Chapter 2 : The Transpose of a Derivative

Will you allow me a little calculus? This is really linear algebra for functions $x(t)$. **The matrix changes to a derivative so $\boldsymbol{A = d/dt}$.** To find the transpose of this unusual A we need to define the inner product between two functions $x(t)$ and $y(t)$.

The inner product changes from the sum of $x_k\, y_k$ to the *integral* of $x(t)\, y(t)$.

Inner product of functions $$\boldsymbol{x^{\mathrm{T}}y = (x, y) = \int_{-\infty}^{\infty} x(t)\, y(t)\, dt}$$

The transpose of a matrix has $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$. The "*adjoint*" of $A = \dfrac{d}{dt}$ has

$$(\boldsymbol{Ax}, \boldsymbol{y}) = \int_{-\infty}^{\infty} \frac{dx}{dt}\, y(t)\, dt = \int_{-\infty}^{\infty} x(t) \left(-\frac{dy}{dt}\right) dt = (\boldsymbol{x}, \boldsymbol{A^{\mathrm{T}}y})$$

I hope you recognize integration by parts. The derivative moves from the first function $x(t)$ to the second function $y(t)$. During that move, a minus sign appears. This tells us that ***the adjoint (transpose) of the derivative is minus the derivative***.

The derivative is *antisymmetric*: $\boldsymbol{A = d/dt}$ ***and*** $\boldsymbol{A^{\mathrm{T}} = -d/dt}$. Symmetric matrices have $S^{\mathrm{T}} = S$, antisymmetric matrices have $A^{\mathrm{T}} = -A$. $S = (d/dt)^2$ is symmetric:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{transposes to} \quad A^{\mathrm{T}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -A.$$

And a forward difference matrix transposes to a backward difference matrix, *multiplied by* -1. In differential equations, the second derivative (acceleration) is symmetric. The first derivative (damping proportional to velocity) is *anti*symmetric.