I.1 Multiplication Ax Using Columns of A

We hope you already know some linear algebra. It is a beautiful subject—more useful to more people than calculus (in our quiet opinion). But even old-style linear algebra courses miss basic and important facts. This first section of the book is about *matrix-vector* multiplication $A\boldsymbol{x}$ and the column space of a matrix and the rank.

We always use examples to make our point clear.

Example 1 Multiply A times $\boldsymbol{x}$ using the three rows of A. Then use the two columns:

$$\textbf{By rows}\qquad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1+3x_2 \\ 2x_1+4x_2 \\ 3x_1+7x_2 \end{bmatrix} = \begin{matrix}\text{inner products} \\ \text{of the rows} \\ \text{with } \boldsymbol{x}=(x_1,x_2)\end{matrix}$$

$$\textbf{By columns}\qquad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{matrix}\text{combination} \\ \text{of the columns} \\ \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_2\end{matrix}$$

You see that both ways give the same result. The first way (a row at a time) produces three inner products. Those are also known as "dot products" because of the dot notation:

$$\textbf{row} \cdot \textbf{column} = (\mathbf{2},\ \mathbf{3}) \cdot (\boldsymbol{x}_1,\ \boldsymbol{x}_2) = \mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2 \qquad (1)$$

This is the way to find the three separate components of $A\boldsymbol{x}$. We use this for computing—but not for understanding. It is low level. Understanding is higher level, using vectors.

The vector approach sees $A\boldsymbol{x}$ as a "linear combination" of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. This is the fundamental operation of linear algebra! A linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ includes two steps:

(1) Multiply the columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ by "scalars" x_1 and x_2

(2) Add vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = A\boldsymbol{x}$.

Thus $A\boldsymbol{x}$ is a linear combination of the columns of A. This is fundamental.

This thinking leads us to the **column space** of A. The key idea is to take **all combinations** of the columns. All real numbers x_1 and x_2 are allowed—the space includes $A\boldsymbol{x}$ for all vectors $\boldsymbol{x}$. In this way we get infinitely many output vectors $A\boldsymbol{x}$. And we can see those outputs geometrically.

In our example, each $A\boldsymbol{x}$ is a vector in 3-dimensional space. That 3D space is called $\mathbf{R}^3$. (The $\mathbf{R}$ indicates real numbers. Vectors with three complex components lie in the space $\mathbf{C}^3$.) We stay with real vectors and we ask this key question:

All combinations $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ produce what part of the full 3D space?

Answer: Those vectors produce a **plane**. The plane contains the complete line in the direction of $\boldsymbol{a}_1 = (2, 2, 3)$, since every vector $x_1\boldsymbol{a}_1$ is included. The plane also includes the line of all vectors $x_2\boldsymbol{a}_2$ in the direction of $\boldsymbol{a}_2$. And it includes the *sum* of any vector on one line plus any vector on the other line. **This addition fills out an infinite plane containing the two lines**. But it does not fill out the whole 3-dimensional space $\mathbf{R}^3$.

Definition **The combinations of the columns** fill out the **column space** of A.

Here the column space is a plane. That plane includes the zero point $(0,0,0)$ which is produced when $x_1 = x_2 = 0$. The plane includes $(5,6,10) = \boldsymbol{a}_1 + \boldsymbol{a}_2$ and $(-1,-2,-4) = \boldsymbol{a}_1 - \boldsymbol{a}_2$. Every combination $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ is in this column space. With probability 1 it does **not** include the random point **rand**$(3,1)$! Which points are in the plane ?

$\boldsymbol{b} = (\boldsymbol{b_1, b_2, b_3})$ is in the column space of $\boldsymbol{A}$ **exactly when** $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $(\boldsymbol{x_1, x_2})$

When you see that truth, you understand the column space $\mathbf{C}(A)$: The solution $\boldsymbol{x}$ shows how to express the right side $\boldsymbol{b}$ as a combination $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ of the columns. For some $\boldsymbol{b}$ this is impossible—they are not in the column space.

Example 2 $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in $\mathbf{C}(A)$. $A\boldsymbol{x} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is unsolvable.

The first two equations force $x_1 = \frac{1}{2}$ and $x_2 = 0$. Then equation 3 fails : $3\left(\frac{1}{2}\right) + 7(0) = \mathbf{1.5}$ (**not 1**). This means that $\boldsymbol{b} = (1,1,1)$ is not in the column space—the plane of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

Example 3 What are the column spaces of $A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$?

Solution. The column space of A_2 is the same plane as before. The new column $(5,6,10)$ is the sum of column 1 + column 2. So $\boldsymbol{a}_3$ = column 3 is already in the plane and adds nothing new. By including this "*dependent*" column we don't go beyond the original plane.

The column space of A_3 is the whole 3D space $\mathbf{R}^3$. Example 2 showed us that the new third column $(1,1,1)$ is not in the plane $\mathbf{C}(A)$. Our column space $\mathbf{C}(A_3)$ has grown bigger. But there is nowhere to stop between a plane and the full 3D space. Visualize the $x - y$ plane and a third vector (x_3, y_3, z_3) out of the plane (meaning that $z_3 \neq 0$). They combine to give **every vector in** $\mathbf{R}^3$.

Here is a total list of all possible column spaces inside $\mathbf{R}^3$. Dimensions $0, 1, 2, 3$:

Subspaces of $\mathbf{R}^3$
- The **zero vector** $(0,0,0)$ by itself
- A **line** of all vectors $x_1\boldsymbol{a}_1$
- A **plane** of all vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$
- The **whole** $\mathbf{R}^3$ with all vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$

In that list we need the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ to be "**independent**". The only combination that gives the zero vector is $0\boldsymbol{a}_1 + 0\boldsymbol{a}_2 + 0\boldsymbol{a}_3$. So $\boldsymbol{a}_1$ by itself gives a line, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ give a plane, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ give every vector $\boldsymbol{b}$ in $\mathbf{R}^3$. The zero vector is in every subspace ! In linear algebra language :

- Three independent columns in $\mathbf{R}^3$ produce an **invertible matrix :** $AA^{-1} = A^{-1}A = I$.
- $A\boldsymbol{x} = \mathbf{0}$ requires $\boldsymbol{x} = (0,0,0)$. Then $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution $\boldsymbol{x} = \boldsymbol{A^{-1}b}$.

You see the picture for the columns of an n by n invertible matrix. Their combinations fill its column space : **all of** $\mathbf{R}^n$. We needed those ideas and that language to go further.

Independent Columns and the Rank of A

After writing those words, I thought this short section was complete. *Wrong*. With just a small effort, we can find a **basis** for the column space of A, we can **factor** A into C times R, and we can prove the **first great theorem** in linear algebra. You will see the rank of a matrix and the dimension of a subspace.

All this comes with an understanding of **independence**. The goal is to create a matrix C whose columns come directly from A—but not to include any column that is a combination of previous columns. The columns of C (as many as possible) will be "independent". Here is a natural construction of C from the n columns of A:

If column 1 of A is not all zero, put it into the matrix C.

If column 2 of A is not a multiple of column 1, put it into C.

If column 3 of A is not a combination of columns 1 and 2, put it into C. *Continue*.

At the end C will have r columns ($\boldsymbol{r} \leq \boldsymbol{n}$).

They will be a "basis" for the column space of A.

The left out columns are combinations of those basic columns in C.

A **basis** for a subspace is a full set of independent vectors: **All vectors in the space are combinations of the basis vectors**. Examples will make the point.

Example 4 If $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ $n = 3$ columns in A $\boldsymbol{r} = \mathbf{2}$ columns in C

Column 3 of A is 2 (column 1) + 2 (column 2). Leave it out of the basis in C.

Example 5 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ then $C = A$. $n = 3$ columns in A $\boldsymbol{r} = \mathbf{3}$ columns in C

This matrix A is invertible. Its column space is all of $\mathbf{R}^3$. Keep all 3 columns.

Example 6 If $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ then $C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $n = 3$ columns in A $\boldsymbol{r} = \mathbf{1}$ column in C

The number $\boldsymbol{r}$ is the **"rank"** of A. It is also the rank of C. **It counts independent columns.** Admittedly we could have moved from right to left in A, starting with its *last* column. This would not change the final count r. *Different basis, but always the same number of vectors*. That number r is the **"dimension"** of the column space of A and C (same space).

The rank of a matrix is the dimension of its column space.

The matrix C connects to A by a third matrix R: $\boldsymbol{A} = \boldsymbol{CR}$. Their shapes are $(m \text{ by } n) = (m \text{ by } r)\,(r \text{ by } n)$. I can show this "factorization of A" in Example 4 above:

$$\boxed{\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{CR}} \qquad (2)$$

When C multiplies the first column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of R, this produces column 1 of C and A.

When C multiplies the second column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of R, we get column 2 of C and A.

When C multiplies the third column $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ of R, we get $2(\text{column } 1) + 2(\text{column } 2)$.

This matches column 3 of A. All we are doing is to put the right numbers in R. Combinations of the columns of C produce the columns of A. Then $A = CR$ stores this information as a matrix multiplication. Actually R is a famous matrix in linear algebra:

$$\boldsymbol{R} = \textbf{rref}(\boldsymbol{A}) = \textbf{row-reduced echelon form of } \boldsymbol{A} \text{ (without zero rows).}$$

Example 5 has $C = A$ and then $R = I$ (identity matrix). Example 6 has only *one* column in C, so it has one row in R:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = CR$$

All three matrices have rank $r = 1$
Column Rank = Row Rank

The number of *independent columns* equals the number of *independent rows*

This rank theorem is true for every matrix. Always columns and rows in linear algebra! The m rows contain the same numbers a_{ij} as the n columns. But different vectors.

The theorem is proved by $A = CR$. Look at that differently—by rows instead of columns. The matrix R has r rows. **Multiplying by $\boldsymbol{C}$ takes combinations of those rows.** Since $A = CR$, we get every row of A from the r rows of R. And those r rows are independent, so they are a **basis for the row space of $\boldsymbol{A}$**. The column space and row space of A both have dimension r, with r basis vectors—columns of C and rows of R.

One minute: Why does R have independent rows? Look again at Example 4.

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \leftarrow \text{independent} \\ \leftarrow \text{rows of } R \end{matrix}$$

$\uparrow\ \uparrow$
ones and zeros

It is those ones and zeros in R that tell me: No row is a combination of the other rows.

The big factorization for data science is the "SVD" of A—when the first factor C has r *orthogonal* columns and the second factor R has r *orthogonal* rows.

Problem Set I.1

1 Give an example where a combination of three nonzero vectors in $\mathbf{R}^4$ is the zero vector. Then write your example in the form $A\boldsymbol{x} = \mathbf{0}$. What are the shapes of A and $\boldsymbol{x}$ and $\mathbf{0}$?

2 Suppose a combination of the columns of A equals a different combination of those columns. Write that as $A\boldsymbol{x} = A\boldsymbol{y}$. Find two combinations of the columns of A that equal the zero vector (in matrix language, find two solutions to $A\boldsymbol{z} = \mathbf{0}$).

3 (Practice with subscripts) The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are in m-dimensional space $\mathbf{R}^m$, and a combination $c_1\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n$ is the zero vector. That statement is at the vector level.

(1) Write that statement at the matrix level. Use the matrix A with the $\boldsymbol{a}$'s in its columns and use the column vector $\boldsymbol{c} = (c_1, \ldots, c_n)$.

(2) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector $\boldsymbol{a}_j$ has components $a_{1j}, a_{2j}, \ldots, a_{mj}$.

4 Suppose A is the 3 by 3 matrix **ones**$(3,3)$ of all ones. Find two independent vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ that solve $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$. Write that first equation $A\boldsymbol{x} = \mathbf{0}$ (with numbers) as a combination of the columns of A. Why don't I ask for a third independent vector with $A\boldsymbol{z} = \mathbf{0}$?

5 The linear combinations of $\boldsymbol{v} = (1,1,0)$ and $\boldsymbol{w} = (0,1,1)$ fill a plane in $\mathbf{R}^3$.

(a) Find a vector $\boldsymbol{z}$ that is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$. Then $\boldsymbol{z}$ is perpendicular to every vector $c\boldsymbol{v} + d\boldsymbol{w}$ on the plane: $(c\boldsymbol{v} + d\boldsymbol{w})^{\mathrm{T}} z = c\boldsymbol{v}^{\mathrm{T}}\boldsymbol{z} + d\boldsymbol{w}^{\mathrm{T}} z = 0 + 0$.

(b) Find a vector $\boldsymbol{u}$ that is not on the plane. Check that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{z} \neq 0$.

6 If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all three of the possible fourth corners? Draw two of them.

7 Describe the column space of $A = [\boldsymbol{v} \;\; \boldsymbol{w} \;\; \boldsymbol{v} + 2\boldsymbol{w}]$. Describe the nullspace of A: all vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ that solve $A\boldsymbol{x} = \mathbf{0}$. Add the "dimensions" of that plane (the column space of A) and that line (the nullspace of A):

dimension of column space + dimension of nullspace = number of columns

8 $\boldsymbol{A = CR}$ is a representation of the columns of A in the basis formed by the columns of C with coefficients in R. If $A_{ij} = j^2$ is 3 by 3, write down A and C and R.

9 Suppose the column space of an m by n matrix is all of $\mathbf{R}^3$. What can you say about m ? What can you say about n ? What can you say about the rank r ?

10 Find the matrices C_1 and C_2 containing independent columns of A_1 and A_2 :

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

11 Factor each of those matrices into $A = CR$. The matrix R will contain the numbers that multiply columns of C to recover columns of A.

This is one way to look at matrix multiplication: $\boldsymbol{C}$ **times each column of** $\boldsymbol{R}$**.**

12 Produce a basis for the column spaces of A_1 and A_2. What are the *dimensions* of those column spaces—the number of independent vectors? What are the *ranks* of A_1 and A_2 ? How many independent rows in A_1 and A_2 ?

13 Create a 4 by 4 matrix A of rank 2. What shapes are C and R ?

14 Suppose two matrices A and B have the same column space.

(a) Show that their row spaces can be different.

(b) Show that the matrices C (basic columns) can be different.

(c) What number will be the same for A and B ?

15 If $A = CR$, the first row of A is a combination of the rows of R. Which part of which matrix holds the coefficients in that combination—the numbers that multiply the rows of R to produce row 1 of A ?

16 The rows of R are a basis for the row space of A. *What does that sentence mean* ?

17 For these matrices with square blocks, find $A = CR$. What ranks ?

$$A_1 = \begin{bmatrix} \text{zeros} & \text{ones} \\ \text{ones} & \text{ones} \end{bmatrix}_{\mathbf{4 \times 4}} \qquad A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{\mathbf{8 \times 4}} \qquad A_3 = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix}_{\mathbf{8 \times 8}}$$

18 If $A = CR$, what are the CR factors of the matrix $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$?

19 "Elimination" subtracts a number ℓ_{ij} times row j from row i : a "row operation." Show how those steps can reduce the matrix A in Example 4 to R (except that this row echelon form R has a row of zeros). The rank won't change !

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \quad \rightarrow \quad \rightarrow \quad R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \textbf{rref}\,(A).$$

This page is about the factorization $A = CR$ and its close relative $A = \boldsymbol{CMR}$. $\boldsymbol{C}$ has the same r independent columns taken from A. *The new matrix* $\boldsymbol{R}$ *has* r *independent rows, also taken directly from* A. The r by r "**mixing matrix**" is $\boldsymbol{M}$. This invertible matrix makes $A = \boldsymbol{CMR}$ a true equation. Here is an example:

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & 3 \\ 2 & 4 & 6 \\ \mathbf{3} & \mathbf{5} & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \end{bmatrix} = \boldsymbol{CMR}$$

How did we find that mixing matrix $\boldsymbol{M}$? We realized that the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is in both $\boldsymbol{C}$ and $\boldsymbol{R}$. It is the overlap of the independent columns $1, 2$ and independent rows $1, 3$. Then the correct mixing matrix $\boldsymbol{M}$ is the *inverse* of this 2 by 2 overlap matrix $\boldsymbol{M^{-1}}$:

$$\boldsymbol{MM^{-1}} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here are extra problems to give practice with all these rectangular matrices of rank r.

20 Find $A = CR$ (R contains I) and also $A = \boldsymbol{C\,M\,R}$ for these matrices.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} (\boldsymbol{M} \text{ is } 1 \text{ by } 1) \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 4 \\ 3 & 6 & 5 \end{bmatrix} (\boldsymbol{M} \text{ is } 2 \text{ by } 2)$$

21 To find a general formula for M, multiply $A = CMR$ by C^{T} on the left and R^{T} on the right. Then multiply by $(C^{\mathrm{T}}C)^{-1}$ on the left and $(RR^{\mathrm{T}})^{-1}$ on the right. *This leaves the formula for* M *that was in earlier printings of this book.*

Inverse of a 2 by 2 matrix
No inverse if $\boldsymbol{ad = bc}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (**)$$

22 Show that this formula $(**)$ breaks down if $\begin{bmatrix} b \\ d \end{bmatrix} = m \begin{bmatrix} a \\ c \end{bmatrix}$: *dependent columns*.

The reason for this page is that the factorizations $A = CR$ and $A = \boldsymbol{CMR}$ have jumped forward in importance for large matrices. When $\boldsymbol{C}$ takes columns directly from A, and $\boldsymbol{R}$ takes rows directly from A, those matrices preserve properties that are lost in the more famous QR and SVD factorizations. Where $A = QR$ and $A = U\Sigma V^{\mathrm{T}}$ involve orthogonalizing the vectors, $\boldsymbol{C}$ and $\boldsymbol{R}$ keep the original data:

If A is nonnegative, so are $\boldsymbol{C}$ and $\boldsymbol{R}$. If A is sparse, so are $\boldsymbol{C}$ and $\boldsymbol{R}$.