These were the problem assignments for the 18.065 course in 2019.

This file contains selected solutions by Tony Tohme.

Part II, Section 2, Problems 3, 5, 9, 10, 11, 12, 22

Part II, Section 4, Problems 2, 4, 6

Part IV, Section 1, Problems 8, 9

Part IV, Section 2, Problems 1, 3, 5, 6

Part IV, Section 2, Problems 3, 6, 7

Part IV, Section 6, Problems 7, 8

Part VI, Section 4, Problem 6

Part VII, Section 1, Problems 9, 15

Tony Tohme
919167634

Matrix Methods
Homework 4

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Problem II.2.3 10/10

$A = \sum \sigma_i u_i v_i^T \qquad A^+ = \sum \frac{v_i u_i^T}{\sigma_i} \qquad A^+A = \sum v_i v_i^T \qquad AA^+ = \sum u_i u_i^T$

We first show that A^+A is correct :

$$A^+A = \sum \frac{v_i u_i^T}{\sigma_i} \sum \sigma_i u_i v_i^T = \sum \frac{v_i u_i^T}{\cancel{\sigma_i}} \cancel{\sigma_i} u_i v_i^T = \sum v_i \underbrace{u_i^T u_i}_{1} v_i^T = \sum v_i v_i^T$$

note : $u_i^T u_i = \|u_i\|^2 = 1^2 = 1$ (u_i is a unit vector)

Therefore, $\boxed{A^+A = \sum v_i v_i^T \text{ and } A^+A \text{ is correct}}$

We then show that $(A^+A)^2 = A^+A =$ projection :

$$(A^+A)^2 = (A^+A)(A^+A) = \sum v_i v_i^T \sum v_i v_i^T = \sum v_i \underbrace{v_i^T v_i}_{1} v_i^T = \sum v_i v_i^T = A^+A$$

note : $v_i^T v_i = \|v_i\|^2 = 1^2 = 1$ (v_i is a unit vector)

Therefore, $\boxed{(A^+A)^2 = A^+A = \text{projection}}$

↑ This problem + some others already done

no change today

problem II.2_5 10/10

Suppose A has independent columns (rank $\overset{m\geq n}{r=n}$; nullspace = zero vector)

a) $$A = U\Sigma V^T = \underset{(m\times m)}{\begin{bmatrix} u_1 \cdots u_n \cdots u_m \end{bmatrix}} \underset{(m\times n)}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix}} \underset{(n\times n)}{\begin{bmatrix} v_1 \cdots v_n \cdots v_n \end{bmatrix}^T}$$

Therefore, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix}$ where $\sigma_1, \dots, \sigma_n \neq 0$ (singular values)

$(\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\underset{r}{\overset{n}{=}}} > 0)$

Thus, $\boxed{\text{there are } n \text{ nonzeros in } \Sigma}$

b) $$\Sigma^T\Sigma = \underset{(n\times m)}{\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]} \underset{(m\times n)}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix}} = \underset{(n\times n)}{\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}}$$

Therefore, $\Sigma^T\Sigma$ is a nonzero diagonal matrix, and $\det(\Sigma^T\Sigma) = \prod_{i=1}^{n} \sigma_i^2 \neq 0$

Thus, $\Sigma^T\Sigma$ is invertible and we find its inverse to be:

$$(\Sigma^T\Sigma)^{-1} = \underset{(n\times n)}{\begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}}$$

c) We write down the n by m matrix $(\Sigma^T\Sigma)^{-1}\Sigma^T$ and we identify it as Σ^+:

$$\Sigma^+ = (\Sigma^T\Sigma)^{-1}\Sigma^T = \underbrace{\begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}}_{(\Sigma^T\Sigma)^{-1}_{n\times n}} \underbrace{\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]}_{\Sigma^T_{n\times m}} = \underbrace{\left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_n} & \end{array}\right]}_{\Sigma^+_{n\times m}}$$

Therefore, $$\boxed{\Sigma^+ = \left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_n} & \end{array}\right]}$$

d) We substitute $A = U\Sigma V^T$ into $(A^TA)^{-1}A^T$ and we identify that matrix as A^+:

$$A^+ = (A^TA)^{-1}A^T = \left[(U\Sigma V^T)^T U\Sigma V^T\right]^{-1}(U\Sigma V^T)^T$$

$$= \left[V\Sigma^T \underbrace{U^TU}_{I}\Sigma V^T\right]^{-1} V\Sigma^T U^T$$

$$= \left(V\Sigma^T\Sigma V^T\right)^{-1} V\Sigma^T U^T$$

$$= \underbrace{(V^T)^{-1}}_{V \text{ (since } V^TV = I)}(\Sigma^T\Sigma)^{-1}\underbrace{V^{-1}V}_{I}\Sigma^T U^T$$

$$= V\underbrace{(\Sigma^T\Sigma)^{-1}\Sigma^T}_{\Sigma^+} U^T = V\Sigma^+U^T$$

$$\Rightarrow \boxed{A^+ = V\Sigma^+U^T}$$

Therefore, $A^+ = V\Sigma^+U^T$

problem II.2-9

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

We complete the Gram-Schmidt process by computing $q_1 = \frac{a}{\|a\|}$ and $A_2 = b - (b^T q_1) q_1$ and $q_2 = \frac{A_2}{\|A_2\|}$ and factoring into QR:

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \|a\| & ? \\ 0 & \|A_2\| \end{bmatrix}$$

We start by computing $q_1 = \frac{a}{\|a\|}$:

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \|a\| = \sqrt{1^2 + 1^2} = \sqrt{2} \Longrightarrow \boxed{q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

We then compute $A_2 = b - (b^T q_1) q_1$:

$$b^T q_1 = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$(b^T q_1) q_1 = 2\cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$b - (b^T q_1) q_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \Longrightarrow \boxed{A_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}}$$

We then compute $q_2 = \frac{A_2}{\|A_2\|}$:

$$A_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \Longrightarrow \|A_2\| = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$q_2 = \frac{A_2}{\|A_2\|} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \Longrightarrow \boxed{q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

We then factor into QR:

$$\underbrace{\begin{bmatrix} a_1 & a_2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} q_1 & q_2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}}_{R = Q^T A}$$

where $a_1 = a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$a_2 = b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ $\quad q_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$r_{ij} = q_i^T a_j$

$$r_{11} = q_1^T \underbrace{a_1}_{a} = \begin{bmatrix}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} = \|a\|$$

$$\Rightarrow r_{11} = \|a\| = \sqrt{2}$$

$$r_{12} = q_1^T \underbrace{a_2}_{b} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$r_{22} = q_2^T \underbrace{a_2}_{b} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \|A_2\|$$

$$\Rightarrow r_{22} = \|A_2\| = 2\sqrt{2}$$

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Therefore, we factor into QR:

$$\begin{bmatrix}1 & 4\\1 & 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 2\sqrt{2}\\ 0 & 2\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 2\sqrt{2}\\ 0 & 2\sqrt{2}\end{bmatrix}$$

<u>Problem II.2_10</u>

If $A = QR$ then $A^TA = (QR)^T QR = R^T \underbrace{Q^T \overset{\text{orthogonal matrix}}{Q}}_{I} R = R^TR$

where $R = Q^TA$ is an upper triangular matrix

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thus, R^T is a lower triangular matrix

Therefore, if $A = QR$ then $A^TA = R^TR$ = lower triangular times upper triangular problem II.2-11

If $Q^TQ = I$ we show that $Q^T = Q^+$

$$Q^+ = (\underbrace{Q^TQ}_{I})^{-1} Q^T = I^{-1}Q^T = IQ^T = Q^T$$

Therefore, $\boxed{\text{if } Q^TQ = I \text{ then } Q^T = Q^+}$

If $A = QR$ for invertible R, we show that $QQ^T = AA^+$

$$AA^+ = A(A^TA)^{-1}A^T = QR((QR)^T QR)^{-1} (QR)^T$$

$$= QR(R^T\underbrace{Q^TQ}_{I}R)^{-1}R^TQ^T$$

$$= QR(R^TR)^{-1}R^TQ^T$$

$$= Q\underbrace{RR^{-1}}_{I}\underbrace{(R^T)^{-1}R^T}_{I}Q^T = QQ^T$$

$$\Rightarrow \boxed{AA^+ = QQ^T}$$

herefore, $\boxed{\text{if } A = QR \text{ for invertible } R\text{, then } QQ^T = AA^+}$

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problem II.2_12

With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ we set up and solve the normal equations $A^TA\hat{x} = A^Tb$

$$A^TA\hat{x} = \begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \Sigma b_i \\ \Sigma b_i t_i \end{bmatrix}$$

$m = 4$ (4 measurements b_i at 4 different times t_i)

$\Sigma t_i = 0 + 1 + 3 + 4 = 8$

$\Sigma t_i^2 = 0^2 + 1^2 + 3^2 + 4^2 = 0 + 1 + 9 + 16 = 26$

$\Sigma b_i = 0 + 8 + 8 + 20 = 36$

$\Sigma b_i t_i = 0 \times 0 + 8 \times 1 + 8 \times 3 + 20 \times 4 = 0 + 8 + 24 + 80 = 112$

Therefore, we get :

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}^{-1}\begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}^{-1} = \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 160 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

therefore, $\boxed{\hat{C} = 1 \text{ and } \hat{D} = 4}$

$\boxed{\text{Closest Line: } b = 1 + 4t}$ the best straight line

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For the best straight line in Figure II.3a, we find its four heights p_i and four errors e_i :

jection b

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + D \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}}_{t} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 + 4 \times 0 \\ 1 + 4 \times 1 \\ 1 + 4 \times 3 \\ 1 + 4 \times 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

therefore, $\boxed{p_1 = 1 \,;\; p_2 = 5 \,;\; p_3 = 13 \,;\; p_4 = 17}$

<u>error vector</u> :

$$e = b - p = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

therefore, $\boxed{e_1 = -1 \,;\; e_2 = 3 \,;\; e_3 = -5 \,;\; e_4 = 3}$

<u>The minimum squared error:</u>

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = (-1)^2 + 3^2 + (-5)^2 + 3^2 = 1 + 9 + 25 + 9 = 44$$

therefore, $\boxed{\text{the minimum squared error } E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = 44}$

problem II.2 - 22

The averages of t_i and b_i are $\bar{t} = 2$ and $\bar{b} = 9$.

We verify that $C + D\bar{t} = \bar{b}$

a) We verify that the best line goes through the center point $(\bar{t}, \bar{b}) = (2, 9)$

in problem II.2-12, we found that the best line has parameters

$C = 1$ and $D = 4$ (i.e. $b = 1 + 4t$)

Thus, $\boxed{C + D\bar{t} = 1 + 4 \times 2 = 9 = \bar{b}}$

Therefore, we verified that $C + D\bar{t} = \bar{b}$

b) Now, we explain why $C + D\bar{t} = \bar{b}$ comes from the first equation in $A^TA\hat{x} = A^Tb$:

$$A^TA\hat{x} = \begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \Sigma b_i \\ \Sigma b_i t_i \end{bmatrix}$$

$$\bar{t} = \frac{\Sigma t_i}{m} \Rightarrow \boxed{\Sigma t_i = m\bar{t}}$$

$$\bar{b} = \frac{\Sigma b_i}{m} \Rightarrow \boxed{\Sigma b_i = m\bar{b}}$$

First Equation in $A^TA\hat{x} = A^Tb$:

$$m\hat{C} + \underbrace{\Sigma t_i}_{m\bar{t}} \hat{D} = \underbrace{\Sigma b_i}_{m\bar{b}} \Rightarrow \cancel{m}\hat{C} + \cancel{m}\bar{t}\hat{D} = \cancel{m}\bar{b}$$

$$\Rightarrow \boxed{\hat{C} + \hat{D}\bar{t} = \bar{b}}$$

Therefore, the solution $\hat{x} = (\hat{C}; \hat{D})$ of the equations $A^TA\hat{x} = A^Tb$ should satisfy: $\hat{C} + \hat{D}\bar{t} = \bar{b}$

Therefore, the first equation of $A^TA\hat{x} = A^Tb$ explains why $C + D\bar{t} = \bar{b}$,

i.e. the best line goes through the center point $(\bar{t}, \bar{b}) = (2, 9)$

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Tony Tohme
919167634

18.0651 _ Matrix Methods
Homework 5

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problem VI.4 - 6 10/10

We derive the gradient descent equation $x_{k+1} = x_k - s_k \nabla f(x_k)$ for the least squares problem of minimizing $f(x) = \frac{1}{2}\|Ax-b\|^2$

In order to do so, we write $f(x)$ as :

$$f(x) = \frac{1}{2}(Ax-b)^T(Ax-b)$$
$$= \frac{1}{2}x^TA^TAx - \frac{1}{2}x^TA^Tb - \frac{1}{2}b^TAx + \frac{1}{2}b^Tb$$

since x^TA^Tb is a scalar quantity, then we know that the transpose of a scalar is the scalar itself, and we get:

$$x^TA^Tb = (x^TA^Tb)^T = b^T(x^TA^T)^T = b^TAx \implies x^TA^Tb = b^TAx$$

Thus, we write $f(x)$ as :

$$f(x) = \frac{1}{2}x^TA^TAx - b^TAx + \frac{1}{2}b^Tb$$

this is a quadratic form (A^TA is a square, symmetric matrix) and therefore, we compute the gradient as follows :

$$\nabla f(x) = A^TAx - A^Tb = A^T(Ax-b)$$

Therefore, we have the following gradient descent equation :

$$x_{k+1} = x_k - s_k\nabla f(x_k) \implies \nabla f(x_k) = A^T(Ax_k - b)$$

$$\implies \boxed{x_{k+1} = x_k - s_k A^T(Ax_k - b)}$$

problem VII.1 - 9

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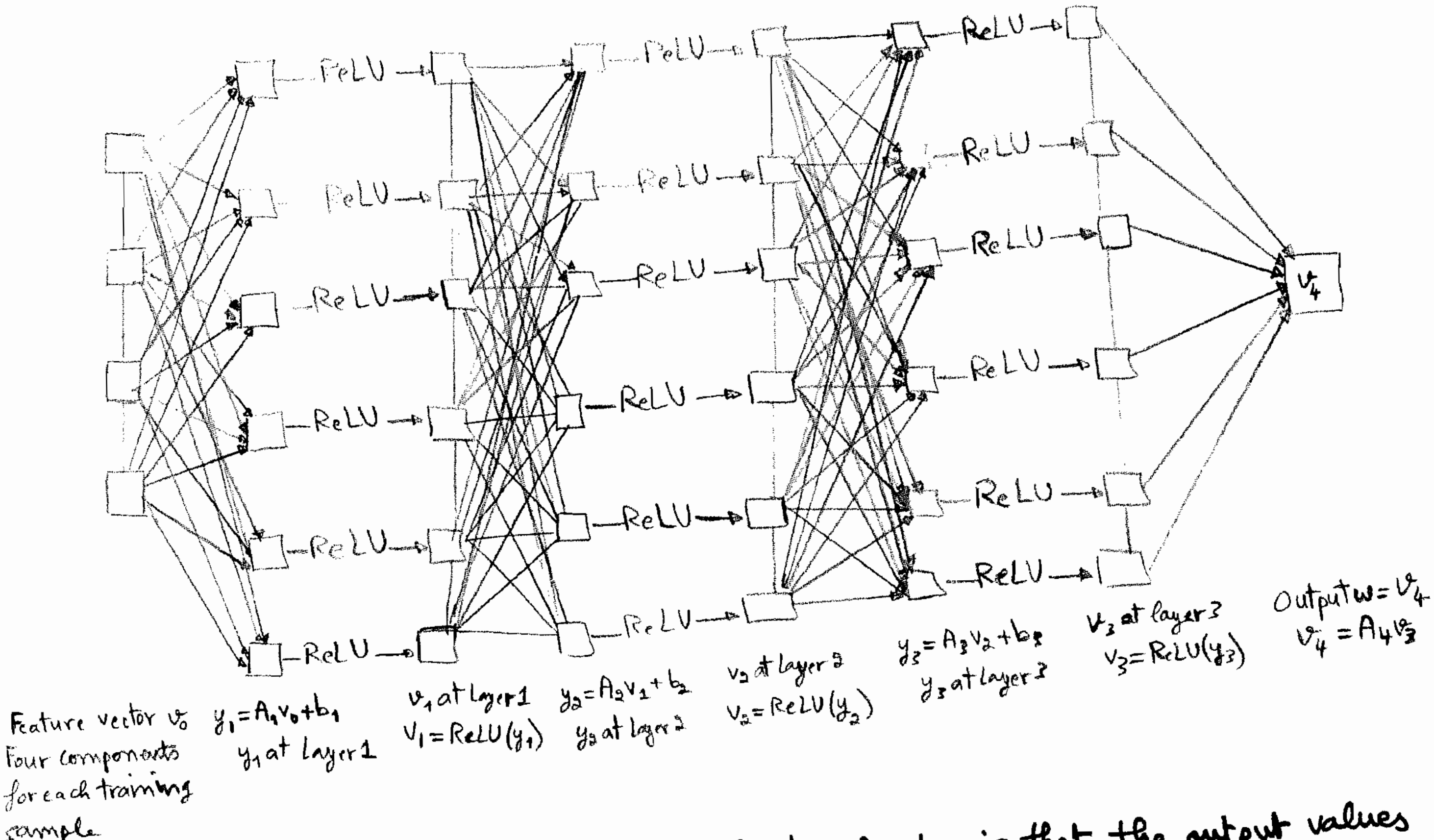
We have a network with $m = N_0 = 4$ inputs in each feature vector v_0 and $N = 6$ neurons on each of the 3 hidden layers

The neural network is shown below :



The goal in optimizing $x = A_1, b_1, A_2, b_2, A_3, b_3, A_4, b_4$ is that the output values $v_\ell = v_4$ at the last layer $\ell = 4$ should correctly capture the important features of the training data v_0.

$A_1 : 6 \times 4$ $\quad$ $b_1 : 6 \times 1$

$A_2 : 6 \times 6$ $\quad$ $b_2 : 6 \times 1$

$A_3 : 6 \times 6$ $\quad$ $b_3 : 6 \times 1$

$A_4 : 1 \times 6$ $\quad$ $b_4 : 1 \times 1$ (not used)

<u>note</u> : usually, there is no bias vector at the final step to the output (no b_4)

$$\boxed{\begin{aligned} \#\text{ weights} &= 120 \\ \#\text{ biases} &= 18 \\ \#\text{ ReLU} &= 18 \end{aligned}}$$

* The number of weights is the total number of elements in $A_1, A_2, A_3, A_4, b_1, b_2, b_3$:

$$\# w = 6 \times 4 + 2 \times 6 \times 6 + 1 \times 6 + 3 \times 6 \times 1 = \underline{\underline{120}}$$

* The number of biases is the total number of elements in the bias vectors b_1, b_2, b_3 :

$$\#\text{ biases} = 3 \times 6 \times 1 = \underline{\underline{18}}$$

* The number of activation functions (ReLU) : There is one (ReLU) for each neuron on the hidden layers :

$$\#(\text{ReLU}) = 6 \times 3 = \underline{\underline{18}}$$

Problem VII.1 - 15

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Example 4 with blue and orange spirals is much more difficult ! With one hidden layer, we explore whether the network learn this training data as N increases. We start with $N = 1$ and we go up to $N = 8$. The results are summarized as follows

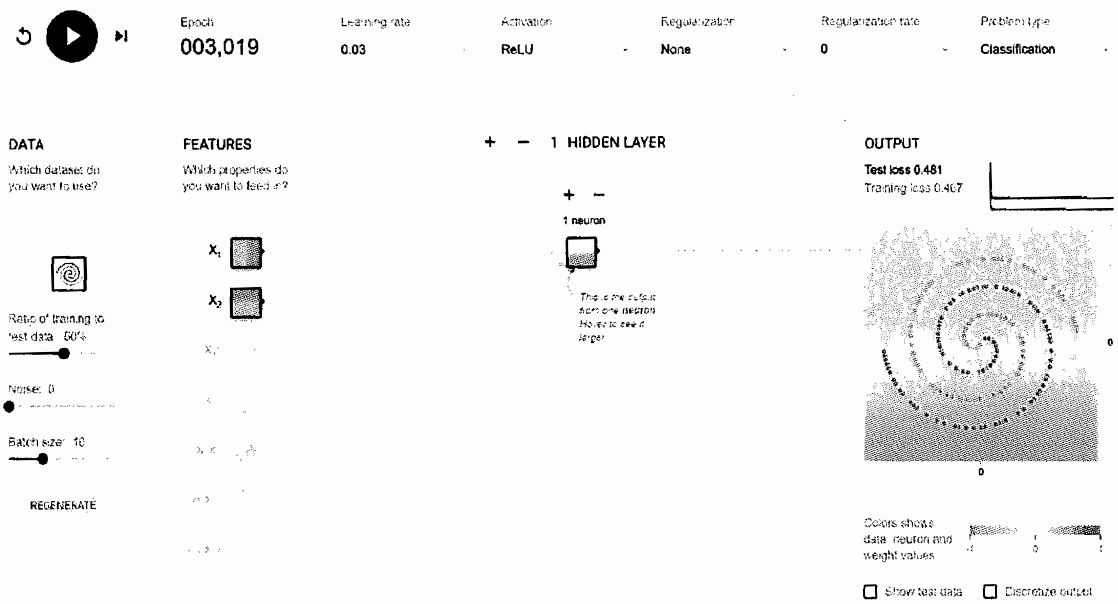

Figure 2: **Example 4:** Blue and Orange Spirals, One Hidden Layer, $N = 1$

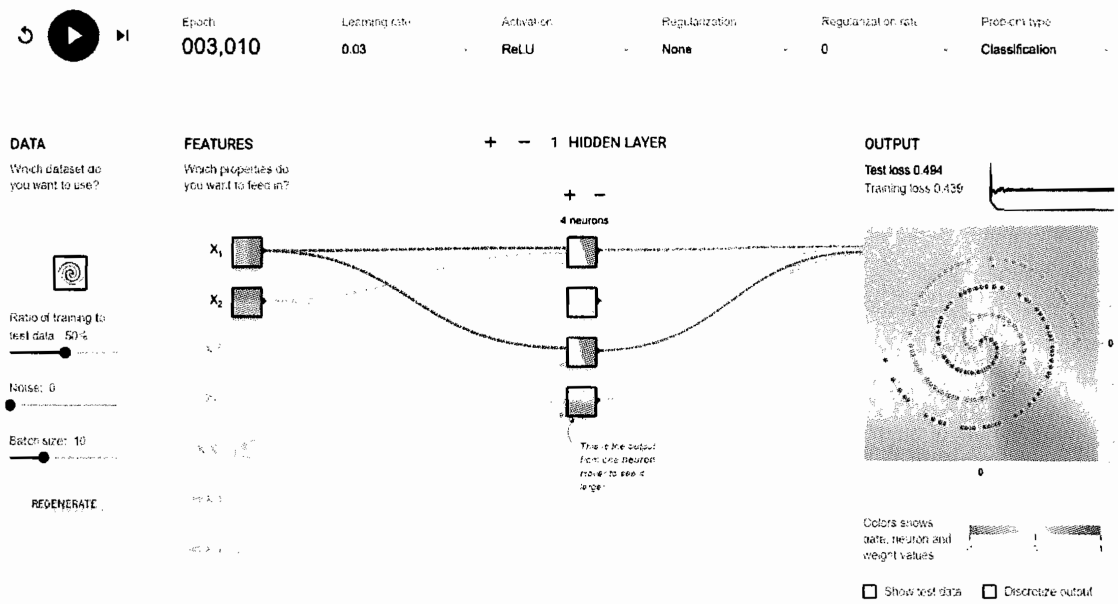

Figure 5: **Example 4:** Blue and Orange Spirals, One Hidden Layer, $N = 4$

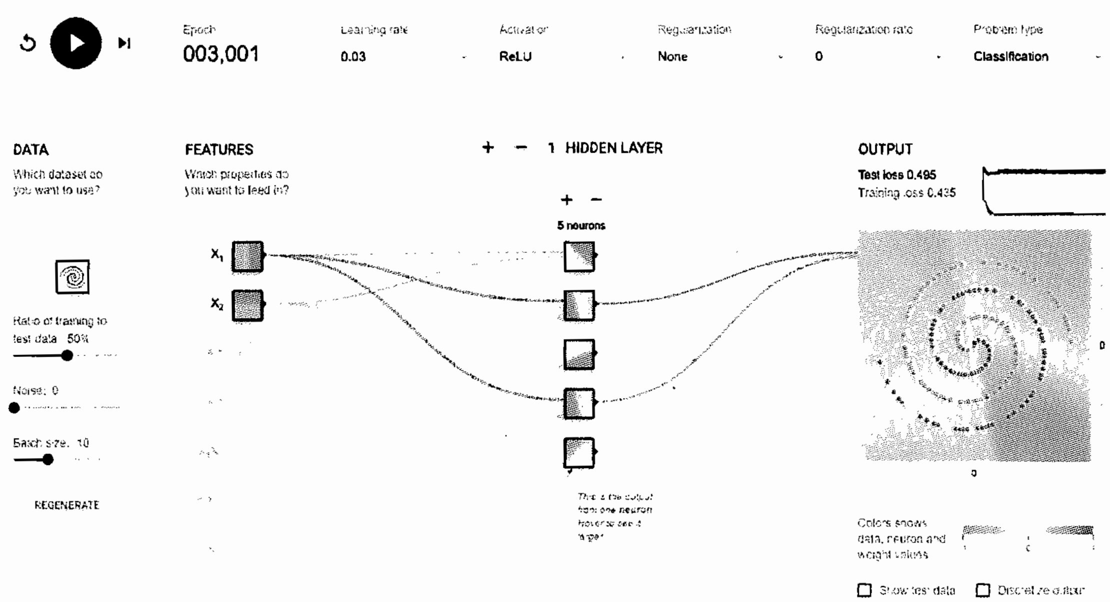

Figure 6: **Example 4:** Blue and Orange Spirals, One Hidden Layer, $N = 5$

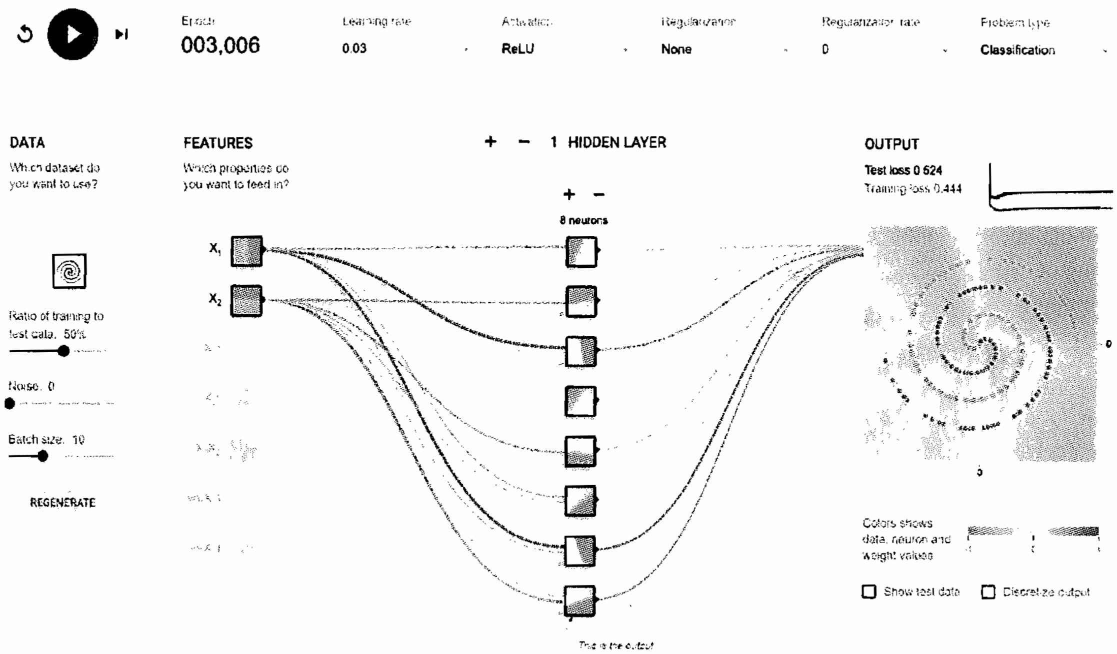

Figure 9: **Example 4:** Blue and Orange Spirals, One Hidden Layer, $N = 8$

No, the network can't learn this training data. As N increases, we observe that the network is not able to classify properly with error being almost the same. This is because the only properties (features) we are feeding in are X_1 and X_2, and we are only using one hidden layer. However, if we use two hidden layers and also feed in the two additional properties X_1^2 and X_2^2, the network is able to learn the training data as shown in Figure 13 in **Problem VII.1 - 16**.

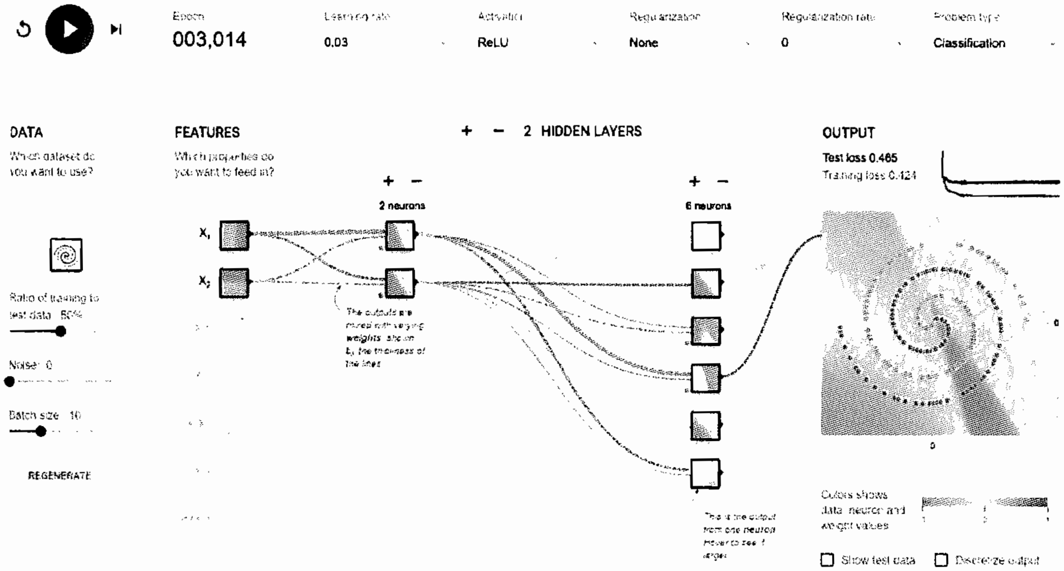

Figure 12: **Example 4:** Blue and Orange Spirals, Two Hidden Layers, $2+6$

As we can see in the figures above, $2+6$ is worse than $6+2$ and it is more unusual.(having higher test and training error) We note that if we use two hidden layers and also feed in the two additional properties X_1^2 and X_2^2, the network is able to learn the training data as shown in the figure below.

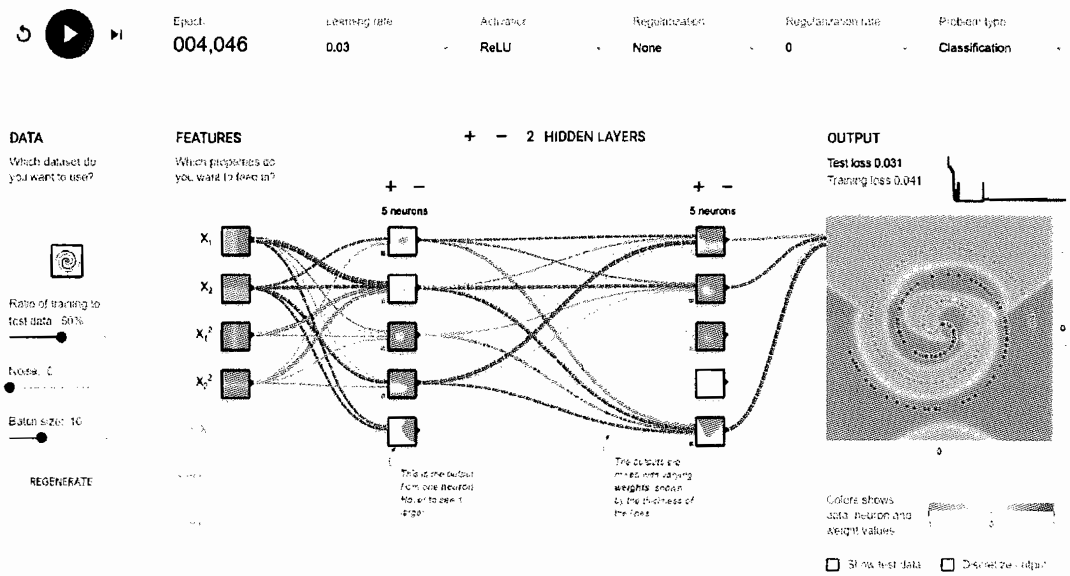

Figure 13: **Example 4:** Blue and Orange Spirals, Two Hidden Layers, $5+5$

Computing Question B

```
x = [1;1];          % Initial Guess
s = @(x)(0.9);      % Learning Rate or Step Size
tol = 1e-10;        % Precision
B = (0.7/1.3)^2;
z_0 = 0;
k = 1;
z(:,k) = gradient_f(x) + B*z_0;
while k <= 10000 && norm(s(k)*z(:,k)) > tol
    x(:,k+1) = x(:,k) - s(k)*z(:,k);
    k = k+1;
    z(:,k) = gradient_f(x(:,k)) + B*z(:,k-1);
end
display('The optimal parameters are: '); display(x(:,end));
```

```
The optimal parameters are:
   1.0e-09 *

   -0.0000
    0.3061
```

```
display('The number of iterations is: '); display(k-1);
```

```
The number of iterations is:
    71
```

```
function g = gradient_f(x)
g(1,1) = 2*x(1,end);
g(2,1) = 2*0.09*x(2,end);
end
```

Therefore, adding momentum to the same gradient descent algorithm results in a faster convergence towards the optimal point (x*,y*) = (0,0).

Tony Tohme
919167634

18.0651 – Matrix Methods
Homework 7

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problem IV.1 – 8 10/10

Every real matrix A with n columns has

$AA^T x = a_1(a_1^T x) + \cdots + a_n(a_n^T x)$

* If A is an orthogonal matrix Q $\quad Q = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix}_{n\times n}$

then its columns $q_1, \dots, q_n$ are orthogonal unit vectors (i.e. orthonormal vectors)

We know that $Q^TQ = QQ^T = I_{n\times n}$

then $AA^Tx = QQ^Tx = q_1(q_1^Tx) + \cdots + q_n(q_n^Tx)$

In order to see what is special about those n pieces, let's consider (WLOG) the j^{th} and the k^{th} piece, and compute their inner product :

$$(q_j q_j^T x)^T (q_k q_k^T x) = x^T q_j \underbrace{q_j^T q_k}_{=0} q_k^T x = 0 \qquad j,k \in \{1,\dots,n\},\ j \neq k$$

Thus, the j^{th} and k^{th} pieces are orthogonal.

Here, we used the fact that the columns q_j and q_k are orthogonal, so their inner product is equal to zero.

Therefore, if A is an orthogonal matrix Q, then those n pieces are orthogonal with each other.

* For the Fourier matrix (complex), it is appropriate to take the conjugate transpose rather than transpose, and the formula becomes :

$A\bar{A}^T x = F_n \bar{F}_n^T = a_1(\bar{a}_1^T x) + \cdots + a_n(\bar{a}_n^T x)$

note that $F_n \bar{F}_n^T = n I_{n\times n}$

problem IV.1 - 9 10/10

We find vector x such that $F_4 x = (1, 0, 1, 0)$

where $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$

Therefore, $x = F_4^{-1}(1, 0, 1, 0)$

where $F_4^{-1} = \frac{1}{4}\Omega_4 = \frac{1}{4}\overline{F}_4 \Rightarrow F_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

Similarly, we find vector y such that $F_4 y = (0, 0, 0, 1)$

therefore, $y = F_4^{-1}(0, 0, 0, 1)$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} 1/4 \\ i/4 \\ -\frac{1}{4} \\ -i/4 \end{bmatrix}$$

Therefore, the vector $x = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ has $F_4 x = (1, 0, 1, 0)$

and the vector $y = \left(\frac{1}{4}, \frac{i}{4}, \frac{-1}{4}, \frac{-i}{4}\right)$ has $F_4 y = (0, 0, 0, 1)$

problem IV.2-1 10/10

We find $c * d$ and $c \circledast d$ for $c=(2,1,3)$ and $d=(3,1,2)$

Ordinary convolution finds the coefficients when we multiply $(2I + 1P + 3P^2)$ times $(3I + 1P + 2P^2)$.

Then cyclic convolution uses the crucial fact that $P^3 = I$

Ordinary Convolution is as follows:

$$
\begin{array}{ccccc}
 & & 2 & 1 & 3 \\
 & & 3 & 1 & 2 \\
\hline
 & & 4 & 2 & 6 \\
 & 2 & 1 & 3 & \\
6 & 3 & 9 & & \\
\hline
6 & 5 & 14 & 5 & 6
\end{array}
= c * d
$$

The cyclic step combines $6+5$ because $P^3 = I$.

It combines $5+6$ because $P^4 = P$.

The result is $(11, 11, 14) = c \circledast d$

In summary, we have:

Convolution: $(2,1,3) * (3,1,2) = (6,5,14,5,6)$

Cyclic Convolution:

$(2,1,3) \circledast (3,1,2) = (6+5, 5+6, 14) = (11,11,14)$

problem IV. 2 - 3   10/10

We show that if $c * d = e$ then $(\Sigma c_i)(\Sigma d_i) = (\Sigma e_i)$

In other words, we show that the sum of the c's times the sum of the d's equals the sum of the outputs.

This is true because every c multiplies every d in $c * d$

In order to see this, we let $c = (c_0, c_1, \ldots, c_{N-1})$ and $d = (d_0, d_1, \ldots, d_{M-1})$ and the output $e = (e_0, e_1, \ldots, e_{N+M-2})$

Ordinary convolution finds the coefficients when we multiply $(c_0 I + c_1 P + \ldots + c_{N-1} P^{N-1})$ times $(d_0 I + d_1 P + \ldots + d_{M-1} P^{M-1})$

Explicitly, we get:

$$\begin{array}{ccccccccc}
 & & & & c_0 & c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} \\
 & & & & & d_0 & d_1 & \cdots & d_{M-2} & d_{M-1} \\
\hline
 & & & & d_{M-1}c_0 & d_{M-1}c_1 & d_{M-1}c_2 & \cdots & d_{M-1}c_{N-2} & d_{M-1}c_{N-1} \\
 & & & d_{M-2}c_0 & d_{M-2}c_1 & d_{M-2}c_2 & d_{M-2}c_3 & \cdots & d_{M-2}c_{N-1} & \\
 & & & & \ddots & & & & & \\
 & & d_2c_0 & \cdots & d_2c_{N-4} & d_2c_{N-3} & d_2c_{N-2} & d_2c_{N-1} & & \\
 & d_1c_0 & d_1c_1 & \cdots & d_1c_{N-3} & d_1c_{N-2} & d_1c_{N-1} & & & \\
d_0c_0 & d_0c_1 & d_0c_2 & \cdots & d_0c_{N-2} & d_0c_{N-1} & & & & \\
\hline
e_0 & e_1 & e_2 & \cdots & \cdots & \cdots & & & e_{N+M-3} & e_{N+M-2}
\end{array}$$

$e_0 = d_0 c_0$;

$e_1 = d_1 c_0 + d_0 c_1$;

$e_2 = d_2 c_0 + d_1 c_1 + d_0 c_2$;

$\vdots$

$e_{N+M-3} = d_{M-1} c_{N-2} + d_{M-2} c_{N-1}$;

$e_{N+M-2} = d_{M-1} c_{N-1}$;

→

thus, we get :

$$\sum_{i=0}^{N+M-2} e_i = e_0 + e_1 + e_2 + \cdots + e_{N+M-3} + e_{N+M-2}$$

$$= d_0 c_0 + d_1 c_0 + d_0 c_1 + d_2 c_0 + d_1 c_1 + d_0 c_2 + \cdots + d_{M-1} c_{N-2} + d_{M-2} c_{N-1} + d_{M-1} c_{N-1}$$

$$= d_0 (c_0 + c_1 + c_2 + \cdots + c_{N-1}) + d_1 (c_0 + \cdots + c_{N-1}) + \cdots + d_{M-1} (c_0 + \cdots + c_{N-1})$$

$$= (c_0 + c_1 + \cdots + c_{N-1})(d_0 + d_1 + \cdots + d_{M-1}) = \sum_{i=0}^{N-1} c_i \sum_{i=0}^{M-1} d_i$$

Therefore, we have shown that if $c * d = e$,

then $(\sum c_i)(\sum d_i) = (\sum e_i)$.

problem IV.2-5 10/10

We find the eigenvalues of the 4 by 4 circulant C

$C = I + P + P^2 + P^3$ where $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow C = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{P^2} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P^3} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{C}$$

therefore, $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{4\times 1} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}_{1\times 4} = uv^T$ where $u = v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Thus, $C = uv^T$ is a rank-1 matrix.

Since C is rank-1 matrix, the dimension of the nullspace $N(C)$ is $4-1=3$ (For any two matrices A and B, the column space of AB always lies in the column space of A). Since the column space of C is one dimensional, and is contained in the line spanned by u, we see that the column space of C is exactly the line spanned by $u = (1,1,1,1)$.

Since $N(C)$ is 3 dimensional, thus 0 is an eigenvalue of C with multiplicity 3. Since the column space of C is one dimensional and contains u, the vector u must be an eigenvector.

Now, we have: $Cu = (uv^T)u = u\underbrace{(v^Tu)}_{\text{scalar}} = (v^Tu)u$

so the corresponding eigenvalue (the last one) is $v^Tu = [1\,1\,1\,1]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4$

Therefore, the eigenvalues of the 4 by 4 circulant C are: $\boxed{\begin{matrix} \lambda_0 = 4 \\ \lambda_1 = \lambda_2 = \lambda_3 = 0 \end{matrix}}$

Connecting those eigenvalues to the discrete transform Fc for $c = (1, 1, 1, 1)$, we find that :

$$\begin{bmatrix} \lambda_0(c) \\ \lambda_1(c) \\ \lambda_2(c) \\ \lambda_3(c) \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i2\pi/4} & e^{i4\pi/4} & e^{i6\pi/4} \\ 1 & e^{i4\pi/4} & e^{i8\pi/4} & e^{i12\pi/4} \\ 1 & e^{i6\pi/4} & e^{i12\pi/4} & e^{i18\pi/4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_0(c) \\ \lambda_1(c) \\ \lambda_2(c) \\ \lambda_3(c) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1+1+1 \\ \boxed{\begin{matrix} 1+i+i^2+i^3 \\ 1+i^2+i^4+i^6 \\ 1+i^3+i^6+i^9 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

these 3 equations will be used to solve : $1+z+z^2+z^3=0$

<u>Note</u> : Remember that $w^k = e^{\frac{2\pi i k}{N}}$ is the k^{th} eigenvalue of P. those numbers are used in the Fourier matrix F. (In our case, $N=4$)

Now, we find three real or complex numbers z such that $1+z+z^2+z^3=0$:
Note from the result above (see the last three rows), we have :

$$1+i+i^2+i^3 = 0 \implies z_1 = i$$

$$1+i^2+i^4+i^6 = 0 \implies 1+i^2+(i^2)^2+(i^2)^3 = 0 \implies z_2 = i^2 = -1$$

$$1+i^3+i^6+i^9 = 0 \implies 1+i^3+(i^3)^2+(i^3)^3 = 0 \implies z_3 = i^3 = -i$$

Therefore, we found three real or complex numbers z such that $1+z+z^2+z^3=0$, namely $\boxed{z_1 = i, \; z_2 = -1, \; z_3 = -i}$

problem IV . 2 - 6 10/10

"A circulant matrix C is invertible when the vector Fc has no zeros". We connect this true statement to this test on the frequency response : $C(e^{i\theta}) = \sum_{0}^{N-1} c_j e^{ij\theta} \neq 0$ at the N points $\theta = \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, 2\pi$

This statement above is true since a circulant matrix C is invertible when its determinant is nonzero. Since the determinant is the product of the eigenvalues, thus the matrix C is invertible when its eigenvalues are nonzero. Since the N eigenvalues of the matrix C are the components of Fc, thus C is invertible when the vector Fc has no zeros.

$$Fc = \begin{bmatrix} \lambda_0(C) \\ \lambda_1(C) \\ \lambda_2(C) \\ \vdots \\ \lambda_{N-1}(C) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

$$Fc = \begin{bmatrix} c_0 + c_1 + \cdots + c_{N-1} \\ c_0 + c_1 w + \cdots + c_{N-1} w^{N-1} \\ c_0 + c_1 w^2 + \cdots + c_{N-1} w^{2(N-1)} \\ \vdots \\ c_0 + c_1 w^{N-1} + \cdots + c_{N-1} w^{(N-1)(N-1)} \end{bmatrix} \neq 0$$

where $w^k = e^{2\pi i k/N}$ is the k^{th} eigenvalue of P.

$$Fc = \begin{bmatrix} c_0 + c_1 e^{i2\pi} + \cdots + e^{i(N-1)2\pi} \\ c_0 + c_1 e^{i2\pi/N} + \cdots + e^{i(N-1)2\pi/N} \\ c_0 + c_1 e^{i4\pi/N} + \cdots + e^{i(N-1)4\pi/N} \\ \vdots \\ c_0 + c_1 e^{i\frac{2(N-1)\pi}{N}} + \cdots + e^{i\frac{(N-1)2(N-1)\pi}{N}} \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{N-1} c_j e^{ij2\pi} \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{2\pi}{N}} \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{4\pi}{N}} \\ \vdots \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{2(N-1)\pi}{N}} \end{bmatrix} \neq 0$$

Therefore, we have :

$$F_c = \begin{bmatrix} \sum_{j=0}^{N-1} c_j e^{ij2\pi} \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{2\pi}{N}} \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{4\pi}{N}} \\ \vdots \\ \sum_{j=0}^{N-1} c_j e^{ij\frac{2(N-1)\pi}{N}} \end{bmatrix} = \begin{bmatrix} C(e^{i2\pi}) \\ C(e^{i2\pi/N}) \\ C(e^{i4\pi/N}) \\ \vdots \\ C\left(e^{i\frac{2(N-1)\pi}{N}}\right) \end{bmatrix} \neq 0$$

This is equivalent to : Test on the frequency response

$$\boxed{C(e^{i\theta}) = \sum_{j=0}^{N-1} c_j e^{ij\theta} \neq 0 \text{ at the } N \text{ points } \theta = \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, 2\pi}$$

Therefore, we have connected the true statement above to this test on the frequency response.

problem IV.3 - 3 10/10

We describe a permutation P so that $P(A \otimes B) = (B \otimes A)P$.
Here, we assume that A and B are square and have same size.

note: Remember that a matrix multiplied by a permutation matrix from the left results in switching the rows of that matrix, while multiplying it from the right results in switching the columns of that matrix.

Let's start with a small example where A and B are 2 by 2.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Thus, we get:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

$$B \otimes A = \begin{bmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{12}a_{11} & b_{12}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{12}a_{21} & b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{21}a_{21} & b_{21}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{bmatrix}$$

Thus, in order to satisfy $P(A \otimes B) = (B \otimes A)P$,
we need to switch the 2^{nd} and 3^{rd} rows of $A \otimes B$ (by multiplying P from left)
and we need to switch the 2^{nd} and 3^{rd} columns of $B \otimes A$ (by multiplying P from right)

→

Therefore, the $2^2 \times 2^2 = 4 \times 4$ matrix P that satisfies $P(A \otimes B) = (B \otimes A)P$ is described below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we notice that $P\,\mathrm{vec}(A) = \mathrm{vec}(A^T)$ and $P\,\mathrm{vec}(B) = \mathrm{vec}(B^T)$ where $\mathrm{vec}(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$ and $\mathrm{vec}(A^T) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$

and $\mathrm{vec}(B) = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix}$ and $\mathrm{vec}(B^T) = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix}$

We can verify that as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix}$$

$\longrightarrow$

In fact, this formula is a property of the commutation matrix, which is a special type of permutation matrix.

therefore, for any $n \times n$ matrix A and $n \times n$ matrix B, finding a permutation matrix P such that

$$P(A \otimes B) = (B \otimes A)P$$

is equivalent to finding a permutation P such that

$$P\,\mathrm{vec}(A) = \mathrm{vec}(A^T) \quad \text{or} \quad P\,\mathrm{vec}(B) = \mathrm{vec}(B^T)$$

In fact, if P satisfies $P\,\mathrm{vec}(A) = \mathrm{vec}(A^T)$, then it also satisfies $P\,\mathrm{vec}(B) = \mathrm{vec}(B^T)$

thus, it is sufficient to solve for one of them.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$$\mathrm{vec}(A) = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{nn} \end{bmatrix} \qquad \mathrm{vec}(B) = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \\ b_{12} \\ \vdots \\ b_{nn} \end{bmatrix}$$

thus, to find the $n^2 \times n^2$ permutation P such that

$$P(A \otimes B) = (B \otimes A)P$$

we find P such that

$$P\,\mathrm{vec}(A) = \mathrm{vec}(A^T)$$

$\longrightarrow$

therefore, we have described a permutation such that

$$P(A \otimes B) = (B \otimes A) P$$

Notice that this equivalent to $P(A \otimes B)P^{-1} = (B \otimes A)$

Since P is invertible, thus the matrices $P(A \otimes B)P^{-1}$ are similar to $(A \otimes B)$ and thus have the same eigenvalues.

since $P(A \otimes B)P^{-1} = (B \otimes A)$, thus the eigenvalues of $(B \otimes A)$ are the same as the eigenvalues of $A \otimes B$.

<u>note</u>: a permutation matrix is orthogonal and thus $P^T P = I$ and $P^{-1} = P^T$

Another way to see this is to suppose x is an eigenvector of A: $Ax = \lambda x$

Suppose y is an eigenvector of B: $By = \mu y$.

then the Kronecker product of x and y is an eigenvector of $A \otimes B$.

the eigenvalue is $\lambda\mu$

$$(A \otimes B)(x \otimes y) = (Ax) \otimes (By) = (\lambda x) \otimes (\mu y) = \underbrace{\lambda\mu}(x \otimes y)$$

and the Kronecker product of y and x is an eigenvector of $B \otimes A$.

the eigenvalue is $\mu\lambda$ $(= \lambda\mu)$

$$(B \otimes A)(y \otimes x) = (By) \otimes (Ax) = (\mu y) \otimes (\lambda x) = \underbrace{\mu\lambda}(y \otimes x)$$

therefore, we have shown that $A \otimes B$ and $B \otimes A$ have the same eigenvalues.

problem IV.3-6 10/10

Let A be an $m \times n$ matrix, and B an $M \times N$ matrix.

Let the $mM \times nN$ X be the Kronecker product of A and B, thus we get

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \dots & b_{1N} \\ \vdots & & \\ b_{M1} & \dots & b_{MN} \end{bmatrix}$$

$$X = A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

If we see A and B as images, we see that each pixel of A (i.e. each element of the matrix A) is multiplied by the entire image B (i.e. entire matrix B) which would change the color intensities of image B, resulting in an image X which consists of repetitions of image B with different color intensities.

In order to illustrate this idea, let's assume a 3 by 3 matrix A such that

$a_{11} = a_{21} = a_{31}$; $a_{12} = a_{22} = a_{32}$; $a_{13} = a_{23} = a_{33}$;

thus, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow X = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{bmatrix}$

In this case, each column in X consists of repetitions of image B with same color intensities, but different than the other columns.

problem IV.3 _ 7 10/10

We intend to apply a 2D Fast Fourier Transform (FFT) to f where f is an $N \times N$ image. The result will be a 2D vector c.
We could think of this process in two steps:

Row by row Apply 1 D FFT to each row of pixels separately

Column by column Rearrange the output by columns and transform each column

The matrix foreach step is N^2 by N^2. First think of the N^2 pixels a row at a time and multiply each row in that long vector by the FFT matrix F_N:
using recursion

$$F_{row} f = \begin{bmatrix} F_N & & & \\ & F_N & & \\ & & \ddots & \\ & & & F_N \end{bmatrix} \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } N \end{bmatrix}$$

note: f has N^2 numbers (entries).
we unfolded those numbers into a long vector of length N^2.
(using vec(f))
thus, f and $F_{row} f$ have length N^2

The matrix is $F_{row} = I_N \otimes F_N$.
It is a Kronecker product of size N^2.
Now the output $F_{row} f$ is rearranged into columns.
the second step of the 2D FFT multiplies each column of that "halfway" image $F_{row} f$ by the FFT matrix F_N. Again we are multiplying by
using recursion
a matrix F_{column} of size N^2. The full 2D FFT is $F_N \otimes F_N$.
That matrix F_{column} is the Kronecker product $F_N \otimes I_N$.
The 2D FFT puts the row and column steps together into $F_N \otimes F_N$.

$$F_{N \times N} = F_{column} F_{row} = (F_N \otimes I_N)(I_N \otimes F_N) = F_N \times F_N$$

We know that 1 D FFT needs $\frac{1}{2} N \log_2 N$ operations (see page 211)
Therefore, for an N by N image, the 2D Fast Fourier Transform needs:
full FFT for each of N rows: $N \frac{1}{2} N \log_2 N$ (step 1) $= \frac{1}{2} N^2 \log_2 N$
full FFT foreach of N columns: $N \frac{1}{2} N \log_2 N$ (step 2) $= \frac{1}{2} N^2 \log_2 N$

The final count is :

\# operations from step 1 + # operations from step 2

$$= \frac{1}{2} N^2 \log_2 N + \frac{1}{2} N^2 \log_2 N = N^2 \log_2 N \text{ operations}$$

<u>note</u> : A reminder on how the 1D Full FFT by recursion works.
The key idea is to connect F_N with the half-size Fourier matrix $F_{N/2}$. We assume that N is a power of 2.

We reduce F_N to $F_{N/2}$.

$$F_N = \begin{bmatrix} I & D_\ell \\ I & -D_\ell \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} P_\ell \qquad N = 2^\ell \quad (\text{where } \ell \text{ is \# levels})$$

then we keep going to $F_{N/4}$.

$$F_N = \begin{bmatrix} I & D_\ell \\ I & -D_\ell \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I & D_{\ell-1} \\ I & -D_{\ell-1} \end{bmatrix} & \\ & \begin{bmatrix} I & D_{\ell-1} \\ I & -D_{\ell-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} F_{N/4} & & & \\ & F_{N/4} & & \\ & & F_{N/4} & \\ & & & F_{N/4} \end{bmatrix} \begin{bmatrix} P_{\ell-1} & \\ & P_{\ell-1} \end{bmatrix} P_\ell$$

That is recursion.
the number of operations for size $N = 2^\ell$ is reduced from N^2 to $\frac{1}{2} N \underbrace{\log_2 N}_{\ell}$
This is the 1D case.

In our case, the direct computation of 2D DFT needs $N^2 \times N^2 = N^4$ operations. However, if we employ 2D FFT (using the 2 steps described before), this count is reduced to $N^2 \log_2 N$.

However, if we compute $F_N \otimes F_N$ directly, then we need $N^2 (\log_2 N)^2$ operations.

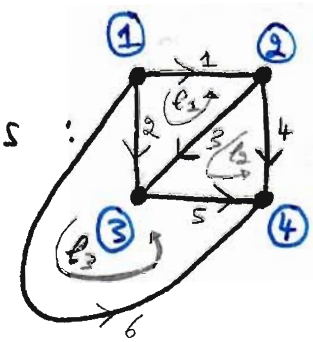
<u>In summary</u> :

$$(F_N \otimes I_N)(I_N \otimes F_N) \implies \sim N^2 \log_2 N$$

VS

$$F_N \otimes F_N \implies \sim N^2 (\log_2 N)^2$$

problem IV.6 - 7 10/10



a) For a complete graph with 4 nodes and 6 edges :

We find the incidence matrix A to be :

$$A = \underset{6\times 4}{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}}$$

Now, we find the matrix A^TA :

$$A^TA = \underset{4\times 6}{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}} \underset{6\times 4}{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Note that,

$$A^TA = D - B = \underbrace{\underset{4\times 4}{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}}_{D} - \underbrace{\underset{4\times 4}{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}}_{B} = \underbrace{\underset{4\times 4}{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}}_{A^TA}$$

where D is the degree matrix and B is the adjacency matrix .

b) We also find $6-4+1 = 3$ solutions (from loops) to Kirchhoff's Law $A^Tw=0$

Kirchhoff's Current Law KCL is $A^Tw = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The key to solving $A^Tw=0$ is to look at the small loops in the graph. A loop is a "cycle" of edges – a path that comes back to the start. Going around those loops are these edges:

loop 1 : Forward on edge 2, backward on edges 3 and 1
loop 2 : Forward on edges 3 and 5, backward on edge 4
loop 3 : Forward on edge 6, backward on edges 5 and 2

Flow around a loop automatically satisfies Kirchhoff's Current Law.
At each node in the loop, the flow into the node goes out to the next node. The three loops in the graph produce three independent solutions to $A^Tw=0$. Each w gives six edge current around a loop:

$$A^T\underline{w}=0 \quad \text{for } \underline{w}_1=\begin{bmatrix}-1\\1\\-1\\0\\0\\0\end{bmatrix} \quad \text{and } \underline{w}_2=\begin{bmatrix}0\\0\\1\\-1\\1\\0\end{bmatrix} \quad \text{and } \underline{w}_3=\begin{bmatrix}0\\-1\\0\\0\\-1\\1\end{bmatrix}$$

There are no more independent solutions even if there are more (larger) loops! The large loop around the whole graph is exactly the sum of the three small loops. So the solution $\underline{w}=(-1,0,0,-1,0,1)$ for that outer loop is exactly the sum $\underline{w}_1+\underline{w}_2+\underline{w}_3$.

therefore, $\underline{w}_1$, $\underline{w}_2$ and $\underline{w}_3$ are three independent solutions to $A^T\underline{w}=0$
therefore, these three vectors are in the nullspace of A^T.
thus, the dimension of the nullspace of A^T is three ($\dim N(A^T)=3$)

problem IV.6 – 8 10/10

We prove Euler's Formula for any connected graph in a plane.

Euler's Formula

Let G be a connected planar graph, and let n, m and ℓ denote, respectively, the number of nodes, edges and small loops in a plane drawing of G.

Then, (number of nodes) – (number of edges) + (number of small loops) = 1

In other words, $n - m + \ell = 1$

Proof

We employ mathematical induction on the number of edges, m, in the graph.

Base: If $m = 0$, the graph consists of a single node and zero small loops (no small loops). In this case, $n = 1$ and $\ell = 0$. So we have $n - m + \ell = 1 - 0 + 0 = 1$ which is clearly right.

Induction: Assume that the formula works for all connected plane graphs with less than m edges, where m is greater than or equal to 1. Let G be a connected graph with m edges.

Case 1: If G doesn't contain a cycle, then G is tree and we know that for a tree (# edges = # nodes – 1). Therefore, $n = m + 1$ and $\ell = 0$ (since there are no cycles, so no small loops).

Thus, we have $n - m + \ell = m + 1 - m + 0 = 1$ and the formula works

Case 2: If G contains at least one cycle (so G is not a tree), pick an edge e that's on a cycle. In other words, let e be a cycle edge of G and consider the graph $G'' = G - e$ that results from removing the cycle edge e from G. Removing the cycle edge will remove one

small loop. So G'' has one fewer small loop than G.
Since G'' has $m-1$ edges (remember we removed one edge from G), then the formula works for G'' by the induction hypothesis. That is, $n''-m''+\ell''=1$. But the connected plane graph G'' has $n''=n$ nodes, $m''=m-1$ edges and $\ell''=\ell-1$ small loops. Substituting, we find that

$$n''-m''+\ell''=1$$

$$n-(m-1)+(\ell-1)=1$$

which implies that

$$n-m+\ell=1$$

This completes the proof of Euler's Formula.

<u>note</u>: Here we assumed that the graph G with m edges contains n nodes and ℓ small loops.

Another method to prove Euler's Formula

Assume a connected planar graph with n nodes, m edges and ℓ small loops.

Let A be the m by n incidence matrix.

the incidence matrix has n columns when the graph has n nodes. Those column vectors add up to the zero vector. In other words, the all-ones vector $x=(1,1,1,1)$ is in the nullspace of the incidence matrix A. the nullspace of A is a single line through that all-one vector. Thus $\boxed{\dim N(A)=1}$. $Ax=0$ requires $x_1=x_2=\dots=x_n$ so that $x=\begin{bmatrix} c\\ c\\ \vdots\\ c\\ c\end{bmatrix}_{n\times 1}$

Now if we look at the equation $A^Ty=0$, we see that this is the Kirchhoff's Current Law (KCL). Since flow around a loop automatically satisfies KCL, then the number of independent solutions is equal to the number of small loops ℓ in the graph (since larger loops are a linear combination of smaller loops). thus the equation $A^Ty=0$ has ℓ independent solutions. thus, $\boxed{\dim N(A^T)=\ell}$

We know that: $\dim N(A)+\dim C(A)=n$

$$1+\dim C(A)=n \Rightarrow \boxed{\dim C(A)=n-1}$$

we also know that: $\dim N(A^T)+\dim C(A^T)=m$

$$\ell+\dim C(A^T)=m \Rightarrow \boxed{\dim C(A^T)=m-\ell}$$

since the row space and the column space have the same dimension and row space of A is column space of A^T, thus $\boxed{\dim C(A)=\dim C(A^T)}$

$$\dim C(A)=\dim C(A^T)$$
$$n-1=m-\ell$$
$$n-m+\ell=1$$

$\Rightarrow$ therefore we proved Euler's Formula: $\boxed{n-m+\ell=1}$ ✓

Computin Problem 1

10/10

```
% Construct Matrix A
temp = diag(2*ones(9,1),1) + diag(-1*ones(10,1)) + diag(-1*ones(8,1),2);
A = temp(1:8,:)
```

```
A = 8×10
    -1     2    -1     0     0     0     0     0     0     0
     0    -1     2    -1     0     0     0     0     0     0
     0     0    -1     2    -1     0     0     0     0     0
     0     0     0    -1     2    -1     0     0     0     0
     0     0     0     0    -1     2    -1     0     0     0
     0     0     0     0     0    -1     2    -1     0     0
     0     0     0     0     0     0    -1     2    -1     0
     0     0     0     0     0     0     0    -1     2    -1
```

```
% Construct Matrix T
T = A(:,2:9)
```

```
T = 8×8
     2    -1     0     0     0     0     0     0
    -1     2    -1     0     0     0     0     0
     0    -1     2    -1     0     0     0     0
     0     0    -1     2    -1     0     0     0
     0     0     0    -1     2    -1     0     0
     0     0     0     0    -1     2    -1     0
     0     0     0     0     0    -1     2    -1
     0     0     0     0     0     0    -1     2
```

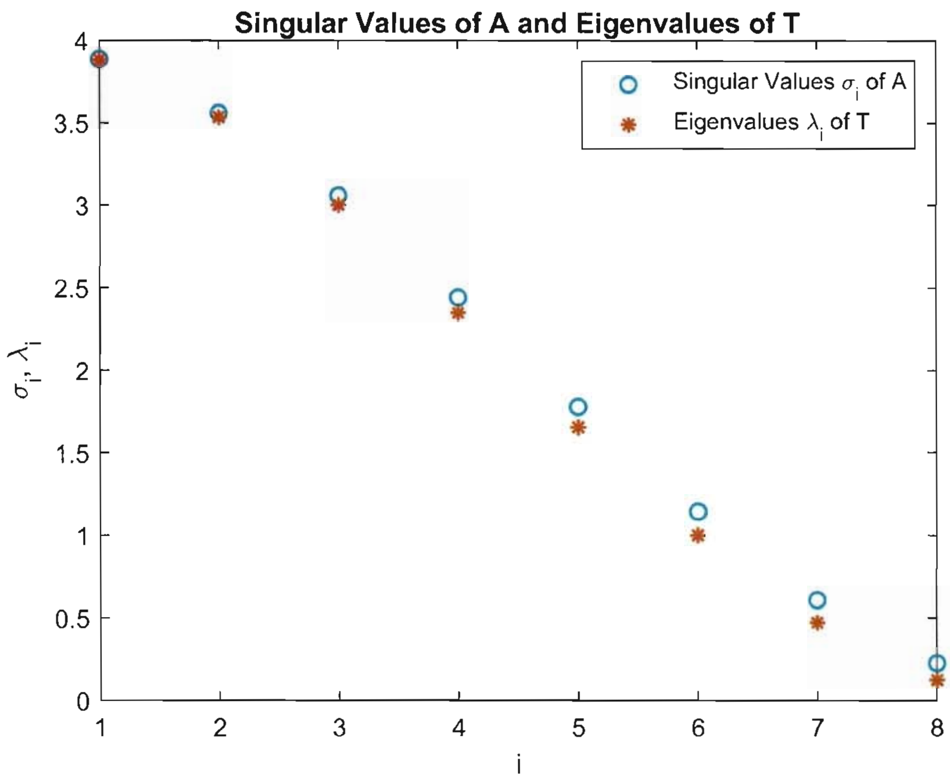
```
s = svd(A) % Singular Values of A
```

```
s = 8×1
    3.8868
    3.5606
    3.0595
    2.4418
    1.7784
    1.1441
    0.6074
    0.2248
```

```
e = sort(eig(T),'descend') % Eigenvalues of T
```

```
e = 8×1
    3.8794
    3.5321
    3.0000
    2.3473
    1.6527
    1.0000
    0.4679
    0.1206
```

```
plot(s,'o','linewidth',1.2);
grid on
hold on
plot(e,'*','linewidth',1.2);
title('Singular Values of A and Eigenvalues of T')
xlabel('i')
ylabel('\sigma_i, \lambda_i')
legend('Singular Values \sigma_i of A', 'Eigenvalues \lambda_i of T')
```



Problem 2

10/10

Using equation (3) page 221 and equation (12) page 224, we show that the singular values of $A \otimes B$ are singular values of A times singular values of B.

Let $A v_A = \sigma_A u_A$ where u_A and v_A are the left and right singular vectors respectively, and σ_A is the singular value of A.

Let $B v_B = \sigma_B u_B$ where u_B and v_B are the left and right singular vectors respectively, and σ_B is the singular value of B.

Then, the Kronecker product of u_A and u_B is a left singular vector of $A \otimes B$ and the Kronecker product of v_A and v_B is a right singular vector of $A \otimes B$. The singular value is $\sigma_A \sigma_B$:

$$\boxed{(A \otimes B)(v_A \otimes v_B) = (A v_A) \otimes (B v_B) = (\sigma_A u_A) \otimes (\sigma_B u_B) = \sigma_A \sigma_B (u_A \otimes u_B)}$$

Therefore, the singular value of $A \otimes B$ is $\sigma_A \sigma_B$.

Interesting observation:

Suppose A has rank r_A, i.e. A has r_A nonzero singular values.
Suppose B has rank r_B, i.e. B has r_B nonzero singular values.
It follows directly that the singular values of $A \otimes B$ are the $r_A r_B$ possible positive products of singular values of A and B (counting multiplicities). Since the rank of a matrix is equal to the number of nonzero singular values, therefore, we find that

$$\text{rank}(A \otimes B) = \text{rank}(B \otimes A) = \text{rank}(A)\,\text{rank}(B) = r_A r_B$$

Tony Tohme
919167634

18.0651 - Matrix Methods
Homework 6

problem II.4 _ 2

We start by solving problem II.4_1.

Given positive numbers $a_1, \ldots, a_n$ we find positive numbers $p_1, \ldots, p_n$ so that $p_1 + p_2 + \cdots + p_n = 1$ and $V = \frac{a_1^2}{p_1} + \cdots + \frac{a_n^2}{p_n}$ reaches its minimum $(a_1 + \cdots + a_n)^2$.

In other words, we find the probabilities $p_1, \ldots, p_n$ that minimize $V = \frac{a_1^2}{p_1} + \cdots + \frac{a_n^2}{p_n}$ subject to the constraint $p_1 + \cdots + p_n = 1$.

We introduce the Lagrange multiplier λ and study the Lagrange function:

$$L(p,\lambda) = V + \lambda(p_1 + \cdots + p_n - 1) = \sum_{i=1}^{n} \frac{a_i^2}{p_i} + \lambda\left(\sum_{i=1}^{n} p_i - 1\right)$$

Take the partial derivatives $\frac{\partial L}{\partial p_i}$ to find the minimizing p_i (the optimal probabilities):

$$\frac{\partial L}{\partial p_i} = -\frac{a_i^2}{p_i^2} + \lambda = 0 \Rightarrow \lambda = \frac{a_i^2}{p_i^2} \Rightarrow p_i^2 = \frac{a_i^2}{\lambda} \Rightarrow p_i = \frac{a_i}{\sqrt{\lambda}}, \quad i = 1, \ldots, n$$

This says that $p_i = \frac{a_i}{\sqrt{\lambda}}$. Choose the Lagrange multiplier λ so that $\sum_{i=1}^{n} p_i = 1$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} p_i - 1 = 0 \Longrightarrow \sum_{i=1}^{n} p_i = 1 \Rightarrow \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{a_i}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{n} a_i = 1$$

This gives $\sqrt{\lambda} = \sum_{i=1}^{n} a_i$ and $\boxed{p_i = \frac{a_i}{\sum_{j=1}^{n} a_j}}$ for all i — discrete optimal solution for problem II.4_1

The minimum can be computed as follows:

$$V = \sum_{i=1}^{n} \frac{a_i^2}{p_i} = \sum_{i=1}^{n} \frac{a_i^2}{a_i} \cdot \sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n} a_i\right)^2 = (a_1 + \cdots + a_n)^2$$

We then move to solve problem II.4_2.

(for functions) Given $a(x) > 0$ we find $p(x) > 0$ by analogy with problem II.4_1, so that $\int_0^1 p(x)\,dx = 1$ and $\int_0^1 \frac{(a(x))^2}{p(x)}\,dx$ is a minimum.

By analogy with problem II.4_1, the optimal probabilities are:

$$\boxed{p(x) = \frac{a(x)}{\int_0^1 a(x)\,dx}}$$

continuous optimal solution for problem II.4_2

The minimum can be computed as follows:

$$\int_0^1 \frac{(a(x))^2}{p(x)}\,dx = \int_0^1 \frac{(a(x))^2}{a(x)} \cdot \left(\int_0^1 a(x)\,dx\right) dx = \int_0^1 a(x)\,dx \cdot \int_0^1 a(x)\,dx$$

$$= \left(\int_0^1 a(x)\,dx\right)^2$$

which is analogous to the minimum of V found in problem II.4_1, i.e. $\left(\sum_{i=1}^{n} a_i\right)^2$

problem II.4_3

We prove that $n(a_1^2+\dots+a_n^2) \geq (a_1+\dots+a_n)^2$ using 2 approaches.

1st approach:

In problem II.4_1, we have shown that $V = \frac{a_1^2}{p_1}+\dots+\frac{a_n^2}{p_n}$ reaches its minimum $(a_1+\dots+a_n)^2$ when $p_i = \frac{a_i}{\sum_{j=1}^{n} a_j}$ for $i=1,\dots,n$ (subject to $\sum_{i=1}^{n} p_i = 1$)

Therefore, we know that $V \geq (a_1+\dots+a_n)^2 \Rightarrow \frac{a_1^2}{p_1}+\dots+\frac{a_n^2}{p_n} \geq (a_1+\dots+a_n)^2$

Now, if $p_i = \frac{1}{n}\ \forall i$ (which are not the optimal probabilities found in problem II.4_1)

we have $V = \frac{a_1^2}{p_1}+\dots+\frac{a_n^2}{p_n} = \frac{a_1^2}{1/n}+\dots+\frac{a_n^2}{1/n} = na_1^2+\dots+na_n^2$

$$= \underbrace{n(a_1^2+\dots+a_n^2)}_{\text{value of } V \text{ when } p_i=\frac{1}{n}\ \forall i} \geq \underbrace{(a_1+\dots+a_n)^2}_{\text{the minimum of } V \text{ at optimum}}$$

the inequality holds since the p_i's are not the optimal probabilities

and note that $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$ (the constraint is satisfied)

therefore, we have shown that $\boxed{n(a_1^2+\dots+a_n^2) \geq (a_1+\dots+a_n)^2}$

2nd approach:

In homework 3, problem I.11_3, we have shown that $\|a\|_1 \leq \sqrt{n}\,\|a\|_2$ by choosing a suitable vector w and applying the Cauchy_Schwarz inequality. Thus, we know that $\|a\|_1 \leq \sqrt{n}\,\|a\|_2$

where $\|a\|_1 = \sum_{i=1}^{n} |a_i| = |a_1| + \dots |a_n| = a_1+\dots+a_n$ since $a_1,\dots,a_n$ are positive

and $\|a\|_2 = \sqrt{\sum_{i=1}^{n} |a_i|^2} = \sqrt{|a_1|^2+\dots+|a_n|^2} = \sqrt{a_1^2+\dots+a_n^2}$

Since both sides of the inequality are nonnegative, we square both sides, we get:

$\|a\|_1^2 \leq n\,\|a\|_2^2$ where $\|a\|_1^2 = (a_1+\dots+a_n)^2$ and $\|a\|_2^2 = a_1^2+\dots+a_n^2$

Therefore, we get: $\boxed{(a_1+\dots+a_n)^2 \leq n(a_1^2+\dots+a_n^2)}$

This completes the proof.

problem II. 4 - 4

If $M = 1\,1^T$ is the n by n matrix of 1's, we prove that $nI - M$ is positive semidefinite by finding the eigenvalues of $nI - M$

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \\ 1 & \cdots & & 1 \end{bmatrix}_{n\times n} \Longrightarrow nI - M = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & \\ -1 & \cdots & & n-1 \end{bmatrix}_{n\times n}$$

we find the eigenvalues of $nI - M$ as follows:

$(nI - M)v = \lambda v$

$$\det(nI - M - \lambda I) = 0 \Longrightarrow \begin{vmatrix} n-\lambda-1 & -1 & \cdots & -1 \\ -1 & n-\lambda-1 & & -1 \\ \vdots & & \ddots & \vdots \\ -1 & & \cdots & n-\lambda-1 \end{vmatrix} = 0$$

Let $D = \begin{bmatrix} n-\lambda & 0 & \cdots & 0 \\ 0 & n-\lambda & & \vdots \\ \vdots & & \ddots & \\ 0 & & \cdots & n-\lambda \end{bmatrix}_{n\times n}$ be a diagonal matrix of diagonal entries $(n-\lambda)$.

and let e be the all-one vector, i.e. $e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$ then $ee^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \\ 1 & \cdots & & 1 \end{bmatrix}_{n\times n}$

Then, $nI - M - \lambda I$ can be written as:

$$nI - M - \lambda I = D - ee^T \Longrightarrow \det(nI - M - \lambda I) = \det(D - ee^T)$$

note:

In linear algebra, the matrix determinant lemma computes the determinant of the sum of an invertible matrix A and the dyadic product, uv^T, of a column vector u and a row vector v^T.

suppose A is an invertible square matrix and u, v are column vectors. Then the matrix lemma states that $\boxed{\det(A + uv^T) = 1 + (v^T A^{-1} u) \det(A)}$

Here, uv^T is the outer product of two vectors u and v.

The theorem can also be stated in terms of the adjugate matrix of A:

$\boxed{\det(A + uv^T) = \det(A) + v^T \operatorname{adj}(A)\, u}$ in which case it applies whether or not the square matrix A is invertible

back to our problem, we want to compute $\det(nI-M-\lambda I)=\det(D-ee^T)$
we can apply the matrix determinant lemma where $A=D$ and $u=v=e$

$$\det(D-ee^T) = \det(D) + e^T \operatorname{adj}(D)\, e = \prod_{j=1}^{n} d_{jj} - \sum_{i=1}^{n} \prod_{j\neq i} d_{jj}$$
$$= (n-\lambda)^n - \sum_{i=1}^{n} (n-\lambda)^{n-1}$$
$$= (n-\lambda)^n - n(n-\lambda)^{n-1}$$
$$= (n-\lambda)^{n-1}\left[(\cancel{n}-\lambda)-\cancel{n}\right]$$
$$= -\lambda(n-\lambda)^{n-1}$$

Now, we find the eigenvalues by setting $\det(nI-M-\lambda I)=\det(D-ee^T)=0$

$\Rightarrow -\lambda(n-\lambda)^{n-1}=0 \quad \Rightarrow \quad \lambda=0$ with algebraic multiplicity of 1

$\lambda=n$ with algebraic multiplicity of $n-1$

Since $n>0$, then $\lambda \geq 0$ (0 or n), therefore, the eigenvalues of $nI-M$ are nonnegative and $nI-M$ is positive semidefinite.

Another way to find the eigenvalues of $nI-M$ is by using the fact that it is a $n\times n$ circulant matrix

note:

A $n\times n$ circulant matrix C takes the form

$$C=\begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

has eigenvalues given by

$$\lambda_j = c_0 + c_{n-1}w_j + c_{n-2}w_j^2 + \dots + c_1 w_j^{n-1}, \quad \text{where } w_j = e^{i\frac{2\pi j}{n}} \quad j=0,\dots,n-1$$

Now, we use the fact that $nI-M$ is a $n\times n$ circulant matrix of the form:

$$nI-M=\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & -1 \\ \vdots & & \ddots & \vdots \\ -1 & & \cdots & n-1 \end{bmatrix}$$

where $c_0 = n-1$
$c_j = -1 \quad j=1,\dots,n-1$

to derive the eigenvalues of $nI-M$ as follows:

$$\lambda_0 = n-1 - e^{i\frac{2\pi 0}{n}} - e^{i\frac{4\pi 0}{n}} - \dots - e^{i\frac{2\pi(n-1)0}{n}} = n-1-\sum_{k=1}^{n-1} e^0 = n-1-\sum_{k=1}^{n-1} 1$$
$$= n-1-(n-1) = 0$$

$$\lambda_1 = n-1-e^{i\frac{2\pi}{n}} - e^{i\frac{4\pi}{n}} - \dots - e^{i\frac{2\pi(n-1)}{n}} = n - \sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}} = n$$

$$\lambda_2 = n-1-e^{i\frac{4\pi}{n}} - e^{i\frac{8\pi}{n}} - \dots - e^{i\frac{4\pi(n-1)}{n}} = n - \sum_{k=0}^{n-1} e^{i\frac{2\pi 2k}{n}} = n$$

$$\vdots$$

$$\lambda_{n-1} = n-1-e^{i\frac{2\pi(n-1)}{n}} - e^{i\frac{2\pi(n-1)2}{n}} - \dots - e^{i\frac{2\pi(n-1)^2}{n}} = n - \sum_{k=0}^{n-1} e^{i\frac{2\pi(n-1)k}{n}} = n$$

Therefore, $nI-M$ has one eigenvalue of value 0 and $n-1$ eigenvalues of value n. Thus, $\lambda \geq 0$ (since $n>0$), and hence $nI-M$ is positive semidefinite

note: we prove that $\sum_{k=0}^{n-1} e^{i\frac{2\pi j k}{n}} = 0, \; j=1,\dots,n-1$

$$\sum_{k=0}^{n-1} e^{i\frac{2\pi jk}{n}} = \sum_{k=0}^{n-1}\Big(\underbrace{e^{i\frac{2\pi j}{n}}}_{r}\Big)^k = \frac{1\left(1-e^{i\frac{2\pi j n}{n}}\right)}{1-e^{i\frac{2\pi j}{n}}} = \frac{1-e^{i2\pi j}}{1-e^{i\frac{2\pi j}{n}}} = \frac{1-1}{1-e^{i\frac{2\pi j}{n}}} = 0$$

sum of geometric series

note that $e^{i2\pi j} = 1, \; j=1,\dots,n-1$

This completes the proof.

problem II . 4_6

We show that $\|AB\|_F^2 \leqslant \left(\Sigma \|a_j\| \|b_j^T\|\right)^2$

note :

We show that the Frobenius norm of the product of a column vector a and a row vector b^T is equal to the product of the Frobenius norms of each of the vectors a and b^T.

In other words, we show that $\|ab^T\|_F = \|a\|_F \, \|b^T\|_F \quad (\mathrm{I})$

$$ab^T = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix}$$

has $\|ab^T\|_F = |a_1|^2\left(|b_1|^2 + \cdots + |b_p|^2\right) + \cdots + |a_m|^2\left(|b_1|^2 + \cdots + |b_p|^2\right)$

$$= \|a\|_F^2 \, \|b\|_F^2 = \|a\|_F^2 \, \|b^T\|_F^2$$

Back to our problem, we get :

$\|AB\|_F = \|a_1 b_1^T + \cdots + a_n b_n^T\|_F$ by column_row multiplication

$\leqslant \|a_1 b_1^T\|_F + \cdots + \|a_n b_n^T\|_F$ by the triangle inequality

$= \|a_1\|_F \|b_1^T\|_F + \cdots + \|a_n\|_F \|b_n^T\|_F$ by equation (I) above

$$= \sum_{j=1}^{n} \|a_j\|_F \|b_j^T\|_F$$

Thus, $\|AB\|_F \leqslant \sum_{j=1}^{n} \|a_j\|_F \|b_j^T\|_F$ since $\|AB\|_F \geqslant 0$ and $\sum_{j=1}^{n} \|a_j\|_F \|b_j\|_F \geqslant 0$

we square both sides of the inequality

Therefore, $\boxed{\|AB\|_F^2 \leqslant \left(\sum_{j=1}^{n} \|a_j\|_F \|b_j^T\|_F\right)^2}$

Thus, we have shown that $\|AB\|_F^2 \leqslant \left(\Sigma \|a_j\| \|b_j^T\|\right)^2$

and hence, the variance computed in equation (7) page 150, cannot be negative !

i.e. $E\left[\|AB - CR\|_F^2\right] = \frac{1}{s}\left(C^2 - \|AB\|_F^2\right) = \frac{1}{s}\Big(\underbrace{\left(\sum_{j=1}^{n} \|a_j\| \|b_j^T\|\right)^2 - \|AB\|_F^2}_{\geqslant 0}\Big) \geqslant 0$

Computing Problem

```
A = triu(ones(1000));
G1 = normrnd(0,1,1000,10);  % 1000 by 10 Gaussian random matrix G1
G2 = normrnd(0,1,1000,100); % 1000 by 100 Gaussian random matrix G2

% Exact or Actual SVD
[u,s,v] = svd(A);

% Randomized or Approximate SVD with G1
Y1 = A*G1;
[Q1,R1] = qr(Y1);
[U1,D1,V1] = svd(Q1'*A); W1 = Q1*U1; % A = (Q1*U1)*D1*V1' = W1*D1*V1'

% Randomized or Approximate SVD with G2
Y2 = A*G2;
[Q2,R2] = qr(Y2);
[U2,D2,V2] = svd(Q2'*A); W2 = Q2*U2; % A = (Q2*U2)*D2*V2' = W2*D2*V2'

s11 = diag(s);  display(s11(1:10)) % The 10 Largest Singular Values From Actual SVD
```

```
 636.93814767091
212.312890336131
127.387943537033
90.9916125292626
70.7714867858701
57.9041816178516
48.9960875295752
42.4635200898666
37.4680581293865
 33.524299918613
```

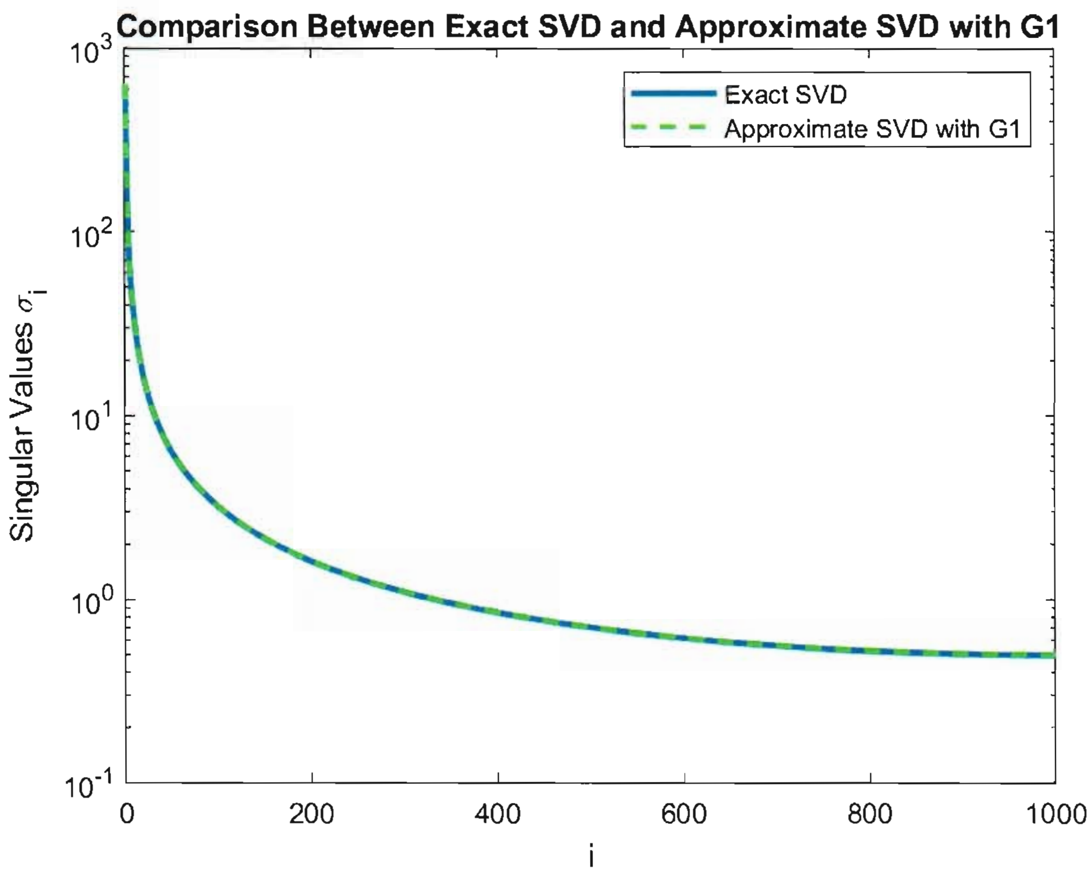
```
D11 = diag(D1); display(D11(1:10)) % The 10 Largest Singular Values From Approximate SVD with G1
```

```
636.938147670908
212.312890336131
127.387943537033
90.9916125292626
70.7714867858701
57.9041816178517
48.9960875295753
42.4635200898666
37.4680581293866
 33.524299918613
```

```
D22 = diag(D2); display(D22(1:10)) % The 10 Largest Singular Values From Approximate SVD with G2
```

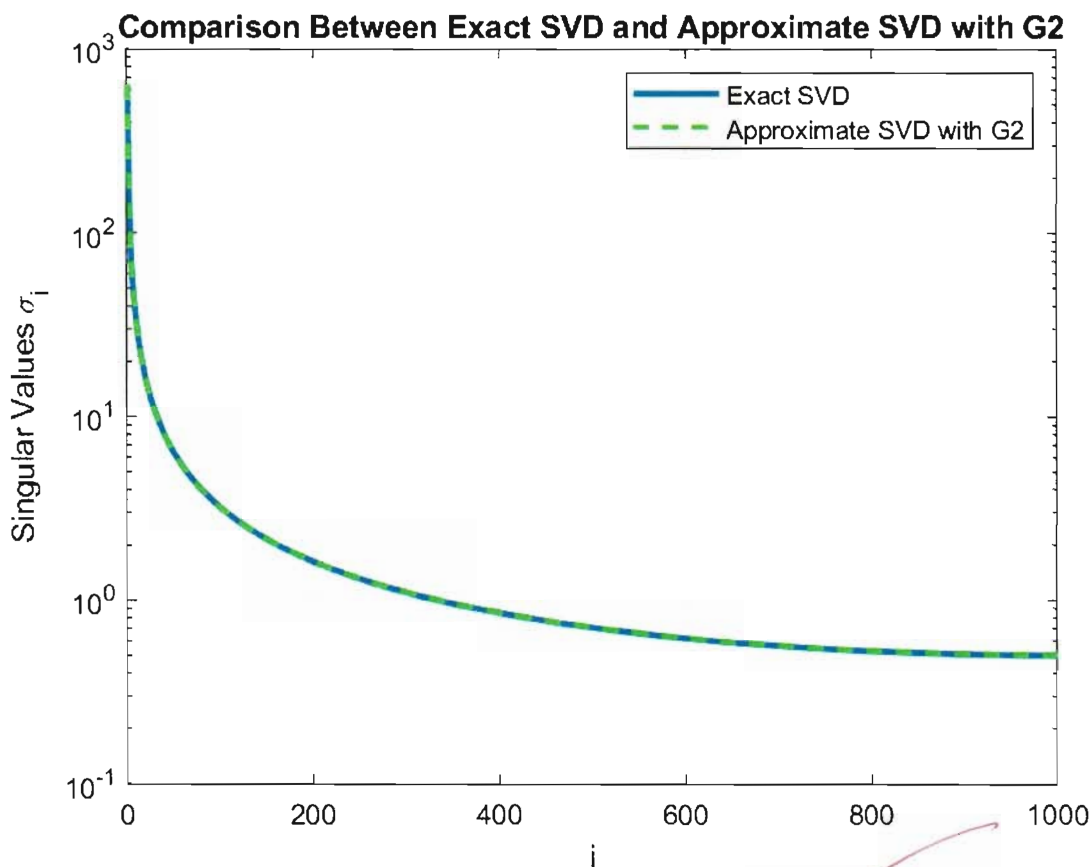
```
636.938147670908
 212.31289033613
127.387943537033
90.9916125292626
70.7714867858701
57.9041816178516
48.9960875295752
42.4635200898666
37.4680581293866
 33.524299918613
```

```
% Comparison Between Actual SVD and Randomized SVD with G1
% Plot of the Singular Values of the Actual SVD and Randomized SVD with G1
figure();
semilogy(s11,'linewidth',2)
hold on
semilogy(D11,'--g','linewidth',1.5)
grid on
title('Comparison Between Exact SVD and Approximate SVD with G1')
legend('Exact SVD','Approximate SVD with G1','location','Northeast')
xlabel('i')
ylabel('Singular Values \sigma_i')
```



As we can see, both SVD's, the actual SVD and randomized SVD with G1, result in approximately same Singular Values.

```
% Comparison Between Actual SVD and Randomized SVD with G2
% Plot of the Singular Values of the Actual SVD and Randomized SVD with G2
figure();
semilogy(s11,'linewidth',2)
hold on
semilogy(D22,'--g','linewidth',1.5)
grid on
title('Comparison Between Exact SVD and Approximate SVD with G2')
legend('Exact SVD','Approximate SVD with G2','location','Northeast')
xlabel('i')
ylabel('Singular Values \sigma_i')
```



As we can see, both SVD's, the actual SVD and randomized SVD with G2, result in approximately same Singular Values.

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