Counting Parameters in the Basic Factorizations

$$A = LU \quad A = QR \quad S = Q\Lambda Q^{\mathrm{T}} \quad A = X\Lambda X^{-1} \quad A = QS \quad A = U\Sigma V^{\mathrm{T}}$$

This is a review of key ideas in linear algebra. The ideas are expressed by those factorizations and our plan is simple: *Count the parameters in each matrix*. We hope to see that in each equation like $A = LU$, the two sides have the same number of parameters.

For $A = LU$, both sides have n^2 parameters.

$\boldsymbol{L}$: **Triangular** $n \times n$ matrix with 1's on the diagonal	$\frac{1}{2}\, n(n-1)$
$\boldsymbol{U}$: **Triangular** $n \times n$ matrix with free diagonal	$\frac{1}{2}\, n(n+1)$
$\boldsymbol{Q}$: **Orthogonal** $n \times n$ matrix	$\frac{1}{2}\, n(n-1)$
$\boldsymbol{S}$: **Symmetric** $n \times n$ matrix	$\frac{1}{2}\, n(n+1)$
$\boldsymbol{\Lambda}$: **Diagonal** $n \times n$ matrix	n
$\boldsymbol{X}$: $n \times n$ matrix of independent **eigenvectors**	$n^2 - n$

Comments are needed for Q. Its first column $\boldsymbol{q}_1$ is a point on the unit sphere in $\mathbf{R}^n$. That sphere is an $\boldsymbol{n-1}$-dimensional surface, just as the unit circle $x^2 + y^2 = 1$ in $\mathbf{R}^2$ has only one parameter (the angle θ). The requirement $||\boldsymbol{q}_1|| = 1$ has used up one of the n parameters in $\boldsymbol{q}_1$. Then $\boldsymbol{q}_2$ has $n-2$ parameters—it is a unit vector and it is orthogonal to $\boldsymbol{q}_1$. The sum $(n-1) + (n-2) + \cdots + 1$ equals $\frac{1}{2}\, \boldsymbol{n(n-1)}$ free parameters in Q.

The eigenvector matrix X has only $n^2 - n$ parameters, not n^2. If $\boldsymbol{x}$ is an eigenvector then so is $c\boldsymbol{x}$ for any $c \neq 0$. We could require the largest component of every $\boldsymbol{x}$ to be 1. This leaves $n-1$ parameters for each eigenvector (and no free parameters for X^{-1}).

The count for the two sides now agrees in all of the first five factorizations.

For the SVD, use the reduced form $\boldsymbol{A_{m\times n} = U_{m\times r}\Sigma_{r\times r}V^{\mathrm{T}}_{r\times n}}$ (known zeros are not free parameters!) Suppose that $m \leq n$ and A is a full rank matrix with $\boldsymbol{r = m}$. The parameter count for A is $\boldsymbol{mn}$. *So is the total count for* U, Σ, *and* V. The reasoning for orthonormal columns in U and V is the same as for orthonormal columns in Q.

$$U \text{ has } \frac{1}{2}\, m(m-1) \quad \Sigma \text{ has } m \quad V \text{ has } (n-1) + \cdots + (n-m) = mn - \frac{1}{2}\, m(m+1)$$

Finally, suppose that A is an m *by* n *matrix of rank* r. **How many free parameters in a rank $\boldsymbol{r}$ matrix?** We can count again for $U_{m\times r}\Sigma_{r\times r}V^{\mathrm{T}}_{r\times n}$:

$$U \text{ has } (m-1) + \cdots + (m-r) = mr - \frac{1}{2}\, r(r+1) \quad V \text{ has } nr - \frac{1}{2}\, r(r+1) \quad \Sigma \text{ has } r$$

The total parameter count for rank r is $\boldsymbol{(m+n-r)\, r}$.

We reach the same total for $A = CR$ in Section I.1. The r columns of C were taken directly from A. The row matrix R includes an r by r identity matrix (not free!). Then the count for CR agrees with the previous count for $U\Sigma V^{\mathrm{T}}$, when the rank is r:

$$C \text{ has } mr \text{ parameters} \qquad R \text{ has } nr - r^2 \text{ parameters} \qquad \text{Total } (m+n-r)\, r.$$