with $m > n$, tell me why AA^T cannot be positive definite. Is A^TA always positive definite? (If not, what is the test on A?)

Answer

AA^T is m by m but its rank is not greater than n (all columns of AA^T are combinations of columns of A). Since $n < m$, AA^T is singular.

A^TA is positive definite if A has full colum rank n. (Not always true, A can even be a zero matrix.)

7. **(11 points)** This problem is to find the determinants of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) Find $\det A$ and give a reason.

Answer:

$\det A = 0$ because two rows are equal.

(b) Find the cofactor C_{11} and then find $\det B$. This is the volume of what region in R^4?

Answer:

The cofactor $C_{11} = -1$. Then $\det B = \det A - C_{11} = 1$. This is the volume of a box in R^4 with edges = rows of B.

(c) Find $\det C$ for any value of x. You could use linearity in row 1.

Answer:

$\det C = xC_{11} + \det B = -x + 1$. Check this answer (zero), for $x = 1$ when $C = A$.

8. **(11 points)**

(a) When A is similar to $B = M^{-1}AM$, prove this statement:

If $A^k \to 0$ when $k \to \infty$, then also $B^k \to 0$.

Answer:

A and B have the same eigenvalues. If $A^k \to 0$ then all $|\lambda| < 1$. Therefore $B^k \to 0$.

(b) Suppose S is a fixed invertible 3 by 3 matrix.

This question is about all the matrices A that are diagonalized by S, so that $S^{-1}AS$ is diagonal. Show that these matrices A form a subspace of 3 by 3 matrix space. (Test the requirements for a subspace.)

Answer:

If A_1 and A_2 are in the space, they are diagonalized by S. Then $S^{-1}(cA_1 + dA_2)S$ is diagonal + diagonal = diagonal.

(c) Give a basis for the space of 3 by 3 *diagonal matrices.* Find a basis for the space in part (b) — all the matrices A that are diagonalized by S.

Answer:

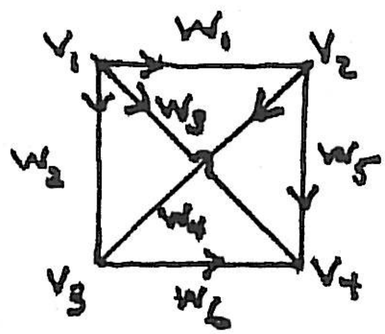
A basis for the diagonal matrices is

$$D_1 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} D_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

Then $SD_1S^{-1}, SD_2S^{-1}, SD_3S^{-1}$ are all diagonalized by S: a basis for the subspace.

9. **(11 points)** This square network has 4 nodes and 6 edges. On each edge, the direction of positive current $w_i > 0$ is from lower node number to higher node number. The voltages at the nodes are $(v_1, v_2, v_3, v_4.)$

Answer:



(a) Write down the incidence matrix A for this network (so that Av gives the 6 voltage differences like $v_2 - v_1$ across the 6 edges). What is the rank of A? What is the dimension of the nullspace of A^T?

Answer:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

has rank $r = 3$. The nullspace of A^T has dimension $6 - 3 = 3$.

(b) Compute the matrix $A^T A$. What is its rank? What is its nullspace?

Answer:

$$A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

has rank 3 like A. The nullspace is the line through $(1, 1, 1, 1)$.

(c) Suppose $v_1 = 1$ and $v_4 = 0$. If each edge contains a unit resistor, the currents $(w_1, w_2, w_3, w_4, w_5, w_6)$ on the 6 edges will be $w = -Av$ by Ohm's Law. Then Kirchhoff's Current Law (flow in = flow out at every node) gives $A^T w = 0$ which means $\boldsymbol{A^T A v = 0}$. Solve $A^T A v = 0$ for the unknown voltages v_2 and v_3. Find all 6 currents w_1 to w_6. How much current enters node 4?

Answer:

Note: As stated there is no solution (my apologies!). All solutions to $A^T A v = 0$ are multiples of $(1, 1, 1, 1)$ which rules out $v_1 = 1$ and $v_4 = 0$.
Intended problem: I meant to solve the reduced equations using KCL only at nodes 2 and 3. In fact symmetry gives $v_2 = v_3 = \frac{1}{2}$. Then the currents are $w_1 = w_2 = w_5 = w_6 = \frac{1}{2}$ around the sides and $w_3 = 1$ and $w_4 = 0$ (symmetry). So $w_3 + w_5 + w_6 = \frac{1}{2}$ is the total current into node 4.

18.06 Professor Edelman Final Exam December 22, 2011

Grading

1

2

3

Your PRINTED name is: ____________________ **4**

5

6

7

Please circle your recitation:

1	T 9	2-132	Kestutis Cesnavicius	2-089	2-1195	kestutis
2	T 10	2-132	Niels Moeller	2-588	3-4110	moller
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	kestutis
4	T 11	2-132	Niels Moeller	2-588	3-4110	moller
5	T 12	2-132	Yan Zhang	2-487	3-4083	yanzhang
6	T 1	2-132	Taedong Yun	2-342	3-7578	tedyun

1 (13 pts.)

Suppose the matrix A is the product

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (3 pts.) What is the rank of A?

A has rank 2. (Since the first matrix is non-singular, it does not affect the rank.)

(b) (5 pts.) Give a basis for the nullspace of A.

$\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Columns 1 and 2 are pivot columns. The other two are free. We assign 1,0 and 0,1 to the free variables.

(c) (5 pts.) For what values of t (if any) are there solutions to $Ax = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$?

$t = 2$. Elimination on $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & t \end{pmatrix}$ yields $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & t-2 \end{pmatrix}$.

2 (12 pts.)

Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.

(a) (3 pts.) Find a basis for the column space of A.

> $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. This matrix is familar from class. The first two columns are pivot columns, the third is free.

(b) (3 pts.) Find a basis for the column space of Σ where $A = U\Sigma V^T$ is the svd of A.

> Σ is diagonal with first two diagonal elements positive. Hence a basis for the column space is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) (3 pts.) Find a basis for the column space of the matrix exponential e^A

> The matrix exponential has full rank, so the three columns of the identity or any linearly independent set of three vectors will do.

(d) (3 pts.) Find a non-zero constant solution (meaning no dependence on t) to $\frac{d}{dt}u(t) = Au(t)$.

> $\frac{d}{dt}u(t) = 0 = Au \implies u(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, the eigenvector corresonding to 0.

3 (12 pts.)

(a) (3 pts.) Give an example of a nondiagonalizable matrix A which satisfies $\det(tI - A) = (4-t)^4$

$$\begin{pmatrix} 4 & 1 & & \\ & 4 & 1 & \\ & & 4 & 1 \\ & & & 4 \end{pmatrix}$$

is a Jordan block hence is non-diagonalizable.

(b) (3 pts.) Give an example of two different matrices that are similar and both satisfy $\det(tI - A) = (1-t)(2-t)(3-t)(4-t)$.

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \text{and} \begin{pmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}$$

(c) (3 pts.) Give an example if possible of two matrices that are not similar and both satisfy $\det(tI - A) = (1-t)(2-t)(3-t)(4-t)$.

All matrices with distinct eigenvalues 1,2,3,4 are similar, so this is impossible.

(d) (3 pts) Give an example of two different 4x4 matrices that have singular values 1,2,3,4.

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \text{and} \begin{pmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & -4 \end{pmatrix}$$

4 (16 pts.)

The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(a) (3 pts.) This matrix has two eigenvalues $\lambda = 2$, and one eigenvalue $\lambda = -2$. Given that, find the fourth eigenvalue.

The trace is $2 - 2i = 2 + 2 - 2 + ?$ so the fourth eigenvalue is $-2i$.

(b) (3 pts.) Find a real eigenvector and show that it is indeed an eigenvector.

$G \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$. One can write down $G - 2I = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2-i & -1 & i \\ 1 & -1 & -1 & -1 \\ 1 & i & -1 & -2-i \end{pmatrix}$ and notice that columns 1 and columns 3 add to 0.

(Problem 4 continued.) The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(c) (4 pts.) Is G a Hermitian matrix? Why or why not. (Remember Hermitian means that $H_{jk} = \bar{H}_{kj}$ where the bar indicates complex conjugate.)

No, the diagonals are not real.

(d) (4 pts.) Give an example of a real non-diagonal matrix X for which $G^H XG$ is Hermitian.

Any real symmetrix non-diagonal X will do, for example $X ==$ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

5 (16 pts.)

The following operators apply to differentiable functions $f(x)$ transforming them to another function $g(x)$. For each one state clearly whether it is linear or not, (expalnations not needed). (2 pts each problem)

(a) $g(x) = \frac{d}{dx} f(x)$ linear (for all linear cases check $cf(x)$ goes to $cg(x)$ and $f_1(x) + f_2(x)$ goes to $g_1(x)+g_2(x)$

(b) $g(x) = \frac{d}{dx} f(x) + 2$ not linear (zero does not go to 0)

(c) $g(x) = \frac{d}{dx} f(2x)$ linear

(d) $g(x) = f(x + 2)$ linear

(e) $g(x) = f(x)^2$ not linear (the function $cf(x)$ should go to $cg(x)$ but it goes to $c^2 g(x)$.)

(f) $g(x) = f(x^2)$ linear

(g) $g(x) = 0$ linear

(h) $g(x) = f(x)+f(2)$ linear (don't be fooled, this one is indeed linear)

6 (20 pts.)

Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) (4 pts.) There are two values of c that make A a projection matrix. Find them by guessing, calculating, or understanding projection matrices. Check that A is a projection matrix for these two c.

$A = A^2 = I - 2cE + 3c^2E \implies 3c^2 = c$ so $c = 0$ or $c = 1/3$. Thus $A = I$ or $A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ which upon squaring is itself.

(b) (4 pts.) There are two values of c that make A an orthogonal matrix. Find them and check that A is orthogonal for these two c.

$I = A^TA = A^2 = I - 2c + 3c^2E \implies 3c^2 = 2c$ so $c = 0$ or $c = 2/3$. Thus $A = I$ or $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$ which upon squaring is the identity.

(c) (4 pts.) For which values of c is A diagonalizable?

The matrix is symmetric, so all values of c make A diagonalizable.

(Problem 6 Continued) Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(d) (4 pts.) Find the eigenvalues of A^{-1} (if it exists) in terms of c. (Hint: find the eigenvalues of E_3 first.)

$E_3$is rank 1 and trace 3 so the eigenvalues are 3,0,0. Then A has eigenvalues 1-3c,1,1. Finally A^{-1} has eigenvalues $\frac{1}{1-3c}, 1, 1$.

(e) (4 pts.) For which values of c is A positive definite?

$\frac{1}{1-3c} > 0$ so $c < 1/3$.

7 (11 pts.)

The general equation of a circle in the plane has the form $x^2 + y^2 + Cx + Dy + E = 0$. Suppose you are trying to fit $n \geq 3$ distinct points (x_i, y_i), $i = 1, ..., n$ to obtain a "best" least squares circle, it is reasonable to write a generally unsolvable equation $Ax = b$.

(a) (7 pts.) Describe A and b clearly, indicating the number of rows and columns of A and the number of elements in b.

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} -x_1^2 - y_1^2 \\ -x_2^2 - y_2^2 \\ \vdots \\ -x_n^2 - y_{n^2} \end{pmatrix}$$

. The matrix A has n rows and 3 columns, while b has n elements.

(b) (4 pts.) When $n = 3$ it is possible to describe when the equation is and is not solvable. You can use your geometric intuition, or a determinant area formula to describe when A is singular. Give a simple geometrical description. (We are looking for a specific word – so only a short answer will be accepted.)

A circle is determined by three points as long as they are not colinear. The matrix A is the area matrix for a triangle, when n=3, so the interpretation is that we can solve the equation, when the area of the triangle is not-zero, i.e. the triangle does not collapse to a line.

18.06 Final Exam Professor Strang May 18, 2011

Your PRINTED name is ____________________ **1.**

Your Recitation Instructor (and time) is ____________________ **2.**

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) **3.**

4.

5.

6.

7.

8.

9.

Please show enough work so we can see your method and give due credit.

1. (a) For this matrix A, find the usual P (permutation) and L and U so that $PA = LU$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix}.$$

(b) Find a basis for the nullspace of A.

(c) The vector (b_1, b_2, b_3) is in the column space of A provided it is orthogonal to ____________ (give a numerical answer).

2. (a) Compute the 4 by 4 matrix P that projects every vector in R^4 onto the column space of A:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(b) What are the four eigenvalues of P? Explain your reasoning.

(c) Find a unit vector u (length 1) that is as far away as possible from the column space of A.

3. Suppose A is an m by n matrix and its pivot columns (not free columns) are $c_1, c_2, \ldots, c_r$. Put these columns into a matrix C.

(a) Every column of A is a ____________ of the columns of C. *How would you produce from this a matrix R so that $A = CR$?* Explain how to construct R.

(b) Using C from part (a) factor the following matrix A into CR, where C has independent columns and R has independent rows.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

4. (a) Find the cofactor matrix C for this matrix A. (The i, j entry of C is the cofactor including $\pm$ sign of the i, j position in A.)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) If a square matrix B is invertible, how do you know that its cofactor matrix is invertible?

(c) True or false **with a reason**, if B is invertible with cofactor matrix C:

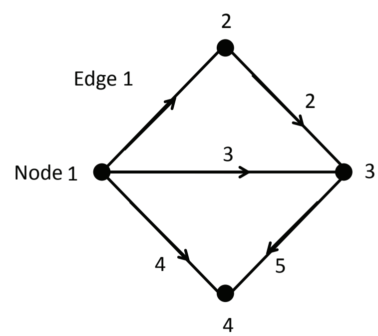
$$\text{determinant of} \quad B^{-1} = \frac{\text{determinant of} \quad C}{\text{determinant of} \quad B}$$

5. (a) Find the eigenvalues λ_1, λ_2, λ_3 and a full set of independent eigenvectors x_1, x_2, x_3 (if possible) for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(b) Suppose $u_0 = 3x_1 + 7x_2 + 5x_3$ is a combination of your eigenvectors of A: Find $A^k u_0$. If $\|v\|$ is the length of v, find the limit of $\frac{\|A^{k+1}u_0\|}{\|A^k u_0\|}$ as $k \to \infty$.

6. (a) For this directed graph, write down the 5 by 4 incidence matrix A. **Describe the nullspace of** A.



(b) Find the matrix $G = A^T A$. Is this matrix G positive definite? (Explain why or why not.) The first entry is $G_{11} = 3$ because the graph has ____________.

(c) What is the sum of the squares of the singular values of A? *Hint:* Remember that those numbers σ^2 are ____________.

7. Suppose A is a positive definite symmetric matrix with n different eigenvalues: $Ax_i = \lambda_i x_i$.

(a) What are the properties of those λ's and x's? How would you find an orthogonal matrix Q so that $A = Q\Lambda Q^T$ with $\Lambda =$ diag $(\lambda_1, \ldots, \lambda_n)$?

(b) I am looking for a symmetric positive definite matrix B with $B^2 = A$ (a square root of A). What will be the eigenvectors and eigenvalues of B? Can you find a formula for B using Q and Λ from part (a)?

(c) What are the eigenvalues and eigenvectors of the matrix e^{-A}? Is this matrix also positive definite and why?

8. Suppose the 2 by 3 matrix A has $Av_1 = 3u_1$ and $Av_2 = 5u_2$ with orthonormal v_1, v_2 in R^3 and orthonormal u_1, u_2 in R^2.

(a) Describe the nullspace of A.

(b) Find the eigenvalues of A^TA.

(c) Find the eigenvalues **and eigenvectors** of AA^T.

9. (a) The **index** of a matrix A is the dimension of its nullspace minus the dimension of the nullspace of A^T. If A is a 9 by 7 matrix of rank r, what is its index?

(b) Suppose M is the vector space consisting of all 2 by 2 matrices. (So those matrices are the "vectors" in M.) Write down a basis for this vector space M: linearly independent and spanning the space M.

(c) $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a specific matrix in M. For every 2 by 2 matrix A, the transformation T produces $T(A) = S^{-1}AS$. Is this a **linear** transformation? What tests do you have to check?

18.06 Professor Edelman Final Exam December 20, 2012

Grading

1

2

3

4

5

6

7

Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	2-132	Andrey Grinshpun	2-349	3-7578	agrinshp
2	T 10	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
3	T 10	2-146	Andrey Grinshpun	2-349	3-7578	agrinshp
4	T 11	2-132	Rosalie Belanger-Rioux	2-331	3-5029	robr
5	T 12	2-132	Geoffroy Horel	2-490	3-4094	ghorel
6	T 1	2-132	Tiankai Liu	2-491	3-4091	tiankai
7	T 2	2-132	Tiankai Liu	2-491	3-4091	tiankai

1 (10 pts.)

What condition on b makes the equation below solvable? Find the complete solution to $\mathbf{x}$ in the case it is solvable.

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}.$$

Solution:

Let's use Gaussian elimination. Starting from

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix},$$

multiply both sides by the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left, which has the effect of subtracting twice the first row from the second:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix}.$$

Then multiply both sides by the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ on the left, which has the effect of subtracting the second row from the third:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ b-1 \end{pmatrix}.$$

Comparing the third row on both sides, we find that $0 = b - 1$. The first and second rows of the matrix on the left-hand side both have pivots, so there are no other restrictions. The equation is solvable precisely when $b = 1$. Let us solve it in this case.

The free variables are x_2 and x_4. To find a *particular solution* to the equation at hand, set both free variables to zero, and solve for the pivot variables; we get $\mathbf{x} = (\frac{1}{2}, 0, \frac{1}{2}, 0)$. To find the complete solution, we must solve the homogeneous equation

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

The two special solutions to this homogeneous equation are found by setting one of the free variables to 1, the other to 0: we get $(-3, 1, 0, 0)$ and $(0, 0, -2, 1)$. Therefore, the complete solution to the original equation when $b = 1$ is

$$\mathbf{x} = \begin{pmatrix} \frac{1}{2} - 3x_2 \\ x_2 \\ \frac{1}{2} - 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

2 (6 pts.)

Let C be the cofactor matrix of an $n \times n$ matrix A. Recall that C satisfies $AC^T = (\det\ A)I_n$. Write a formula for $\det C$ in terms of $\det A$ and n.

Solution:

Since $AC^T = (\det\ A)I_n$, we get $\det(AC^T) = \det((\det\ A)I_n)$. The left hand side simplifies to $\det A \det C$ and the right hand side is equal $(\det A)^n$. This gives $\det C = (\det A)^{n-1}$.

The conscientious may object that we have divided both sides of the equation $\det A \det C = (\det A)^n$ by $\det A$, which is invalid if $\det A = 0$. So we still have to prove that, if $\det A = 0$, then C must also be singular. Well, assume for the sake of contradiction that $\det A = 0$ but C is invertible. Then C^T is also invertible, and we may multiply the original equation $AC^T = (\det A)I_n$ by $(C^T)^{-1}$:

$$A = (\det A)(C^T)^{-1} = 0(C^T)^{-1} = 0.$$

So A is the zero matrix, but in this case obviously so is its cofactor matrix C. This contradiction shows that indeed $\det C = (\det A)^{n-1}$ in all cases.

3 (25 pts.)

The matrix $A = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$ satisfies $A^2 = 6A$.

(a) (4 pts.) The eigenvalues of A are $\lambda_1 =$______, $\lambda_2 =$______, and $\lambda_3 =$______.

(b) (5 pts) Find a basis for the nullspace of A and the column space of A.

(c) (16 pts.) Circle all that apply. The matrix $M = \frac{1}{6}A$ is

1.orthogonal   3. a projection   5. singular   7. a Fourier matrix

2. symmetric   4. a permutation   6. Markov   8. positive definite

4 (12 pts.)

The matrix $G = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2-i & -1 & i \\ 1 & -1 & -1 & -1 \\ 1 & i & -1 & -2-i \end{pmatrix}$, where $i = \sqrt{-1}$.

(a) (6 pts) Use elimination or otherwise to find the rank of G.

(b) (6 pts) Find a real nonzero solution to $\frac{d}{dt}x(t) = Gx(t)$.

5 (12 pts.)

Given a vector x in $\mathbb{R}^n$, we can obtain a new vector $y = \text{cumsum}(x)$, the cumulative sum, by the following recipe:

$$y_1 = x_1$$

$$y_j = y_{j-1} + x_j, \text{ for } j = 2, ..., n.$$

(a) (7 pts) What is the Jordan form of the matrix of this linear transformation?

Solution:

Let's first find the matrix A representing the linear transformation cumsum, and then worry about finding the Jordan form of A. Note that cumsum maps $(1, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$, so the first column of A should be $(1, 1, \ldots, 1)$. The other columns of A can be found in a similar way, and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

If $n = 1$, then A is already in Jordan form. *So, for the rest of this solution, let's assume* $n \geqslant 2$.

To find the Jordan form of A, let's first find its eigenvalues. We see that A is a lower triangular matrix, so its eigenvalues are just its diagonal entries, which are all equal to 1. Thus, 1 is the only eigenvalue of A, and it occurs with arithmetic multiplicity n. This fact alone is not enough to determine the Jordan form of A, however. In fact, there are $p(n)$ non-similar $n \times n$ matrices whose only eigenvalue is 1, where $p(n)$ denotes the number of *partitions* of n — the number of distinct ways of writing n as a sum of positive integers, if order is irrelevant. Of the $p(n)$ possible distinct Jordan forms of such matrices, which one is actually the Jordan form of A?

One possibility that we can immediately eliminate is the identity matrix I_n. The only matrix similar to I_n is I_n itself: $MI_nM^{-1} = I_n$ for any invertible matrix M. Since A isn't the identity matrix, it isn't similar to I_n either, and its Jordan form is not I_n.

The key is to consider $A - I_n$: this matrix has rank $n-1$, because (for example) its transpose is in row-echelon form with $n-1$ pivots. Thus, $A - I_n$ has a 1-dimensional kernel, which is to say A has a *1-dimensional eigenspace* for the eigenvalue 1. This means that the Jordan form of A consists of a single Jordan block, which must therefore be

$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}$$

(the empty entries are zeroes).

(b) (5 pts) For every n,find an eigenvector of cumsum.

Solution:

An eigenvector for A is the same thing as a vector in the nullspace of

$$A - I_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}.$$

Row operations or mere inspection quickly leads to the conclusion that only the nth column is free, while all other columns have pivots. So all eigenvectors for A are scalar multiples of $(0, \ldots, 0, 1)$. It is easy to check that $(0, \ldots, 0, 1)$ is unchanged by the transformation cumsum, so this makes sense.

6 (20 pts.)

This problem concerns matrices whose entries are taken from the values +1 and -1. In other words, the general form of this matrix is $\begin{pmatrix} \pm 1 & \pm 1 & ... & \pm 1 \\ \pm 1 & \pm 1 & ... & \pm 1 \\ \vdots & \vdots & \ddots & \\ \pm 1 & \pm 1 & ... & \pm 1 \end{pmatrix}$. We will call these matrices ± 1matrices.

One $3x3$ example of such a matrix is $\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

a) (5 pts.) Find a two by two example of a ± 1 matrix with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$ or prove it is impossible.

b) (5 pts.) Suppose A is a 10×10 example of a ± 1 matrix. Compute $\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{10}^2$

c) (5 pts.) The big determinant formula for a 5×5 A has exactly ________ terms. A computer package for matrices computes that the determinant of a ± 1 matrix that is 5×5 is an odd integer. If this is possible exhibit such a ± 1 matrix, if not argue clearly why the package must not be giving the right answer for this 5×5 matrix.

d) (5 pts.) For every n, construct a ± 1matrix $A_n$with $(n-1)$ eigenvalues exactly equal to 2. (Hint: Think about $A_n - 2I$.)

7 (15 pts.)

Let V be the six dimensional vector space of functions $f(x, y)$ of the form $ax^2 + bxy + cy^2 + dx + ey + f$. Let W be the three dimensional vector space of (at most) second degree quadratics in x.

a) (6 pts.) Write down a basis for V and a basis for W.

Solution:

A basis for V is $1, x, y, xy, x^2, y^2$. A basis for W is $1, x, x^2$.

b (9 pts.) In your chosen basis, what is the matrix of the linear transformation from V to W that takes $f(x, y)$ to $g(x) = f(x, x)$?

Recall that the ith column of the matrix simply describes the image of the ith basis vector of V as a linear combination of the basis vectors of W. Therefore, the transformation is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Fall 2012 18.06 final solutions (questions ≥3, excluding 5, 7)

3. a) If $Av=\lambda v$ then $\lambda^2=6\lambda$, so $\lambda=0$ or 6.

Sum evals = trace $A=12$, so evals must be $6, 6, 0$.

b) Basis for nullspace (dim 1); so just find a nullspace vector):

$$\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}.$$

c) Column space of a symmetric matrix is orthogonal to its nullspace, so a basis is e.g.

$$\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}.$$

c) $M=\frac{1}{6}A$ is symmetric, a projection, singular, Markov and none of the others.

4. a) Elimination:

$$G \Rightarrow \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & -1-i & 0 & 1+i \\ 0 & 0 & 0 & 0 \\ 0 & 1+i & 0 & -1-i \end{pmatrix}$$

2nd row = − 4th row, so now clear that rank $G = 2$.

b) Note that $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \in$ null G, so the constant function

$x(t) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ works, since $\frac{d}{dt} x(t) = G x(t) = 0$.

---

5. Already present in pdf.

6.

a) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has $\lambda_1 = \lambda_2 = 0$.

b) $\sigma_1^2 + \sigma_2^2 + \dots + \sigma_{10}^2$ = sum of squares of entries of matrix.

All 100 entries square to 1, so sum of squares = 100.

c) The big determinant formula has $5! = 120$ terms. It is not possible for the determinant of a 5×5 matrix with entries ± 1 to be odd, because then every term in the big formula is ± 1, and so odd. But the sum of an even number of odd numbers is always even.

18.06 Professor Strang Final Solutions May 23rd, 2012

Grading

1

2

3

4

5

6

7

8

Your PRINTED name is: ____________________

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	`ailsa`
r02	T 11	36-153	Rune Haugseng	`haugseng`
r03	T 12	4-159	Jennifer Park	`jmypark`
r04	T 12	36-153	Rune Haugseng	`haugseng`
r05	T 1	4-153	Dimiter Ostrev	`ostrev`
r06	T 1	4-159	Uhi Rinn Suh	`ursuh`
r07	T 1	66-144	Ailsa Keating	`ailsa`
r08	T 2	66-144	Niels Martin Moller	`moller`
r09	T 2	4-153	Dimiter Ostrev	`ostrev`
r10	ESG		Gabrielle Stoy	`gstoy`

1 (12 pts.)

(a) - Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution. The eigenvalues are:

$$\boxed{\lambda = 0, 3, 6}$$

The corresponding eigenvectors are:

$$\boxed{\lambda = 0 : \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}}$$

$$\boxed{\lambda = 3 : \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{\lambda = 6 : \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}}$$

□

(b) - Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of A.

- Find the vector $A^{10}\mathbf{v}$.

Solution. We have that, forming $T = [\, v_1 \mid v_2 \mid v_3 \,]$ (with columns = the three vectors),

$$\boxed{\mathbf{y} = T^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \end{bmatrix}}$$

Or in other words:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

Therefore, we also see:

$$A^{10}\mathbf{v} = -3^{10}\frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6^{10}\frac{1}{3} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} 100737594 \\ 60466176 \\ 60466176 \end{bmatrix}$$

(*) Required for mental arithmetics wizards only. □

(c) If you solve $\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$ (notice the minus sign), with $\mathbf{u}(0)$ a given vector, then as $t \to \infty$ the solution $\mathbf{u}(t)$ will always approach a multiple of a certain vector $\mathbf{w}$.

- Find this steady-state vector $\mathbf{w}$.

Solution. Since the eigenvalues of $-A$ are $0, -3, -6$, we see that this steady state is:

$$\mathbf{w} = v_1 = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}$$

□

2 (12 pts.)

Suppose A has rank 1, and B has rank 2 (A and B are both 3×3 matrices).

(a) - What are the possible ranks of $A + B$?

Solution. Of course, $0 \leq \text{rank}(A + B) \leq 3$. But the only ranks that are possible are:

$$\boxed{\text{rank}(A + B) = 1, 2, 3.}$$

The reason 0 is not an option is: It implies $A + B = 0$, i.e. that $A = -B$. But $\text{rank}(-B) = \text{rank}(B)$, so for that to happen A and B should have had the same rank. □

(b) - Give an example of each possibility you had in (a).

Solution. Here are some simple examples:

Example w/ $\text{rank}(A + B) = 1$: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Example w/ $\text{rank}(A + B) = 2$: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Example w/ $\text{rank}(A + B) = 3$: Take e.g.

$$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

□

(c) - What are the possible ranks of AB?

- Give an example of each possibility.

Solution. As a general rule, recall $0 \leq \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) = 1$. In this case, both possibilities do happen:

$$\boxed{\text{rank}(AB) = 0, 1.}$$

Diagonal examples suffice:

<u>Example w/ $\text{rank}(AB) = 0$</u>: Take e.g.

$$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

<u>Example w/ $\text{rank}(AB) = 1$</u>: Take e.g.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

□

3 (12 pts.)

(a) - Find the three pivots and the determinant of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution. We see that

$$\boxed{A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$$

Thus,

$$\boxed{\text{The pivots are } 1, 1, -2}$$

Since we reduced A without any *row switches* (permutation P's), or row scalings, we have:

$$\boxed{\det A = 1 \cdot 1 \cdot (-2) = -2}$$

□

(b) - The rank of $A - I$ is ______, so that $\lambda =$______ is an eigenvalue.

- The remaining two eigenvalues of A are $\lambda =$____________.

- These eigenvalues are all ________________________, because $A^T = A$.

Solution. We see that

$$\boxed{\operatorname{rank}(A - I) = 2}$$

So $\dim N(A - I) = 1$. Thus,

$$\boxed{\lambda = 1}$$

is an eigenvalue of algebraic and geometric multiplicity one.

The other two eigenvalues of A are:

$$\boxed{\lambda = -1, 2.}$$

$$\boxed{\text{The eigenvalues are all real values, because } A \text{ is symmetric.}}$$

□

(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Solution. As suggested, we check that A does the correct thing on the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2\}$.

$$\boxed{\left(\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T\right) \mathbf{x_i} = \lambda_i (\mathbf{x}_i^T \mathbf{x}_i) \mathbf{x}_i = \lambda_i \mathbf{x}_i = A \mathbf{x}_i}$$

Having checked this, then by linearity of matrix multiplication, the two expressions agree always (and hence the matrices are identical).

For the record, the three vectors are:

$$\mathbf{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{x_3} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

□

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $\mathbf{0}$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix A?

$A =$

- Find an orthonormal basis for that nullspace (that plane).

Solution.

$$\boxed{A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}$$

□

We could identify a basis of $N(A)$ as usual, then apply Gram-Schmidt to make it an orthonormal basis.

But if we can find two orthonormal vectors in $N(A)$, we are done. Here, one can first easily guess one vector in $N(A)$:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in N(A)$$

Then anything of the form $[a \quad 1 \quad 1]$ will be orthogonal to $\mathbf{v}_1$, and we pick the one that is in the null space:

$$\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \in N(A)$$

Then an orthonormal basis is:

$$\boxed{\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}}$$

(b) That plane is the column space of many matrices B.

- Give two examples of B.

Solution. We can use the basis vectors from above as columns, and (independent) linear combinations of them. Or filling in a zero column:

$$\boxed{B_1 = [\mathbf{v}_1 \quad \mathbf{v}_2]}$$

$$\boxed{B_2 = [\mathbf{v}_1 \quad 2\mathbf{v}_1 + \mathbf{v}_2]}$$

$$\boxed{B_3 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{0}]}$$

□

Then $c(B_i) = N(A)$.

(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)

- What is the rank of P?

Solution. It can be computed using a matrix B from above (if it has *independent* columns: So B_1, B_2 but not B_3 here), via the usual formula:

$$\boxed{P = B(B^T B)^{-1} B^T}$$

For a projection, $c(P)$ is always the subpace it projects on, in this case it is the two-dimensional plane:

$$\boxed{\text{rank}(P) = \dim c(P) = 2}$$

□

5 (12 pts.)

Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^3$. This question is about the matrix H.

$$H = I - 2\mathbf{v}\mathbf{v}^T.$$

(a) - Multiply H times H to show that $H^2 = I$.

Solution.

$$\boxed{H^2 = (I - 2\mathbf{v}\mathbf{v}^T)^2 = I^2 + 4(\mathbf{v}\mathbf{v}^T)^2 - 4\mathbf{v}\mathbf{v}^T = I + 4\mathbf{v}\mathbf{v}^T - 4\mathbf{v}\mathbf{v}^T = I}$$

□

(b) - Show that H passes the tests for being a symmetric matrix <u>and</u> an orthogonal matrix.

Solution. Transpose is linear, $I^T = I$, and anything of the form AA^T is symmetric:

$$\boxed{(I - 2\mathbf{v}\mathbf{v}^T)^T = I - 2(\mathbf{v}^T)^T\mathbf{v}^T = I - 2\mathbf{v}\mathbf{v}^T}$$

For orthogonality, we use (a) and symmetry:

$$\boxed{HH^T = H^2 = I}$$

□

(c) - What are the eigenvalues of H?

You have enough information to answer for any unit vector $\mathbf{v}$, but you can choose one $\mathbf{v}$ and compute the λ's.

Solution. Note first that (since $\|\mathbf{v}\| = 1$):

$$H\mathbf{v} = \mathbf{v} - 2(\mathbf{v}^T\mathbf{v})\mathbf{v} = -\mathbf{v},$$

so that

$$\boxed{\lambda = -1}$$

is an eigenvalue (with a one-dimensional eigenspace spanned by $\mathbf{v}$).

Let on the other hand $\mathbf{u} \in \left(\text{span}\{\mathbf{v}\}\right)^\perp$ be any vector orthogonal to $\mathbf{v}$. Then we have:

$$H\mathbf{u} = \mathbf{u} - 2(\mathbf{v}^T\mathbf{u})\mathbf{v} = \mathbf{u},$$

so that

$$\boxed{\lambda = 1}$$

is also an eigenvalue.

Since $\left(\text{span}\{\mathbf{v}\}\right)^{\perp}$ is two-dimensional, we have found all eigenvalues.

$\square$

6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0),\ \ (-1, 0),\ \ (0, 1),\ \ (1, 1),\ \ (2, 1).$$

Solution. We insert all points into the equation:

$$\begin{aligned} -2C + D &= 0 \\ -C + D &= 0 \\ 0 + D &= 1 \\ 1 + D &= 1 \\ 2C + D &= 1. \end{aligned}$$

Written as a matrix system:

$$A\mathbf{x} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{b}$$

We consider instead $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. We compute:

$$A^TA = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}, \qquad A^T\mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

and

$$(A^TA)^{-1} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}.$$

Thus finally:

$$\begin{bmatrix} C \\ D \end{bmatrix} = \hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 3/10 \\ 3/5 \end{bmatrix}.$$

So, the closest line to the five points is:

$$\boxed{y = \frac{3}{10}t + \frac{3}{5}.}$$

□

(b) - The word "closest" means that you minimized which quantity to find your line?

Solution. $\boxed{\text{It means that the sum of squares deviation } \|A\mathbf{x} - \mathbf{b}\|^2 \text{ was minimized.}}$

□

(c) If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated). Since $A^T A$ is symmetric and $\mathbf{x} \cdot (A^T A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$ always, it is positive semi-definite. Since $N(A^T A) = \{0\}$, zero is not eigenvalue. Hence:

$$\boxed{\text{The eigenvalues of } A^T A \text{ are positive, if } A^T \text{ is invertible}}$$

By symmetry:

$$\boxed{\text{Eigenvectors belonging to different eigenvalues are orthogonal}}$$

7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad H^2 = 4I.$$

(a) What is the determinant of H?

Solution. By row reduction, we get the pivots $1, -2, -2, 4$, so:

$$\boxed{\det H = 16}$$

□

(b) What are the eigenvalues of H? (Use $H^2 = 4I$ and the trace of H).

Solution. By $H^2 = 4I$, the eigenvalues are all either ± 2. They sum up to $\mathrm{tr} H = 0$. Hence:

$$\boxed{\text{Two eigenvalues must be } +2, \text{ and two eigenvalues be } -2}$$

Note also that this shows $\det H = 16$ as in (a) □

(c) What are the singular values of H?

$$\boxed{\text{The singular values of } H \text{ are } 2, 2, 2, 2}$$

8 (16 pts.)

In this TRUE/FALSE problem, you should *circle* your answer to each question.

(a) Suppose you have 101 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{101} \in \mathbb{R}^{100}$.

- Each v_i is a combination of the other 100 vectors: TRUE – (FALSE)

- Three of the v_i's are in the same 2-dimensional plane: TRUE – (FALSE)

(b) Suppose a matrix A has repeated eigenvalues $7, 7, 7$, so $\det(A - \lambda I) = (7 - \lambda)^3$.

- Then A certainly cannot be diagonalized ($A = S\Lambda S^{-1}$): TRUE – (FALSE)

- The Jordan form of A must be $\mathcal{J} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$: TRUE – (FALSE)

(c) Suppose A and B are 3×5.

- Then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$: (TRUE) – FALSE

(d) Suppose A and B are 4×4.

- Then $\det(A + B) \leq \det(A) + \det(B)$: TRUE – (FALSE)

(e) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthonormal, and call the vector $\mathbf{b} = 3\mathbf{u} + \mathbf{v}$. Take V to be the line of all multiples of $\mathbf{u} + \mathbf{v}$.

- The orthogonal projection of $\mathbf{b}$ onto V is $2\mathbf{u} + 2\mathbf{v}$: (TRUE) – FALSE

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$, for a fixed function f. The input is x, the output is $T(x)$.

- Then T is always a linear transformation: TRUE – (FALSE)

18.06 Professor Edelman Final Exam December 19, 2013

Grading

1

2

3

4

5

6

Your PRINTED name is: ____________________

Please circle your recitation:

1	T 9	Dan Harris	E17-401G	3-7775	dmh
2	T 10	Dan Harris	E17-401G	3-7775	dmh
3	T 10	Tanya Khovanova	E18-420	4-1459	tanya
4	T 11	Tanya Khovanova	E18-420	4-1459	tanya
5	T 12	Saul Glasman	E18-301H	3-4091	sglasman
6	T 1	Alex Dubbs	32-G580	3-6770	dubbs
7	T 2	Alex Dubbs	32-G580	3-6770	dubbs

This page intentionally blank.

1 (6 pts.)

Project b onto the column space of A:

(a) (3 pts.) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(b) (3 pts.) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$.

2 (20 pts.)

The matrix $A = \begin{pmatrix} z & z & z \\ z & z+c & z-c \\ z & z-c & z+c \end{pmatrix}$.

a) (4 pts) Under what conditions on z and c would A be positive semidefinite?

b) (4 pts) Under what conditions on z and c would A be Markov?

c) (4 pts) Under what conditions on z and c would the first column of A be a free column?

d) (4 pts) Under what conditions on z and c does A have rank $r = 2$.

e) (4 pts) Assuming A has rank 2, for which b in R^3 does the equation $Ax = b$ have a solution?

3 (22 pts.)

a) (5 pts.) What are the two eigenvalues of $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$?

b) (5 pts.) What is κ, the ratio of the maximum and minimum values of $\|Ax\|$ over the unit circle $\|x\| = 1$?

c) (6 pts.) What are the maximum and minimum value of $v^T x$, for x over the unit circle $\|x\| = 1$, when $v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$? Same question when $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$? (Hint: the minimum is negative)

d) (6 pts.) When every point on the unit circle is multiplied by A, the result is an ellipse. Find the dimensions of the tightest rectangle with sides parallel to the coordinate axes that encloses the ellipse. (Hint: the previous maximum and minimum question is meant to be a warm-up. Another hint: singular values are not so useful here.)

4 (23 pts.)

The 8×8 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

is symmetric and satisfies $A^2 = 8I$. The diagonal elements add to 0.

a) (3 pts.) What are the eigenvalues of A and the determinant of A?

b) (2 pts.) What is the condition number $\kappa(A) = \sigma_1(A)/\sigma_8(A)$? (Hint $\kappa \geq 1$ always.)

c) (3 pts.) Can A have any Jordan blocks of size greater than 1? Explain briefly.

d) (2 pts.) What is the rank of the matrix consisting of the first four columns of A?

e) (3 pts.) The subspace of R^8 spanned by the first four columns of A is the ____________ ____________ of the last four columns of A. Fill in the blank with the best two words and explain briefly.

f) (5 pts.) Let P project onto the first column of A. P is an 8×8 matrix all of whose 64 entries are the same number. This number is ____________________.

g) (5 pts.) Projection onto the last seven columns of A (we are dropping only the first column) gives an 8×8 projection matrix whose 8 diagonal entries are the same number. This number is ____________________.

5 (15 pts.)

Consider the vector space of symmetric 2×2 matrices of the form

$$S = \begin{pmatrix} x & z \\ z & y \end{pmatrix}.$$

a) (6 pts.) Write down a basis for this vector space.

b) (6 pts.) If D is the diagonal matrix $D = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}$, is $T(S) = DSD$ a linear transformation from this space to itself? If so, write down the matrix of this transformation in your chosen basis.

c) (3 pts.) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we want to know if $T(S) = ASA$ os necessarily a linear transformation from symmetric 2×2 matrices to symmetric 2×2matrices? If yes, explain why. If not, explain why not.

6 (14 pts.)

Choose the best choice from “must", “might", "can’t" (and explain briefly.)

a) (2 pts.) A $3x3$ matrix M with rank $r = 2$ ________________ have a non-zero solution to $Mx = 0$.

b) (2 pts.) A $3x3$ matrix M with rank $r = 3$ ________________ have a non-zero solution to $Mx = 0$ with $x_1 = 0$.

c) (2 pts.) A $3x4$ matrix M with rank $r = 3$ ________________ have a non-zero solution to $Mx = 0$ with $x_1 = 1$.

d) (2 pts.) A permutation matrix ________________ be singular.

e) (2 pts.) A projection matrix ________________ be singular.

f) (2 pts.) $M - \lambda I$ ________________ be singular, if λ is an eigenvalue of M.

g) (2 pts.) We recall the number of columns of an incidence matrix for a graph is the number of nodes and every row has one entry $+1$, one entry -1, and the remaining entries 0. Such an incidence matrix ________________ have full column rank.

Thank you for taking linear algebra. We hope you enjoyed it. Linear algebra will serve you well in the future. Have a happy holiday!

18.06 F13 Final: solutions

Q1. a) $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in the column space of A, and

$$b - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

is orthogonal to the column space of A, so $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is the projection.

b)

$$b = 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$ is already in the column space of A, so the projection is just b itself.

Q2 a) By elimination,

A becomes $\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & -c & c \end{pmatrix}$

then $\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & 0 & 0 \end{pmatrix}$

Its pivots are z, c. So A is +ve semidefinite
$\Leftrightarrow z \geq 0, c \geq 0$.

b) We need $3z = 1 \quad \therefore \quad z = \frac{1}{3}$.

We then have all columns summing to 1, but need all entries non-negative

i.e. $z + c = \frac{1}{3} + c \geq 0, \ \frac{1}{3} - c \geq 0$

So $-\frac{1}{3} \leq c \leq \frac{1}{3}$.

Q2. cont

c) A column is free if it is a linear combination of the columns before it. This is only true for the first column if it is 0 $\iff z=0$.

d) As we saw in part a), the rank is always ≤ 2. For rank $=2$, we must have $z \neq 0$, $c \neq 0$. (If either is 0, the matrix is rank 1, but if both are $\neq 0$, then

$$(z\ z\ z) \quad \text{and} \quad (0\ c\ -c) \quad \text{are LI.})$$

e) Suppose $Ax = b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Then

$$\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & -c & c \end{pmatrix} x' = \begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_1 \end{pmatrix} \quad \text{some } x'$$

and

$$\begin{pmatrix} z & z & z \\ 0 & c & -c \\ 0 & 0 & 0 \end{pmatrix} x'' = \begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_2 + b_3 - 2b_1 \end{pmatrix} \quad \text{some } x''$$

So there is a solution if and only if

$$b_2 + b_3 - 2b_1 = 0.$$

Q3 a) They are the solutions to the eqn

$$0 = -\lambda(3-\lambda) - 4 = \lambda^2 - 3\lambda - 4$$
$$= (\lambda+1)(\lambda-4)$$

$\lambda = -1$ or 4.

b) Since A is symmetric, singular values are |eigenvalues|. Smallest SV = min value of $\|Ax\|$ on unit circle $= 1$

Largest SV = max value of $\|Ax\|$ on unit circle

$= 4$

$\therefore\ \kappa = 4/1 = 4$.

c) Greatest when x is in same direction as v, least when in opposite direction, so

$v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ max: $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v^T x = 2$

min: $x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $v^T x = -2$

$v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ max: $x = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $v^T x = \sqrt{13}$

min: $x = -\frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $v^T x = -\sqrt{13}$

d) Max/min $v^T x$ when $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ = ~~largest~~

max/min x-coordinate of ellipse

Max/min $v^T x$ when $v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ =

max/min y-coordinate of ellipse

So: tightest rectangle has corners

$(\pm\sqrt{13}, \pm 2)$

Dimensions: $2\sqrt{13} \times 4$.

Q4

a) Since $A^2 = 8I$, any eigenvalue λ of A must satisfy $\lambda^2 = 8$

$\iff \lambda = \pm\sqrt{8}$.

~~We~~ Trace $A = 0$, so eigenvalues sum to 0

$\therefore$ must have 4 eigenvalues of $\sqrt{8}$ and 4 eigenvalues of $-\sqrt{8}$.

Det A = product of eigenvalues

$= (\sqrt{8})^4(-\sqrt{8})^4$

$= 8^4$.

b) All singular values ~~of~~ are $\sqrt{8}$ (since A is symmetric, so singular values $= |\text{eigenvalues}|$)

So $K(A) = \sqrt{8}/\sqrt{8} = 1$.

c) No. No symmetric matrix can have Jordan blocks of size greater than 1.

d) Since $\det A \neq 0$, all columns of A are LI

$\Rightarrow$ first 4 columns are LI

$\Rightarrow$ Matrix of 1st 4 columns has rank 4.

Q4 cont

e) Orthogonal complement, since all columns of A are orthogonal to one another.

f) If this number is c, then

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}c\\c\\c\\c\\c\\c\\c\\c\end{pmatrix} \text{ must be orthogonal to } \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

So $1-8c=0 \Rightarrow c=\frac{1}{8}$.

g) If P is projection onto the first column, then I-P is projection onto last 7 columns. So diagonal entries are $1-\frac{1}{8}=\frac{7}{8}$.

Q5

a)

$$\left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

b) Yes, it's linear: if S is symmetric, then

$$(DSD)^T = D^T S^T D^T = DSD$$ so DSD symmetric

$$D(S+T)D = DSD + DTD$$
$$D(\lambda S)D = \lambda DSD$$
so linear.

$$De_1D = \begin{pmatrix} d^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$De_2D = \begin{pmatrix} 0 & de \\ de & 0 \end{pmatrix}$$

$$De_3D = \begin{pmatrix} 0 & 0 \\ 0 & e^2 \end{pmatrix}$$

So matrix of T relative to $\{e_1, e_2, e_3\}$ is $\begin{pmatrix} d^2 & 0 & 0 \\ 0 & de & 0 \\ 0 & 0 & e^2 \end{pmatrix}$.

c) No, since ASA need not be symmetric.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $T(I) = A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ Not symmetric

Q6 a) Must. M has nullspace of dim 1, and any nonzero nullspace vector x has $Mx = 0$.

b) Can't. M has full rank, so there are no nonzero solutions to $Mx = 0$.

c) Might. M has dim 1 nullspace, but this may or may not contain vectors with $x_1 = 1$.

d) Can't. det (permutation matrix) = (sign of permutation) $\neq 0$

(there is exactly one nonzero term in the big formula, and it is ± 1)

e) Might ~~(any vector orthogonal to the col. spa~~

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ singular

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ not singular

f) Must. This is the definition of eigenvalue.

$(M - \lambda I)x = 0 \iff Mx = \lambda x$.

g) ~~Might. e.g.~~

~~$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$~~

~~not full column rank~~

~~$\bullet \rightarrow \bullet$ $M = \begin{pmatrix} 1 & -1 \end{pmatrix}$~~

~~Full column~~ rank

Can't. Sum of all columns = 0, so not full column rank.

Please PRINT your name ______________________

Please Circle Your Recitation

r1	T	10	36-156	Russell Hewett	r7	T	1	36-144	Vinoth Nandakumar
r2	T	11	36-153	Russell Hewett	r8	T	1	24-307	Aaron Potechin
r3	T	11	24-407	John Lesieutre	r9	T	2	24-307	Aaron Potechin
r4	T	12	36-153	Stephen Curran	r10	T	2	36-144	Vinoth Nandakumar
r5	T	12	24-407	John Lesieutre	r11	T	3	36-144	Jennifer Park
r6	T	1	36-153	Stephen Curran					

1.
2.
3.
4.
5.
6.
7.
8.
9.

(1) **(7+7 pts)**

(a) Suppose the nullspace of a square matrix A is spanned by the vector $v = (4, 2, 2, 0)$. Find the reduced echelon form $R = \text{rref}\,(A)$.

(b) Suppose S and T are subspaces of $\mathbf{R}^5$ and Y and Z are subspaces of $\mathbf{R}^3$. When can they be the four fundamental subspaces of a 3 by 5 matrix B? Find any required conditions to have $S = C(B^T),\ T = N(B),\ Y = C(B),$ and $Z = N(B^T)$.

(2) **(6+6 pts.)**

(a) Find bases for all four fundamental subspaces of this R.

$$R = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find U, Σ, V in the Singular Value Decomposition $A = U\Sigma V^T$:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

(3) **(5+5 pts.)**

Suppose $q_1, \ldots q_5$ are orthonormal vectors in $\mathbf{R}^5$.

The 5 by 3 matrix A has columns q_1, q_2, q_3.

(a) If $b = q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5$, find the best least squares solution $\widehat{x}$ to $Ax = b$.

(b) If terms of q_1, q_2, q_3 find the projection matrix P onto the column space of A.

(4) **(3+4+3 pts.)**

The matrix A is symmetric and also orthogonal.

(a) How is A^{-1} related to A?

(b) What number(s) can be eigenvalues of A and why?

(c) Here is an example of A. What are the eigenvalues of this matrix? I don't recommend computing with $\det(A - \lambda I)$! Find a way to use part (b).

$$A = \begin{bmatrix} .5 & \text{-}.5 & \text{-}.1 & \text{-}.7 \\ \text{-}.5 & .5 & \text{-}.1 & \text{-}.7 \\ \text{-}.1 & \text{-}.1 & .98 & \text{-}.14 \\ \text{-}.7 & \text{-}.7 & \text{-}.14 & .02 \end{bmatrix}.$$

(5) **(4+5+3 pts.)**

Suppose the real column vectors q_1 and q_2 and q_3 are orthonormal.

(a) Show that the matrix $A = q_1\, q_1^T + 2q_2\, q_2^T + 5q_3\, q_3^T$ has the eigenvalues $\lambda = 1,\, 2,\, 5$.

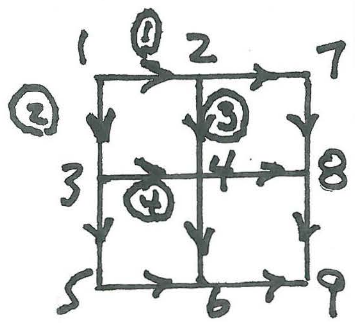
(b) Solve the differential equation $du/dt = Au$ starting at any vector $u(0)$. Your answer can involve the matrix Q with columns $q_{1,}\, q_{2,}\, q_{3,}$.

(c) Solve the differential equation $du/dt = Au$ starting from $u(0) = q_1 - q_3$.

(6) **(4+3+3 pts.)**

This graph has $m = 12$ edges and $n = 9$ nodes. Its 12 by 9 incidence matrix A has a single -1 and $+1$ in every row, to show the start and end nodes of the corresponding edge in the graph.

(a) Write down the 4 by 4 submatrix S of A that comes from the 4-node graph (a loop) in the corner. Find a vector x in the nullspace $N(S)$ and a vector y in $N(S^T)$.



(b) For the whole matrix A, find a vector Y in $N(A^T)$. You won't need to write A or to know more edge numbers.

(c) The all-ones vector $(1, 1, \ldots, 1)$ spans $N(A)$. Find the dimension of the left nullspace $N(A^T)$ (give a number).

(7) **(3+3+3+3 pts.)**

The equation $y_{n+2} + By_{n+1} + Cy_n = 0$ has the solution $y_n = \lambda^n$ if $\lambda^2 + B\lambda + C = 0$. In most cases this will give two roots λ_1, λ_2 and the complete solution is $y_n = c_1\,\lambda_1^n + c_2\,\lambda_2^n$.

Now solve the same problem the matrix way (slower). Create this vector unknown and vector equation $u_{n+1} = A\,u_n$.

$$u_n = \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+1} \end{bmatrix}.$$

(a) What is the matrix A in that equation?

(b) What equation gives the eigenvalues λ_1 and λ_2 of A?

(c) If λ_1 is an eigenvalue, show directly that

$$A \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{so we have the eigenvector.}$$

(d) If $\lambda_1 \neq \lambda_2$, what is now the complete solution u_n (including constants c_1 and c_2) to our equation $u_{n+1} = A\,u_n$? ((Then y_n is the first component of u_n.))

(8) **(5+5 pts.)**

(a) Find the determinant of this matrix A, using the cofactors of row 1.

$$A = \begin{bmatrix} 1 & b & 0 & 0 \\ b & 1 & b & 0 \\ 0 & b & 1 & b \\ 0 & 0 & b & 1 \end{bmatrix}.$$

(b) Find the determinant of A by the BIG formula with 24 terms. This means to find all the nonzero terms in that formula with their correct signs.

(9) **(6+4 pts.)**

(a) Suppose $\mathbf{v} = (v_1, v_2, v_3)$ is a column vector, so $A = \mathbf{v}\mathbf{v}^T$ is a symmetric matrix. Show that A is positive semidefinite, using one of these tests:

1. The eigenvalue test
2. The determinant test
3. The energy test on $x^T Ax$.

(b) Suppose A is m by n of rank r. What conditions on m and n and r guarantee that $A^T A$ is positive definite? If those conditions fail, prove that $A^T A$ **will not be positive definite.**

Spring '13 18.06 Final Solutions

1. a)

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- dim nullspace $= 1$ $\Rightarrow$ bottom row 0s
- last entry of nullspace vector is 0, so last column should be a pivot column

b) We must have

S is orthogonal to T and $\dim S + \dim T = 5$

Y is orthogonal to Z and $\dim Y + \dim Z = 3$

Also, $\dim Y = \dim S$

$\therefore \dim T = \dim Z + 2$

---

2. a) Column space: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Right nullspace: $\left\{ \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{3} \end{pmatrix} \right\}$

Row space: $\{(1\ 2\ 0\ 4),\ (0\ 0\ 1\ 3)\}$

Left nullspace: $\{(0\ 0\ 1)\}$

2. cont

b) We have

$$A^T A = V \Sigma^T \Sigma V^T = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

So we may take $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$.

Then $3u_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $\quad u_1 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$3u_2 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

u_3 ortho to u_1 and u_2 $\Rightarrow$ can take $u_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

$$U = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

3.

a) Projection of b onto column space of A

$= q_1 + 2q_2 + 3q_3$

∴ If $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $Ax = q_1 + 2q_2 + 3q_3$

∴ $\hat{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ best least squares approx

b) $P = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$

---

4. a) $A^{-1} = A^T = A$ (symmetric)

~~(symm)~~ (orthogonal)

b) Only ± 1 can be eigenvalues of A, since it has real eigenvalues (since symmetric) with absolute value 1 (since orthogonal).

c) Trace A = Sum of eigenvalues = 2

∴ eigenvalues must be (1, 1, 1, −1)

5. a)

$$Aq_1 = q_1, \quad Aq_2 = 2q_2, \quad Aq_3 = 5q_3$$

b) Write $u(0) = a_1 q_1 + a_2 q_2 + a_3 q_3$

Then $u(t) = e^{t} a_1 q_1 + e^{2t} a_2 q_2 + e^{5t} a_3 q_3$

c) Thus if $u(0) = q_1 - q_3$,

$$u(t) = e^{t} q_1 - e^{5t} q_3.$$

---

6. a)

$$S = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a vector in $N(S)$

$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is a vector in $N(S^T)$.

6. cont

b) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a vector in $N(A^T)$.

c) Rank $A = 9 - \dim(\text{null } A) = 8$

Dim null $(A^T) = 12 - \text{rank } A = 4$.

---

7. a) ~~$A = \begin{pmatrix} C & B \end{pmatrix}$~~ $A = \begin{pmatrix} 0 & 1 \\ C & B \end{pmatrix}$

b) $\det(A - \lambda I) = 0$

that is $\det\begin{pmatrix} -\lambda & 1 \\ C & B-\lambda \end{pmatrix} = -\lambda(B-\lambda) - C$

$= \lambda^2 - B\lambda - C = 0$

c) $A\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ C & B \end{pmatrix}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ C + B\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \end{pmatrix}$ by the equation in b)

so $A\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \lambda_1\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$.

7. cont

d) $\lambda_1 \neq \lambda_2 \Rightarrow$ can write $u_0 = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

So $u_n = c_1 \lambda_1^n \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \lambda_2^n \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

and $y_n = c_1 \lambda_1^n + c_2 \lambda_2^n$.

8.

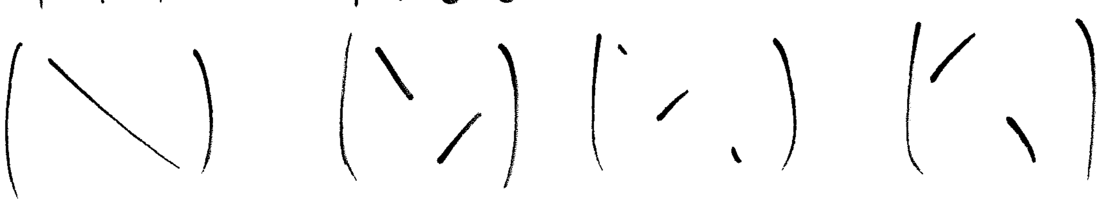
a) $\det A = \det \begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix} - b \det \begin{pmatrix} b & b & 0 \\ 0 & 1 & b \\ 0 & b & 1 \end{pmatrix}$

$= (1 - b^2 - b^2) - b(b - b^3)$

$= 1 - 3b^2 + b^4$

b) $\det A =$

$1\cdot1\cdot1\cdot1 \quad - 1\cdot1\cdot b\cdot b \quad - 1\cdot b\cdot b\cdot 1 \quad - b\cdot b\cdot 1\cdot 1$



$+ b\cdot b\cdot b\cdot b$ all other terms 0

$= 1 - 3b^2 + b^4$

9. a)

$$x^TAx = x^Tvv^Tx$$
$$= (v^Tx)^T(v^Tx)$$
$$= |v^Tx|^2 \geq 0.$$

b) We must have $r = n$ (i.e. null $A = 0$)
This implies $m \geq n$ since $r \leq m$ always.

Then if $x \neq 0$

$$x^TA^TAx = (Ax)^T(Ax)$$
$$= |Ax|^2 > 0$$ since null $A = 0$

On the other hand, if $r < n$ then there is some $v \neq 0$ in null A. Then

$$v^TA^TAv = 0$$ so not positive definite.

18.06 Final Exam **Professor Strang** **May 21, 2014**

Your PRINTED Name is: ______________________________

Please circle your section:

R01	T	10	36-144	Qiang Guang
R02	T	10	35-310	Adrian Vladu
R03	T	11	36-144	Qiang Guang
R04	T	11	4-149	Goncalo Tabuada
R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
R09	T	1	36-144	Jui-En (Ryan) Chang
R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu
ESG	T	3		Gabrielle Stoy

Grading 1:

2:

3:

4:

5:

6:

7:

8:

____

Thank you for taking **18.06**. I hope you have a great summer. You could look at **18.085** (Computational Science and Engineering) which starts with applied linear algebra.

This exam has 20 parts, worth 5 points each.

For each problem, explain your answer as much as you can.

1. (15 points)

(a) If A is a 3 by 4 matrix, what does this tell us about its nullspace?

(b) If we also know that

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution, what do we know about the rank of A?

(c) If $Ax = b$ and $A^T y = 0$, find $y^T b$ by using those equations. This says that the ____ space of A and the ____________________ are _________.

2. (15 points) Suppose $Ax = b$ reduces by the usual row operations to $Ux = c$:

$$Ux = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_2 + b_1 \end{bmatrix} = c.$$

(a) Give a basis for the nullspace of A (that matrix is not shown) and a basis for the row space of A.

(b) When does $Ax = b$ have a solution? Give a basis for the column space of A.

(c) Give a basis for the nullspace of A^T.

3. (10 points)

(a) Suppose q_1, q_2 are orthonormal in $\mathbb{R}^4$, and v is NOT a combination of q_1 and q_2. Find a vector q_3 by Gram-Schmidt, so that q_1, q_2, q_3 is an orthonormal basis for the space spanned by q_1, q_2, v.

(b) If p is the projection of b onto the subspace spanned by q_1 and q_2 and v, find p as a combination of q_1, q_2, q_3. (You are solving the least squares problem $Ax = b$ with $A = [q_1, q_2, q_3]$.)

4. (10 points)

(a) To solve a square system $Ax = b$ when $\det A \neq 0$, Cramer's Rule says that the first component of x is

$$x_1 = \frac{\det B}{\det A} \quad \text{with} \quad B = [b\, a_2 \ldots a_n]\,.$$

So b goes into the first column of A, replacing a_1. If $b = a_1$, this formula gives the right answer $x_1 = \frac{\det A}{\det A} = 1$.

1. If $b =$ a different column a_j, show that this formula gives the right answer, $x_1 =$ ____.
2. If b is any combination $x_1a_1 + \cdots + x_na_n$, why does this formula give the right answer x_1?

(b) Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}.$$

Scrap Paper

5. (15 points)

(a) Suppose an n by n matrix A has n independent eigenvectors $x_1, \ldots, x_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. What matrix equation would you solve for $c_1, \ldots, c_n$ to write the vector u_0 as a combination $u_0 = c_1 x_1 + \cdots + c_n x_n$?

(b) Suppose a sequence of vectors $u_0, u_1, u_2, \ldots$ starts from u_0 and satisfies $u_{k+1} = A u_k$. Find the vector u_k as a combination of $x_1, \ldots, x_n$.

(c) State the exact requirement on the eigenvalues λ so that $A^k u_0 \to 0$ as $k \to \infty$ for every vector u_0. Prove that your condition **must hold**.

6. (10 points)

(a) Find the eigenvalues of this matrix A (the numbers in each column add to zero).

$$A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 1 & -1 & 1 \\ 0 & \frac{1}{2} & -1 \end{bmatrix}.$$

(b) If you solve $\frac{du}{dt} = Au$, is (1) or (2) or (3) true as $t \to \infty$?

(1) $u(t)$ goes to zero?

(2) $u(t)$ approaches a multiple of (what vector?)

(3) $u(t)$ blows up?

7. (10 points) Every invertible matrix A equals an orthogonal matrix Q times a positive definite matrix S. This famous fact comes directly from the SVD for the square matrix $A = U\Sigma V^T$, by choosing $Q = UV^T$.

(a) How can you prove that $Q = UV^T$ is orthogonal?

(b) Substitute Q^{-1} and A to write $S = Q^{-1}A$ in terms of U, V and Σ. How can you tell that this matrix S is symmetric positive definite?

8. (15 points) A 4-node graph has all six possible edges. Its incidence matrix A and its Laplacian matrix A^TA are

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(a) Describe the nullspace of A.

(b) The all-ones matrix $B = \text{ones}\,(4)$ has what eigenvalues? Then what are the eigenvalues of $A^TA = 4I - B$?

(c) For the Singular Value Decomposition $A = U\Sigma V^T$, can you find the nonzero entries in the diagonal matrix Σ and one column of the orthogonal matrix V?

Scrap Paper

1. (15 points)

(a) If A is a 3 by 4 matrix, what does this tell us about its nullspace?

Solution: $dim\, N(A) \geq 1$, since $rank(A) \leq 3$.

(b) If we also know that

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution, what do we know about the rank of A?

Solution: $C(A)$ does not span the entire $\mathbf{R}^3$, so $rank(A) \leq 2$.

(c) If $Ax = b$ and $A^T y = 0$, find $y^T b$ by using those equations. This says that the _____ space of A and the _________________________ are __________.

Solution: $y^T b = y^T (Ax) = (A^T y)x = 0$. This says that the **column** space of A and the **null space** of A^T are **orthogonal**.

2. (15 points) Suppose $Ax = b$ reduces by the usual row operations to $Ux = c$:

$$Ux = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_2 + b_1 \end{bmatrix} = c.$$

(a) Give a basis for the nullspace of A (that matrix is not shown) and a basis for the row space of A.

Solution: $N(A) = N(U)$, $R(A) = R(U)$. Therefore we can read the bases directly from U:

$$N(A) = span \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$R(A) = span \left\{ \begin{bmatrix} 2 \\ 6 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \end{bmatrix} \right\}$$

(b) When does $Ax = b$ have a solution? Give a basis for the column space of A.

Solution: $b \in C(A)$, equivalent to $c \in C(U)$.

By looking at c as a function of b we can reconstruct A. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$. We have $U = EA$, $c = Eb$. Hence $A = E^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 & 8 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 2 & 6 & 8 & 12 \\ 2 & 6 & 12 & 16 \end{bmatrix}$

From here we see that

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

(c) Give a basis for the nullspace of A^T.

Solution: $N(A^T) \perp C(A)$. So $N(A^T) = span \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ (we only need to find a vector orthogonal to both basis vectors we gave for $C(A)$).

3. (10 points)

(a) Suppose q_1, q_2 are orthonormal in $\mathbb{R}^4$, and v is NOT a combination of q_1 and q_2. Find a vector q_3 by Gram-Schmidt, so that q_1, q_2, q_3 is an orthonormal basis for the space spanned by q_1, q_2, v.

Solution:

$$q_3 = \frac{v - (v^T q_1)q_1 - (v^T q_2)q_2}{\|v - (v^T q_1)q_1 - (v^T q_2)q_2\|}$$

(b) If p is the projection of b onto the subspace spanned by q_1 and q_2 and v, find p as a combination of q_1, q_2, q_3. (You are solving the least squares problem $Ax = b$ with $A = [q_1, q_2, q_3]$.)

Solution:

$$p = A(A^T A)^{-1} A^T b = AA^T b = q_1(q_1^T b) + q_2(q_2^T b) + q_3(q_3^T b)$$

where we used the fact that the columns of A are orthonormal for $A^T A = I$. (It is easy to see the final result just by thinking that we are actually projecting onto an orthonormal basis).

4. (10 points)

(a) To solve a square system $Ax = b$ when $\det A \neq 0$, Cramer's Rule says that the first component of x is

$$x_1 = \frac{\det B}{\det A} \quad \text{with} \quad B = [b\, a_2 \dots a_n]\,.$$

So b goes into the first column of A, replacing a_1. If $b = a_1$, this formula gives the right answer $x_1 = \frac{\det A}{\det A} = 1$.

1. If $b =$ a different column a_j, show that this formula gives the right answer, $x_1 =$ _____.
2. If b is any combination $x_1 a_1 + \cdots + x_n a_n$, why does this formula give the right answer x_1?

Solution:

1. $x_1 = 0$, since $\det(B) = 0$. Let us check that it gives the correct answer: the solution to $Ax = a_j$ is $x = e_j$ (it is unique, since $\det(A) \neq 0$, so A's columns are linearly independent). Hence $x_1 = 0$.
2. For the same reason as above the first component of x is indeed x_1, the coefficient of a_1.

   Now let us see what Cramer's Rule gives. In this case $\det(B) = \det([b\, a_2\, \dots a_n]) = \det(x_1 \cdot a_1\, a_2\, \dots a_n) = x_1 \det(A)$. So, indeed, $x_1 = \det(B)/\det(A)$.

(b) Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}.$$

Cofactor expansion by the first row:

$$C_{11} = \det \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 7 \\ 0 & 8 & 9 \end{bmatrix} = 4 \cdot 6 \cdot 9 - 4 \cdot 8 \cdot 7 = -8$$

$$C_{12} = -\det \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 7 \\ 0 & 8 & 9 \end{bmatrix} = -(3 \cdot 6 \cdot 9 - 3 \cdot 7 \cdot 8) = 6$$

So $\det(A) = 1 \cdot (-8) + 2 \cdot 6 = 4$

5. (15 points)

(a) Suppose an n by n matrix A has n independent eigenvectors $x_1, \ldots, x_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. What matrix equation would you solve for $c_1, \ldots, c_n$ to write the vector u_0 as a combination $u_0 = c_1 x_1 + \cdots + c_n x_n$?

Solution:

$$[x_1\, x_2 \ldots x_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = u_0$$

(b) Suppose a sequence of vectors $u_0, u_1, u_2, \ldots$ starts from u_0 and satisfies $u_{k+1} = Au_k$. Find the vector u_k as a combination of $x_1, \ldots, x_n$.

Solution: Let $u_0 = c_1 x_1 + \cdots + c_n x_n$. Then $u_k = A^k u_0 = \sum_{i=1}^{n} (\lambda_i^k c_i) x_i$.

(c) State the exact requirement on the eigenvalues λ so that $A^k u_0 \to 0$ as $k \to \infty$ for every vector u_0. Prove that your condition **must hold**.

Solution: $|\lambda_i| < 1$, for all i. Clearly, if this holds, all the coefficients of x_i in $A^k u_0$ go to 0 as $k \to \infty$.

For the converse, we require that the coefficient $\lambda_i^k c_i$ of x_i in $A^k u_0$ to go to 0, for any choice of u_0. Equivalently, we need $\lambda_i^k c_i \to 0$ for any c_i. Hence $|\lambda_i| < 1$.

6. (10 points)

(a) Find the eigenvalues of this matrix A (the numbers in each column add to zero).

$$A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 1 & -1 & 1 \\ 0 & \frac{1}{2} & -1 \end{bmatrix}.$$

Solution: The number in each column add to zero, hence $\mathbf{1} \in N(A^T)$, so $\dim N(A^T) > 0$, and thus $rank(A) = rank(A^T) < 3$. So $\lambda_1 = 0$.

We can easily spot $\lambda_2 = -1$ as another eigenvalue, since subtracting $A + I$ has two equal columns, and hence $\det(A + I) = 0$.

Looking at the trace we get that $\lambda_3 = tr(A) - \lambda_1 - \lambda_2 = -2$.

(b) If you solve $\frac{du}{dt} = Au$, is (1) or (2) or (3) true as $t \to \infty$?

(1) $u(t)$ goes to zero?

(2) $u(t)$ approaches a multiple of (what vector?)

(3) $u(t)$ blows up?

Solution: Approaches a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Observe that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector for the 0 eigenvalue, and that the only nonzero eigenvalue of e^{At} as $t \to \infty$ is $e^{0 \cdot t} = 1$, with the same eigenvector.

Also, $u(t) = e^{At}u(0)$, and that the only nonzero eigenvalue of e^{At} (as $t \to \infty$) is 1, with the same eigenvector. So $\lim_{t \to \infty} u(t)$ is a projection of $u(0)$ on the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, hence a multiple of it.

7. (10 points) Every invertible matrix A equals an orthogonal matrix Q times a positive definite matrix S. This famous fact comes directly from the SVD for the square matrix $A = U\Sigma V^T$, by choosing $Q = UV^T$.

(a) How can you prove that $Q = UV^T$ is orthogonal?

Solution: Q is orthogonal if and only if $Q^TQ = I$. Notice that since A is invertible, U and V are both square matrices of full rank.

$$Q^TQ = (UV^T)^T(UV^T) = VU^TUV^T = V(U^TU)V^T = VV^T = I$$

We used the fact that U is orthogonal, hence $U^TU = I$, and that V^T is orthogonal because V's columns are eigenvectors of a symmetric matrix A^TA, so $V^T = V^{-1}$.

(b) Substitute Q^{-1} and A to write $S = Q^{-1}A$ in terms of U, V and Σ. How can you tell that this matrix S is symmetric positive definite?

Solution:

$$S = Q^{-1}A = (UV^T)^{-1}A = (V^T)^{-1}U^{-1}U\Sigma V^T = V\Sigma V^T = (\Sigma^{1/2}V^T)^T(\Sigma^{1/2}V^T)$$

8. (15 points) A 4-node graph has all six possible edges. Its incidence matrix A and its Laplacian matrix A^TA are

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(a) Describe the nullspace of A.

Solution: $N(A) = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (Recall that applying A to a vector of potentials gives the potential drops along edges, so in order for a vector of potentials to be in the null space, all the potentials within one connected component must be the same.)

(b) The all-ones matrix $B = \text{ones}\,(4)$ has what eigenvalues? Then what are the eigenvalues of $A^TA = 4I - B$?

Solution: $B = \vec{1} \cdot \vec{1}^T = 4(\vec{1}/2)(\vec{1}/2)^T$, where $\vec{1}$ is the all-ones vector in $\mathbb{R}^4$. So B has eigenvalues $4, 0, 0, 0$.

I and B diagonalize in the same eigenbasis, so $\lambda_i(4I - B) = \lambda_i(4I) - \lambda_i(B) = 4\lambda_i(I) - \lambda_i(B)$ for all i. So the eigenvalues of A^TA are $0, 4, 4, 4$.

(c) For the Singular Value Decomposition $A = U\Sigma V^T$, can you find the nonzero entries in the diagonal matrix Σ and one column of the orthogonal matrix V?

Solution: $\sigma_i = \sqrt{\lambda_i(A^TA)}$, so the nonzero singular values are $2, 2, 2$. We only need to find one eigenvector of A^TA. An obvious one is $\vec{1}/2$, since all the rows sum up to 0.

18.06 Final Exam **Professor Strang** **May 18, 2015**

Your PRINTED Name is: ____________________

Please CIRCLE your section:

R01	T10	26-302	Dmitry Vaintrob
R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
R04	T11	26-322	Francesco Lin
R05	T11	26-328	Laszlo Lovasz
R06	T12	36-144	Michael Andrews
R07	T12	26-302	Netanel Blaier
R08	T12	26-328	Laszlo Lovasz
R09	T1pm	26-302	Sungyoon Kim
R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading

1:

2:

3:

4:

5:

6:

7:

8:

____

Thank you for taking 18.06! I hope you have a wonderful summer!

EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form $R = \text{rref}(A)$ for this matrix A :

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(b) Find a basis for the column space $C(A)$.

(c) Find all solutions (and first tell me the conditions on b_1, b_2, b_3 for solutions to exist!).

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

2. (a) What is the 3 by 3 projection matrix P_a onto the line through $a = (2, 1, 2)$?

(b) Suppose P_v is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_a P_v$ (product of 2 projections)

3. Suppose I give you an orthonormal basis $q_1, \ldots, q_4$ for $\mathbf{R}^4$ and an orthonormal basis $z_1, \ldots, z_6$ for $\mathbf{R}^6$. From these you create the 6 by 4 matrix $A = z_1 q_1^T + z_2 q_2^T$.

(a) Find a basis for the nullspace of A.

(b) Find a particular solution to $Ax = z_1$ and find the complete solution.

(c) Find $A^T A$ and find an eigenvector of $A^T A$ with $\lambda = 1$.

4. Symmetric positive definite matrices H and orthogonal matrices Q are the most important. Here is a great theorem: *Every square invertible matrix A can be factored into $A = HQ$.*

(a) Start from $A = U\Sigma V^T$ (the SVD) and *choose* $Q = UV^T$. Find the other factor H so that $U\Sigma V^T = HQ$. Why is your H symmetric and why is it positive definite?

(b) Factor this 2 by 2 matrix into $A = U\Sigma V^T$ and then into $A = HQ$:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = U\Sigma V^T = HQ$$

5. (a) Are the vectors $(0,1,1),(1,0,1),(1,1,0)$ independent or dependent?

(b) Suppose T is a linear transformation with input space = output space = $\mathbf{R}^3$. We have a basis u, v, w for $\mathbf{R}^3$ and we know that $T(u) = v + w, T(v) = u + w, T(w) = u + v$. Describe the transformation T^2 by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.

6. Suppose A is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors x_1, x_2, x_3.

(a) What is the rank of A? Describe all vectors in its column space $C(A)$.

(b) How would you solve $du/dt = Au$ with $u(0) = (1, 1, 1)$?

(c) What are the eigenvalues and determinant of e^A?

7. (a) Find a 2 by 2 matrix such that

$$A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix} \text{ and also } A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

or say why such a matrix can't exist.

(b) The columns of this matrix H are orthogonal but not orthonormal:

$$H=\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & 1\\ 0 & -2 & 1 & 1\\ 0 & 0 & -3 & 1\end{bmatrix}$$

Find H^{-1} by the following procedure. First multiply H by a diagonal matrix D that makes the columns orthonormal. Then invert. Then account for the diagonal matrix D to find the 16 entries of H^{-1}.

8. (a) Factor this symmetric matrix into $A = U^T U$ where U is upper triangular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Show by two different tests that A is symmetric positive definite.

(c) Find and explain an upper bound on the eigenvalues of A. Find and explain a (positive) lower bound on those eigenvalues if you know that

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Scrap Paper

18.06 Final Exam   Professor Strang   May 18, 2015

SOLUTIONS

Your PRINTED Name is: ____________________

Please CIRCLE your section:

R01	T10	26-302	Dmitry Vaintrob
R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
R04	T11	26-322	Francesco Lin
R05	T11	26-328	Laszlo Lovasz
R06	T12	36-144	Michael Andrews
R07	T12	26-302	Netanel Blaier
R08	T12	26-328	Laszlo Lovasz
R09	T1pm	26-302	Sungyoon Kim
R10	T1pm	36-144	Tanya Khovanova
R11	T1pm	26-322	Jay Shah
R12	T2pm	36-144	Tanya Khovanova
R13	T2pm	26-322	Jay Shah
R14	T3pm	26-322	Carlos Sauer
ESG			Gabrielle Stoy

Grading
1:
2:
3:
4:
5:
6:
7:
8:

Thank you for taking 18.06! I hope you have a wonderful summer!

EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form $R = \text{rref}(A)$ for this matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution. We have

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last matrix is the RREF.

(b) Find a basis for the column space $C(A)$.

Solution. We can see that the pivot columns are columns 1 and 3, so these columns from the *original* matrix form a basis,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Find all solutions (and first tell me the conditions on b_1, b_2, b_3 for solutions to exist!).

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution. We can see that we need $b_2 = b_3$. First, let us find a particular solution. Since x_2, x_4 are free variables, we can set them to 0, and then we can solve to get

$$\begin{pmatrix} b_1 - b_2 \\ 0 \\ b_2 \\ 0 \end{pmatrix}.$$

Now, we need a basis for the nullspace, the special solutions. Setting each free variable to 1 and the other to 0, we obtain the special solutions

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

So, the general solutions are given by vectors

$$\begin{pmatrix} b_1 - b_2 \\ 0 \\ b_2 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

2. (a) What is the 3 by 3 projection matrix P_a onto the line through $a = (2, 1, 2)$?

Solution.

$$P_a = \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}}{\begin{pmatrix} 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} = \frac{1}{9} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

(b) Suppose P_v is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_a P_v$ (product of 2 projections)

Solution. $P_a P_v v = P_a v = \frac{5}{9} a$ and so $a \in C(P_a P_v) \subset C(P_a)$. Since $C(P_a)$ is spanned by a, a basis for $C(P_a P_v)$ is given by $\{a\}$.

3. Suppose I give you an orthonormal basis $q_1, \ldots, q_4$ for $\mathbf{R}^4$ and an orthonormal basis $z_1, \ldots, z_6$ for $\mathbf{R}^6$. From these you create the 6 by 4 matrix $A = z_1 q_1^T + z_2 q_2^T$.

(a) Find a basis for the nullspace of A.
Solution. The matrix has SVD ZJQ^T where J is the 6 by 4 matrix with diagonal entries $(1, 1, 0, 0)$. This means that its nullspace consists of the q's in columns of Q corresponding to zero singular values, which is q_3, q_4.

(b) Find a particular solution to $Ax = z_1$ and find the complete solution.
Solution. One particular solution to Ax_1 is q_1, since $(z_1 q_1^T)q_1 = z_1(q_1^T z_1) = z_1(q_1 \cdot q_1) = z_1$, by and $(z_2 q_2^T)q_1 = z_2(q_2^T q_1) = z_1(q_2 \cdot q_1) = 0$ by orthonormality of q_i. The complete solution is obtained by adding an element of the nullspace, i.e. a linear combination of basis vectors of the nullspace: $q_1 + cq_2 + dq_4$ for scalars c, d.

(c) Find $A^T A$ and find an eigenvector of $A^T A$ with $\lambda = 1$.
Solution. $A^T A = (q_1 z_1^T + q_2 z_2^T)(z_1 q_1^T + z_2 q_2^T)$. Expanding and reparenthezising gives $A^T A = q_1(z_1^T z_1)q_1^T + q_1(z_1^T z_2)q_2^T + q_2(z_2 z_1^T)q_2^T + q_2(z_2 z_2^T)q_2^T$. In every term, the parenthesized scalar in the middle is a dot product: $z_1 \cdot z_2 = 0$ for the middle two terms and 1 for the first and fourth terms, leaving $A^T A = q_1 q_1^T + q_2 q_2^T$. We see that $A^T A q_1 = q_1(q_1 \cdot q_1) + q_2(q_2 \cdot q_1) = q_1$ and, for the same reason, $A^T A q_2 = q_2$. So q_1 and q_2 (or any nonzero linear combination) are all eigenvectors with eigenvalue 1.

4. Symmetric positive definite matrices H and orthogonal matrices Q are the most important. Here is a great theorem: *Every square invertible matrix A can be factored into $A = HQ$.*

(a) Start from $A = U\Sigma V^T$ (the SVD) and *choose* $Q = UV^T$. Find the other factor H so that $U\Sigma V^T = HQ$. Why is your H symmetric and why is it positive definite?
Solution. By definition we need $U\Sigma V^T = A = HQ = HUV^T$ so we get by inverting U and V^T (which are orthogonal hence invertible) that $H = U\Sigma U^{-1}$. The last item can also be written as $U\Sigma U^T$ because U is orthogonal. This matrix is symmetric because $H^T = (U\Sigma U^T)^T = U\Sigma^T U^T = H$ as Σ is diagonal so it is equal to its own transpose. To see that it is positive definite we can use the eigenvalue test: the eigenvalues of H are given by the diagonal elements of Σ, i.e. the singular values of A. They are all *nonnegative* because they are the eigenvalues of $A^T A$, and they cannot be zero because A is invertible by assumption. Hence the eigenvalues of H are all positive.

(b) Factor this 2 by 2 matrix into $A = U\Sigma V^T$ and then into $A = HQ$:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = U\Sigma V^T = HQ$$

Solution. We have $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix}$ so the singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = \sqrt{2}$ and the corresponding eigenvectors are $v_1 = (0, 1)$ and $v_2 = (1, 0)$ so that $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We then have

$$u_1 = Av_1/\sigma_1 = (1/\sqrt{2}, 1/\sqrt{2}) \qquad u_2 = Av_2/\sigma_2 = (1/\sqrt{2}, -1/\sqrt{2}),$$

so the SVD is

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Finally

$$H = U\Sigma U^T = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2\sqrt{2} \end{bmatrix} \qquad Q = UV^T \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

5. (a) Are the vectors $(0,1,1),(1,0,1),(1,1,0)$ independent or dependent?

Solution. These vectors are independent. One way to see this is that

$$\det\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}=2\neq 0$$

(b) Suppose T is a linear transformation with input space = output space = $\mathbf{R}^3$. We have a basis u,v,w for $\mathbf{R}^3$ and we know that $T(u)=v+w, T(v)=u+w, T(w)=u+v$. Describe the transformation T^2 by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.

Solution. We have

$$T^2(u)=T(v+w)=T(v)+T(w)=2u+v+w$$

$$T^2(v)=T(u+w)=T(u)+T(w)=u+2v+w$$

$$T^2(w)=T(u+v)=T(u)+T(v)=u+v+2w$$

Note that this means that in the basis (u,v,w), the matrix of T^2 is

$$\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$$

6. Suppose A is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors x_1, x_2, x_3.

(a) What is the rank of A? Describe all vectors in its column space $C(A)$.

Solution. Vectors x_1, x_2, and x_3 are independent. Any vector y in $\mathbf{R}^3$ can be represented as a linear combination of the eigenvectors: $y = ax_1 + bx_2 + cx_3$. Applying A we get $Ay = bx_2 - cx_3$. Thus x_2 and x_3 form a basis in the column space and the rank of A is 2.

(b) How would you solve $du/dt = Au$ with $u(0) = (1, 1, 1)$?

Solution. By the formula $u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$, where λ_i are eigenvalues and x_i the corresponding eigenvectors. We are given λ_i and x_i, so we can plug them in to get: $u(t) = c_1 e^{0t} x_1 + c_2 e^t x_2 + c_3 e^{-t} x_3 = c_1 x_1 + c_2 e^t x_2 + c_3 e^{-t} x_3$. To find the coefficients c_1, c_2, and c_3, we need to use the initial conditions, that is to solve the equation: $u(0) = (1, 1, 1) = c_1 x_1 + c_2 x_2 + c_3 x_3$.

(c) What are the eigenvalues and determinant of e^A?

Solution. The eigenvalues of e^A are the same as the eigenvalues of e^Λ, where Λ is the diagonalization of A. Therefore, the eigenvalues of e^A equal e to the power of the eigenvalues of A: $e^0 = 1$, $e^1 = e$ and $e^{-1} = 1/e$. The determinant is the product of the eigenvalues and is equal to $1 \cdot e \cdot 1/e = 1$.

7. (a) Find a 2 by 2 matrix such that

$$A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix} \text{ and also } A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

or say why such a matrix can't exist.

Solution. $A=\begin{pmatrix}1 & 1\\ 4/3 & 4/3\end{pmatrix}$ is the 2 by 2 matrix such that $A\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$ and $A\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$. One way to arrive at A is to let $B=\begin{pmatrix}3 & 3\\ 4 & 4\end{pmatrix}$ be the matrix which sends the standard basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ both to $\begin{pmatrix}3\\4\end{pmatrix}$ and let $C=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ be the change of basis matrix which sends the standard basis vectors to $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$. Then $A=BC^{-1}$.

(b) The columns of this matrix H are orthogonal but not orthonormal:

$$H=\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & 1\\ 0 & -2 & 1 & 1\\ 0 & 0 & -3 & 1\end{bmatrix}$$

Find H^{-1} by the following procedure. First multiply H by a diagonal matrix D that makes the columns orthonormal. Then invert. Then account for the diagonal matrix D to find the 16 entries of H^{-1}.

Solution. To normalize the columns of H, we let D be the diagonal matrix with diagonal entries $1/\sqrt{2}$, $1/\sqrt{6}$, $1/\sqrt{12}$, and $1/2$, and we multiply H by D on the right: $H'=HD$. Because H' is an orthogonal matrix, $H'^{-1}=H'^{T}$. Then $H^{-1}=D(HD)^{-1}=DH'^{T}$.

Computing, we obtain $H^{-1}=\begin{pmatrix}1/2 & -1/2 & 0 & 0\\ 1/6 & 1/6 & -1/3 & 0\\ 1/12 & 1/12 & 1/12 & -1/4\\ 1/4 & 1/4 & 1/4 & 1/4\end{pmatrix}$.

8. (a) Factor this symmetric matrix into $A = U^T U$ where U is upper triangular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution. By applying row operations we find the factorization $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $L = U^T$.

(b) Show by two different tests that A is symmetric positive definite.
Solution. Unfortunately it is hard to compute the eigenvalues explicitly, but nevertheless one can apply one of these tests:

i. $A = U^T U$ for U invertible;
ii. the energy test, $x^T A x = x U^T U x = \|Ux\|^2 \geq 0$ of $x \neq 0$ because U is invertible;
iii. the pivots of A are the pivots of U which are all positive;
iv. the upper left determinants of A are all 1 hence positive;
v. the eigenvalues satisfy the equation $-(\lambda^3 - 6\lambda^2 + 5\lambda - 1)$ which cannon be zero for negative λ by checking the signs in the sum.

(c) Find and explain an upper bound on the eigenvalues of A. Find and explain a (positive) lower bound on those eigenvalues if you know that

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are positive and they sum to the trace, which is 6, so they can be at most 6. The inverses of the eigenvalues $1/\lambda_1, 1/\lambda_2, 1/\lambda_3$ are the eigenvalues of A^{-1}, which has trace 5, so this tells us that each of the λ_i is at least 1/5.

Scrap Paper