LINEAR ALGEBRA

FOR EVERYONE

MANUAL FOR INSTRUCTORS

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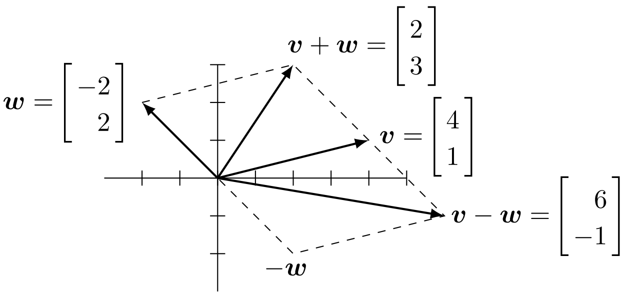
Problem Set 1.1, page 8

1 $c = ma$ and $d = mb$ lead to $ad = amb = bc$. With no zeros, $ad = bc$ is the equation for a 2×2 matrix to have rank 1.

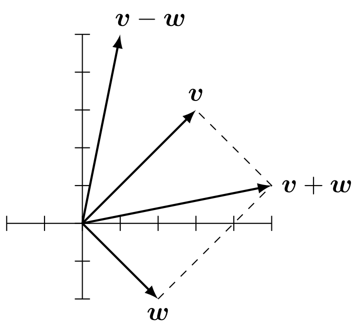
2 The three edges going around the triangle are $\boldsymbol{u} = (5, 0), \boldsymbol{v} = (-5, 12), \boldsymbol{w} = (0, -12)$. Their sum is $u + v + w = (0, 0)$. Their lengths are $||\boldsymbol{u}|| = 5, ||\boldsymbol{v}|| = 13, ||\boldsymbol{w}|| = 12$. This is a $5 - 12 - 13$ right triangle with $5^2 + 12^2 = 25 + 144 = 169 = 13^2$—the best numbers after the $3 - 4 - 5$ right triangle.

3 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $\boldsymbol{v} + \boldsymbol{w} = (2, 3)$ and $\boldsymbol{v} - \boldsymbol{w} = (6, -1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0, 0)$.



5 This problem gives the diagonals $\boldsymbol{v} + \boldsymbol{w} = (5, 1)$ and $\boldsymbol{v} - \boldsymbol{w} = (1, 5)$ of the parallelogram and asks for the sides $\boldsymbol{v}$ and $\boldsymbol{w}$: The opposite of Problem 4. In this example $\boldsymbol{v} = (3, 3)$ and $\boldsymbol{w} = (2, -2)$. Those come from $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$.

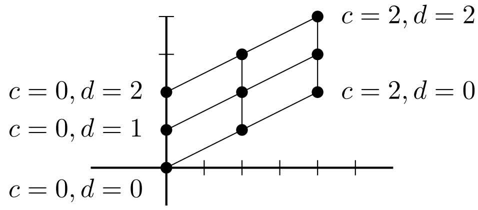


6 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

7 $\boldsymbol{u}+\boldsymbol{v} = (-2, 3, 1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = (0, 0, 0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w} =$ (add first answers) $= (-2, 3, 1)$. The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane because a combination $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ gives $(0, 0, 0)$. Stated another way : $\boldsymbol{u} = -\boldsymbol{v} - \boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

8 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v} = (1, -2, 1)$ and of $\boldsymbol{w} = (0, 1, -1)$ add to zero. $c = 3$ and $d = 9$ give $3\boldsymbol{v} + 9\boldsymbol{w} = (3, 3, -6)$. There is no solution to $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, 6)$ because $3 + 3 + 6$ is not zero.

9 The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = 0, 1, 2$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.



10 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Three possible parallelograms!

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 The combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$ fill the xy plane in xyz space.

13 (a) Sum = zero vector. (b) Sum = −2:00 vector = 8:00 vector.

(c) 2:00 is 30° from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from **0** to $12\boldsymbol{j} = (0, 12)$.

15 First part : $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in the same direction.

Second part : Some combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ gives the zero vector but those 3 vectors are not on a line.

16 The two equations are $c + 3d = 14$ and $2c + d = 8$. The solution is $c = 2$ and $d = 4$.

17 The point $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

18 The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $0 \le c \le 1$ and $0 \le d \le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$ then $c\boldsymbol{v} + d\boldsymbol{w}$ fills the unit square. In a special case like $\boldsymbol{v} = (a, 0)$ and $\boldsymbol{w} = (b, 0)$ these combinations only fill a segment of a line.

With $c \ge 0$ and $d \ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, then the cone is the whole first quadrant $x \ge 0$, $y \ge 0$. *Question*: What if $\boldsymbol{w} = -\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (-1, 0)$ only fill a line.

19 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies halfway between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \ge 0$, $d \ge 0$, $e \ge 0$, and $c + d + e = \mathbf{1}$.

20 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

21 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

22 *All* vectors in 3D are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding all the vectors $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$. Different answer when $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane.

23 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges.

24 Fact : For any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, some combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ is the zero vector (beyond the obvious $c = d = e = 0$). So if there is one combination $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w}$ that produces $\boldsymbol{b}$, there will be many more—just add c, d, e or $2c, 2d, 2e$ to the particular solution C, D, E.

The example has $3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w} = 3(1,3) - 2(2,7) + 1(1,5) = (0,0)$. It also has $-2\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{b} = (0,1)$. Adding gives $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w} = (0,1)$. In this case c, d, e equal $3, -2, 1$ and $C, D, E = -2, 1, 0$.

Could another example have $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that could NOT combine to produce $\boldsymbol{b}$? Yes. The vectors $(1,1), (2,2), (3,3)$ are on a line and no combination produces $\boldsymbol{b}$. We can easily solve $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = 0$ but not $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w} = \boldsymbol{b}$.

25 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0), (0,1,0,0), (0,0,1,0)$, and $(0,0,0,1)$.

26 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} \mathbf{2c} & \mathbf{-d} & & \mathbf{= 1} \\ \mathbf{-c} & \mathbf{+2d} & \mathbf{-e} & \mathbf{= 0} \\ & \mathbf{-d} & \mathbf{+2e} & \mathbf{= 0} \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 16

1 $\boldsymbol{u} \cdot \boldsymbol{v} = -2.4 + 2.4 = \mathbf{0}$, $\boldsymbol{u} \cdot \boldsymbol{w} = -.6 + 1.6 = \mathbf{1}$, $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} = \mathbf{0} + \mathbf{1}, \boldsymbol{w} \cdot \boldsymbol{v} = 4 + 6 = \mathbf{10} = \boldsymbol{v} \cdot \boldsymbol{w}$.

2 The lengths are $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = \sqrt{5}$. Then $|\boldsymbol{u} \cdot \boldsymbol{v}| = 0 < (1)(5)$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{4}{5}, \frac{3}{5}) = (0.8, 0.6)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (1/\sqrt{5}, 2/\sqrt{5})$. The vectors $\boldsymbol{w}, (2, -1)$, and $-\boldsymbol{w}$ make $0\,°, 90\,°, 180\,°$ angles with $\boldsymbol{w}$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = 10/5\sqrt{5} = 2/\sqrt{5}$.

4 For unit vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$: (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = \mathbf{-1}$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = \mathbf{0}$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = \mathbf{-3}$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (1, 3)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2, 1, 2)/3$. $\boldsymbol{U}_1 = (3, -1)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-3, 1)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1, -2, 0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v} = (2, -1)$. They lie on a line. All vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*. All vectors perpendicular to both $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line* in 3-dimensional space.

7 (a) $\cos\theta = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60°$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90°$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60°$ or $\pi/3$ (d) $\cos\theta = -5/\sqrt{10}\,\sqrt{5} = -1/\sqrt{2}$ so $\theta = 135°$ or $3\pi/4$ radians.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True : $\boldsymbol{u} \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + 2\boldsymbol{u} \cdot \boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v})$ splits into $\boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u} = 0$.

9 If $v_2 w_2/v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = \boldsymbol{v} \cdot \boldsymbol{w} = 0$: perpendicular ! The vectors $(1, 4)$ and $(1, -\frac{1}{4})$ are perpendicular because $1 - 1 = 0$.

10 Slopes $2/1$ and $-1/2$ multiply to give -1. Then $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the two vectors (the arrow directions) are perpendicular.

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90^\circ$; these $\boldsymbol{w}$'s fill half of 3-dimensional space. Draw a picture to show $\boldsymbol{v}$ and the $\boldsymbol{w}$'s.

12 $(1,1)$ is perpendicular to $(1,5) - c(1,1)$ if $(1,1) \cdot (1,5) - c(1,1) \cdot (1,1) = 6 - 2c = 0$ (then $\boldsymbol{c = 3}$). $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w}/\boldsymbol{v} \cdot \boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to constructing a perpendicular vector $\boldsymbol{w} - c\boldsymbol{v}$.

13 One possibility among many: $\boldsymbol{u} = (1,-1,0,0), \boldsymbol{v} = (0,0,1,-1), \boldsymbol{w} = (1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in their 3D hyperplane and they will stay perpendicular."

14 $\frac{1}{2}(x+y) = (2+8)/2 = \mathbf{5}$ and $\mathbf{5} > \mathbf{4}$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = \mathbf{8/10}$.

15 $\|\boldsymbol{v}\|^2 = 1 + 1 + \cdots + 1 = 9$ so $\|\boldsymbol{v}\| = \mathbf{3}$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3}, \ldots, \frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

16 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ the cosines with the 3 axes are $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

17 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20 + 5$ for the length of the hypotenuse $\boldsymbol{v} + \boldsymbol{w} = (3,4)$.

18 $||\boldsymbol{v} + \boldsymbol{w}||^2 = (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{w} \cdot (\boldsymbol{v} + \boldsymbol{w})$. This expands to $\boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} = ||\boldsymbol{v}||^2 + 2||\boldsymbol{v}||\,||\boldsymbol{w}|| \cos\theta + ||\boldsymbol{w}||^2$.

19 We know that $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta$ for $\boldsymbol{v} \cdot \boldsymbol{w}$. Here θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When $\theta < 90^\circ$ this $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive, so in this case $\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$.

Pythagoras changes from equality $a^2 + b^2 = c^2$ to *inequality* when $\theta < 90^\circ$ or $\theta > 90^\circ$.

20 $2\boldsymbol{v} \cdot \boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

21 $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \le v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \ge 0$.

22 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True : $.96 < 1$.

23 The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than $1 : x^2/(x^2 + y^2) \le 1$.

24 These two lines add to $2||\boldsymbol{v}||^2 + 2||\boldsymbol{w}||^2$:

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$$

$$||\boldsymbol{v} - \boldsymbol{w}||^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$$

25 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

26 Three vectors in the plane could make angles greater than $90°$ with each other: for example $(1, 0), (-1, 4), (-1, -4)$. Four vectors could *not* do this ($360°$ total angle). How many can can be perpendicular to each other in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n+1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

27 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} \text{The columns have} \\ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1. \\ \text{Their dot products} \\ \text{are all zero.} \end{matrix} .$$

28 The commands $V = \textbf{randn}\ (3, 30); D = \textbf{sqrt}\ (\textbf{diag}\ (V' * V));\ U = V \backslash D$; will give 30 random unit vectors in the columns of U. Then $\boldsymbol{u}' * U$ is a row matrix of 30 dot products whose average absolute value should be close to $2/\pi$.

29 The four vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ must add to zero. Then the four corners of the quadrilateral could be 0 and $\boldsymbol{v}_1$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$. We are allowing the side vectors $\boldsymbol{v}$ to cross each other—can you answer if that is not allowed ?

Problem Set 1.3, page 26

1 The column space $\mathbf{C}(A_1)$ is a plane in $\mathbf{R}^3$: the two columns of A_1 are independent

The column space $\mathbf{C}(A_2)$ is all of $\mathbf{R}^3$

The column space $\mathbf{C}(A_3)$ is a line in $\mathbf{R}^3$

2 The combination $A\boldsymbol{x} =$ column $1 - 2$ (column 2) $+$ column 3 is zero for both matrices. This leaves 2 independent columns. So $\mathbf{C}(A)$ is a (2-dimensional) plane in $\mathbf{R}^3$.

3 B has 2 independent columns so its column space is a plane. The matrix C has the same 2 independent columns and the same column space as B.

4 $A\boldsymbol{x} = \begin{bmatrix} 14 \\ 28 \\ 2 \end{bmatrix}$ Typical dot product is $2(1) + 1(2) + 2(5) = 14$ $\quad B\boldsymbol{y} = \begin{bmatrix} 4 \\ 8 \\ 18 \end{bmatrix} \quad I\boldsymbol{z} = \boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

5 $$A\boldsymbol{x} = 1\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \\ 2 \end{bmatrix}$$

$$B\boldsymbol{y} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 18 \end{bmatrix}$$

$$I\boldsymbol{z} = z_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

6 A has **2** independent columns, B has **3**, and $A + B$ has **3**. These are the ranks of A and B and $A + B$. The rule is that rank$(A + B) \leq$ rank$(A) +$ rank(B).

7 (a) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad A + B = \begin{bmatrix} 4 & 4 \\ 6 & 6 \end{bmatrix} =$ rank 1

(b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix} \quad A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$ rank 0

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad A + B = I =$ rank 4

8 The column space of A is all of $\mathbf{R}^3$. The column space of B is a **line** in $\mathbf{R}^3$. The column space of C is a 2-dimensional plane in $\mathbf{R}^3$. If C had an additional row of zeros, its column space would be a 2-dimensional plane **in** $\mathbf{R}^4$.

9 $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ **Seven ones** is the maximum for rank 3. With eight ones, two columns will be equal

10 $A = \begin{bmatrix} 3 & \mathbf{9} \\ 5 & 15 \end{bmatrix}$ has rank $1 : 1$ independent column, 1 independent row

$B = \begin{bmatrix} 1 & 2 & -5 \\ 4 & 8 & \mathbf{-20} \end{bmatrix}$ has 1 independent column in $\mathbf{R}^2$, 1 independent row in $\mathbf{R}^3$

11 (a) If B has an extra zero column, A and B have the **same** column space. Different row spaces because of different row lengths !

(b) If column $3 =$ column $2 -$ column 1, A and B have the same column spaces.

(c) If the new column 3 in B is $(1, 1, 1)$, then the column space is not changed or changed depending whether $(1, 1, 1)$ was already in $\mathbf{C}(A)$.

12 If $\boldsymbol{b}$ is in the column space of A, then $\boldsymbol{b}$ is a combination of the columns of A and *the numbers in that combination* give a solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$. The examples are solved by $(x_1, x_2) = (1, 1)$ and $(1, -1)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

13 $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -2 \end{bmatrix}$ $\quad A + B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ -1 & -3 \end{bmatrix}$ has the same column space as A and B (other examples could have a smaller column space : for example if $B = -A$ in which case $A + B =$ zero matrix).

14 $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & \mathbf{10} \end{bmatrix}$ has column $3 = 2$ (column 1) $+ 3$ (column 2)

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \mathbf{9} \end{bmatrix}$ has column $3 = -1$ (column 1) $+ 2$ (column 2)

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & \boldsymbol{q} \end{bmatrix}$ has 2 independent columns if $\boldsymbol{q} \neq \mathbf{0}$

15 If $A\boldsymbol{x} = \boldsymbol{b}$ then the extra column $\boldsymbol{b}$ in $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ is a combination of the first columns, so the column space and the rank are not changed by including the $\boldsymbol{b}$ column.

16 (a) *False* : B could be $-A$, then $A + B$ has rank zero.

(b) *True* : If the n columns of A are independent, they could not be in a space $\mathbf{R}^m$ with $m < n$. Therefore $m \geq n$.

(c) *True* : If the entries are random and the matrix has $m = n$ (or $m \geq n$), then the columns are almost surely independent.

17 **rank 2** : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **rank 1** : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

rank 0 : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

18 $3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix} = S\boldsymbol{x} = \boldsymbol{b}$

$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and the 3 dot products in $S\boldsymbol{x}$ are $3, 7, 12$

19 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ leads to $y_1 = 1, y_2 = 0, y_3 = 0$ since $\boldsymbol{b} =$ column 1.

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$ leads to $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ first 3 odd numbers.

The sum of the first 3 odd numbers is $3^2 = 9$. The sum of the first 10 is $10^2 = 100$.

20 $S\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is solved by $\boldsymbol{y} = \begin{bmatrix} c_1 \\ c_2 - c_1 \\ c_3 - c_2 \end{bmatrix}$. This is

$\boldsymbol{y} = S^{-1}\boldsymbol{c} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. S is square with independent columns. So S has an inverse with $SS^{-1} = S^{-1}S = I$.

21 To solve $A\boldsymbol{x} = \mathbf{0}$ we can simplify the 3 equations (this is the subject of Chapter 2).

Start from $A\boldsymbol{x} = \mathbf{0}$ $\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_1 + 5x_2 + 6x_3 &= 0 \\ 4x_1 + 7x_2 + 9x_3 &= 0 \end{aligned}$ $\quad$ Row 2 $-$ 3(row 1), row 3 $-$ 4(row 1) $\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_2 - 3x_3 &= 0 \\ -x_2 - 3x_3 &= 0 \end{aligned}$

If $x_3 = 1$ then $x_2 = -3$ and $x_1 = 3$. Any answer $\boldsymbol{x} = (3c, -3c, c)$ is correct.

22 $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c = \mathbf{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & c = \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{2} & 1 \\ 4 & \mathbf{2} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ have dependent columns

23 The equation $A\boldsymbol{x} = \mathbf{0}$ says that $\boldsymbol{x}$ is perpendicular to each row of A (three dot products are zero). So $\boldsymbol{x}$ is perpendicular to all combinations of those rows. In other words, $\boldsymbol{x}$ is perpendicular to the row space (here a plane).

An important fact for linear algebra: Every $\boldsymbol{x}$ in the nullspace of A (meaning $A\boldsymbol{x} = \mathbf{0}$) is perpendicular to every vector in the row space.

Problem Set 1.4, page 35

1 Here are the 4 ways to multiply AB and the operation counts. A is m by n, B is n by p.

Row i times column k	mp dot products, n multiplications each
Matrix A times column k	p columns, mn multiplications each
Row i times matrix B	m rows, np multiplications each
Column j of A times row j of B	n (columns) (rows), mp multiplications each

2 $A = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \end{bmatrix}$ factors into $CR = \begin{bmatrix} \boldsymbol{a} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

3 $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

4 (a) $$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$$

5 A has 7 columns and 4 rows. Those columns are vectors in 4-dimensional space. We cannot have 5 independent column vectors because we cannot have 5 independent vectors in 4-dimensional space. (This is really just a restatement of the problem. The proof comes in Section 3.2 : Every m by n matrix C, with $m < n$ has a nonzero solution to $C\boldsymbol{x} = \mathbf{0}$. Here $m = 4$ and $n = 5$ and 5 columns of C cannot be independent.)

6 $A = \begin{bmatrix} 2 & -2 & 1 & 6 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 3 & -3 & 0 & 6 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

7 $CR = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$ in Problem 6.

8 $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = AI$ $\quad A = C$ and $R = I$

$B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 4 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = CR$

9 A random 4 by 4 matrix has independent columns ($C = A$ and $R = I$) with probability 1. (We could be choosing the 16 entries of A between 0 and 1 with uniform probability by $A = \textbf{rand}(4, 4)$. We could be choosing those 16 entries of A from a "bell-shaped" normal distribution by $A = \textbf{rand}(4, 4)$. If we were choosing those 16 entries from a finite list of numbers, then there is a nonzero probability that the columns of A are *dependent*. In fact a nonzero probability that all 16 numbers are the same.)

10 If A is a random 4 by 5 matrix, then (using **rand** or **randn** as above) with probability 1 the first 4 columns are independent and go into C. With probability zero (this does not mean it can't happen !) the first 4 columns will be dependent and C will be different (C will have r columns with $r \leq 4$).

11 $A = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} = CR$. Many other possibilities !

12 $A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$A_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 1.5 \end{bmatrix}$ $\qquad A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

13 $C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 4 \end{bmatrix}$ have $CR = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$ and $RC = \begin{bmatrix} 14 \end{bmatrix}$

and $CRC = \begin{bmatrix} 14 \\ 42 \end{bmatrix}$ and $RCR = \begin{bmatrix} 28 & 56 \end{bmatrix}$.

Here is an interesting fact when A is m by n and B is n by m. The m numbers on the main diagonal of AB have the same total as the n numbers on the main diagonal of BA. Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \quad AB = \begin{bmatrix} 8 & 26 \\ 17 & 62 \end{bmatrix} \quad BA = \begin{bmatrix} 12 & 15 & 18 \\ 17 & 22 & 27 \\ 22 & 29 & 36 \end{bmatrix}$$

$8 + 62 = 12 + 22 + 36$

14 $\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$ $\quad \begin{bmatrix} 6 & -7 \\ 7 & 6 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 0 \\ 3 & 6 \end{bmatrix}$ $\quad \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

rank one — orthogonal columns — rank 2 — $A^2 = I$

15

1. Column j of A equals the matrix C times column j of R.

   This is a combination of the **columns** of C.

2. Row i of A is row i of C times the matrix R.

   This is a combination of the **rows** of R.

3. (row i of C) $\cdot$ (column j of R) gives A_{ij}

   That dot product requires the number of columns of C to equal the number of rows of R.

4. C has r columns so R has r rows (to multiply CR). Those columns of C are independent (by construction). Those rows of R are independent (because R contains the r by r identity matrix).

16 (a) The vector $AB\boldsymbol{x}$ is the matrix A times the vector $B\boldsymbol{x}$. So it is a combination of the columns of A. Therefore $\mathbf{C}(AB) \subseteq \mathbf{C}(A)$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ give $AB =$ zero matrix and $\mathbf{C}(AB) =$ zero vectors.

17 (a) If A and B have rank 1, then AB has rank 1 or 0. $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ and $B = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ give $AB = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x})\boldsymbol{y}^{\mathrm{T}}$ so $AB =$ zero matrix if the dot product $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x}$ happens to be zero.

(b) If A and B are 3 by 3 matrices of rank 3, then it is **true** that AB has rank 3. *One approach*: If $AB\boldsymbol{x} = \mathbf{0}$ then $B\boldsymbol{x} = \mathbf{0}$ because A has 3 independent columns. But $B\boldsymbol{x} = \mathbf{0}$ only when $\boldsymbol{x} = \mathbf{0}$, because B has 3 independent columns.

(c) Suppose $AB = BA$ for all 2 by 2 matrices B. Choose $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $AB = \begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ e & f \end{bmatrix}$. This tells us that $\begin{bmatrix} c & 0 \\ e & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ and therefore $d = e = 0$. Now choose $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that $AB = \begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$. This tells us that $\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ and $c = f$ and $A = cI$.

18 (a) $AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $BC = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

(b) $(AB)C =$ column exchange of $AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$A(BC) =$ row exchange of $BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} =$ same result ABC.

19 $AB = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

20 $AB = (4 \times 3)\,(3 \times 2)$ needs $mnp = (4)\,(3)\,(2) = 24$ multiples.

Then $(AB)C = (4 \times 2)\,(2 \times 1)$ needs $(4)\,(2)\,(1) = 8$ more: TOTAL 32.

$BC = (3 \times 2)\,(2 \times 1)$ needs $mnp = (3)\,(2)\,(1) = 6$ multiplies.

Then $A(BC) = (4 \times 3)\,(3 \times 1)$ needs $(4)\,(3)\,(1) = 12$ more: TOTAL 18.

Best to start with $\boldsymbol{C}$ = vector. Multiply by B to get the vector BC, and then the vector $A(BC)$. Vectors need less computing time than matrices !

Problem Set 2.1, page 46

1 Multiply equation 1 by $\ell_{21} = \frac{10}{2} = \mathbf{5}$ and subtract from equation 2 to find $2x + 3y = 1$ (unchanged) and $-6y = 6$. The pivots to circle are 2 and -6. Back substitution in $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$.

2 The row picture and column picture and coefficient matrix are changed. The solution has not changed.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $\boldsymbol{3y = 3}$. Then $y = 1$ and $x = 5$. If the right sides change sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1 from equation 2. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$. Notice the "determinant of A" $= ad - bc$. It must be nonzero for this division.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. The two lines in the row picture are the same line, containing all solutions.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines $2x + 4y = 16$ and $4x + 8y = 32$ become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line in the row picture). The column picture has both columns along the same line.

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = 20 - 4 = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or row 3, that avoids a row operation.

$$\begin{aligned} 2x - 3y \quad\;\; &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

13 Subtract 2 times row 1 from row 2 to reach $(d - 10)y - z = 2$ along with $y - z = 3$. If $d = \mathbf{10}$ exchange rows 2 and 3. If $d = \mathbf{11}$ the system becomes singular.

14 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. But equation (3) is the same so there is a *line of solutions* $(x, y, z) = (1, 1, -1)$ when $b = -1$.

15 (a) **Example of 2 exchanges**

$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then breakdown**

$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(rows 1 and 3 are not consistent)

16 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3. The new row 3 has no pivot. If column 2 = column 1, then column 2 has no pivot.

17 *Example* $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions to $A\boldsymbol{x} = \mathbf{0}$ but almost surely no solution to $A\boldsymbol{x} = \boldsymbol{b}$ for a random $\boldsymbol{b}$.

18 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$ which allows infinitely many solutions. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

19 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = \mathbf{2}$ or $a = \mathbf{0}$. (You could notice that the determinant $a^2 - 2a$ is zero for $a = 2$ and $a = 0$.)

20 $a = 2$ gives equal columns, $a = 4$ gives equal rows, $a = 0$ gives a zero column.

21 Solvable for $s = 10$ (add the two pairs of equations to get $a + b + c + d$ on the left sides, 12 and $2 + s$ on the right sides). So 12 must agree with $2 + s$, which makes $s = 10$. The four equations for a, b, c, d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 8 \\ s \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

22 $A(2,:) = A(2,:) - 3 * A(1,:)$ subtracts 3 times all of row 1 from all of row 2.

23 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! *With row exchanges* in MATLAB's **lu** code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for **randn** with normal instead of uniform probability distribution for the numbers in A).

24 If $A(5, 5)$ is 7 not 11, then the last pivot will be 0 not 4.

25 Row j of U is a combination of rows $1, \ldots, j$ of A (when there are no row exchanges). If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U just keeps the diagonal of A when A is *lower triangular,* all entries below that diagonal go to zero.

26 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 **columns** is **the column of zeros**.

(b) A very singular matrix has all ones: $A =$ **ones** (100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(c) The row picture has 100 planes **meeting along a common line through** $\mathbf{0}$. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.2, page 53

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{7} & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \leftarrow E_{21}, E_{31}E_{32}$ $\quad E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-4} & 1 & 0 \\ \mathbf{10} & \mathbf{-2} & 1 \end{bmatrix}$.

Those E's are in the right order to give $EA = U$.

$$E^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{4} & 1 & 0 \\ \mathbf{-2} & \mathbf{2} & 1 \end{bmatrix}$$

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \overset{E_{21}}{\to} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \overset{E_{31}}{\to} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \overset{E_{32}}{\to} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b} = (1, 0, 0)$ has become $U\boldsymbol{x} = \boldsymbol{c} = (\mathbf{1}, \mathbf{-4}, \mathbf{10})$. Then back substitution gives $\boldsymbol{z} = \mathbf{-5}, y = \frac{\mathbf{1}}{\mathbf{2}}, x = \frac{\mathbf{1}}{\mathbf{2}}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mathbf{7} & 0 & 1 \end{bmatrix} \text{ is } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}. \text{ Multiplication confirms } EE^{-1} = I.$$

8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M^* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. $\det M^* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$! Subtracting row 1 from row 2 doesn't change $\det M$.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ for both parts (a) and (b).

After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 At the same time $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

12 For the first, a simple row exchange has $P^2 = I$ so $P^{-1} = P$. For the second, $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} = "transpose" of P, coming in Section 2.4.

13 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \dfrac{1}{10}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination.

14 An upper triangular U with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any a. And also $-U$.

15 (a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

16 (a) If $A\boldsymbol{x} = (0, 0, 1)$ then equation 1 + equation 2 − equation 3 is $0 = 1$

(b) Right sides must satisfy $b_1 + b_2 = b_3$

(c) In elimination, Row 3 becomes a row of zeros—no third pivot.

17 (a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

18 Yes, B is invertible (A was just multiplied by a permutation matrix P). If you exchange rows 1 and 2 of A to reach B, you exchange **columns** 1 and 2 of A^{-1} to reach B^{-1}. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this P.

19 (a) If $B = -A$ then A, B can be invertible but $A + B$ = zero matrix is not invertible.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

20 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $\boldsymbol{A^{-1} = BC^{-1}}$.

21 $M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by C and the right by A : $B^{-1} = CM^{-1}A$.

22 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract *column* 2 of A^{-1} from *column* 1.

23 If A has a column of zeros, so does BA. Then $BA = I$ is impossible. There is no A^{-1}.

24 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

25 $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E.$

Reverse the order and change -1 to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} =$ $L = E^{-1}$. The off-diagonal 1's are unchanged by multiplying inverses in this order.

26 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

27 $A * \mathsf{ones}(4,1) = \begin{bmatrix} 4 & 4 & 4 & 4 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} 4 & 4 & 4 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ so A cannot be invertible.

28 Six of the sixteen $0-1$ matrices are invertible : I and P and all four with three 1's.

29 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix};$

$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$

30 A can be invertible with diagonal zeros (example to find). B is singular because each row adds to zero. The all-ones vector $\boldsymbol{x} = (1,1,1,1)$ has $B\boldsymbol{x} = \mathbf{0}$.

31 $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{\mathbf{-1}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}; \quad B\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

so B^{-1} does not exist.

32 $\begin{bmatrix} U & I \end{bmatrix} = \begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & U^{-1} \end{bmatrix}.$

33 (a) True (If A has a row of zeros, then so does every AB, and $AB = I$ is impossible).

(b) False (the matrix of all ones is singular even with diagonal 1's).

(c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

34 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

The matrix C is not invertible if $c = 0$ or $c = 7$ or $c = 2$.

35 $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. When the triangular A alternates 1 and -1 on its diagonals, A^{-1} has 1's on the main diagonal and next diagonal.

36 $\boldsymbol{x} = (1, 1, \ldots, 1)$ has $\boldsymbol{x} = P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P - Q)\boldsymbol{x} = \mathbf{0}$. Permutations do not change this all-ones vector. Then $P - Q$ is not invertible.

37 The block inverses are $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

38 A is invertible when elimination (with row exchanges allowed) produces 3 nonzero pivots.

Problem Set 2.3, page 61

1 $\ell_{21} = \mathbf{1}$ multiplied row 1 and subtracted from row 2; **in reverse** $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ times

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \boldsymbol{c} \text{ is } A\boldsymbol{x} = \boldsymbol{b} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

In letters, **L multiplies $U\boldsymbol{x} = \boldsymbol{c}$ to give $A\boldsymbol{x} = \boldsymbol{b}$.**

2 $L\boldsymbol{c} = \boldsymbol{b}$ is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, solved by $\boldsymbol{c} = \begin{bmatrix} \mathbf{5} \\ \mathbf{2} \end{bmatrix}$ as elimination goes forward.

$U\boldsymbol{x} = \boldsymbol{c}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, solved by $\boldsymbol{x} = \begin{bmatrix} \mathbf{3} \\ \mathbf{2} \end{bmatrix}$ in back substitution.

3 $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & 1 & 0 \\ 0 & \mathbf{4} & 2 \\ 0 & 0 & \mathbf{5} \end{bmatrix} = U.$

With E^{-1} as L, $A = LU = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$

4 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U.$ Then $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U$ is the same as $E_{21}^{-1} E_{32}^{-1} U = LU$. The multipliers $\ell_{21} = \ell_{32} = 2$ fall into place in L.

5 $E_{32} E_{31} E_{21}\, A = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\mathbf{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -\mathbf{3} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -\mathbf{2} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$. This is

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$. Put those multipliers $2, 3, 2$ into L. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{3} & \mathbf{2} & 1 \end{bmatrix} U = LU.$

6 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 0 & \mathbf{1} \end{bmatrix} = LDU$; $\boldsymbol{U}$ **is** $\boldsymbol{L}^{\mathrm{T}}$

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & & \\ \mathbf{4} & \mathbf{1} & \\ 0 & \mathbf{-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{4} & 0 \\ 0 & \mathbf{1} & \mathbf{-1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} = LDL^{\mathrm{T}}.$$

7 $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$. Need $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$. All of the multipliers are $\ell_{ij} = 1$ for this A

8 **Correction**: Problem 8 has the same L as **Problem 7**.

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}. \text{ Need } \begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$$

9 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Eliminate to $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boldsymbol{c}$.

10 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Those are forward elimination and back substitution for $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

11 (a) L goes to I (b) I goes to L^{-1} (c) LU goes to U. **Elimination multiplies by** $\boldsymbol{L^{-1}}$.

12 (a) Multiply $LDU = L_1 D_1 U_1$ by inverses to get $L_1^{-1} LD = D_1 U_1 U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.

(b) L, U, L_1, U_1 have diagonal 1's so $D = D_1$. Then $L_1^{-1} L$ and $U_1 U^{-1}$ are both I.

13 $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU;\quad \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = L \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} U.$

A tridiagonal matrix A has **bidiagonal factors** L and U.

14 For the first matrix A, L keeps the 3 zeros at the start of rows. But U may not have the upper zero where $A_{24} = 0$. For the second matrix B, L keeps the bottom left zero at the start of row 4. U keeps the upper right zero at the start of column 4. **One zero in A and two zeros in B are filled in**.

15 The 2 by 2 upper submatrix A_2 has the first two pivots $5, 9$. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

16 $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 6 & 4 \end{bmatrix}$

17 $L^{\mathrm{T}}L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $LL^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Problem Set 2.4, page 71

1 $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} =$ $\begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$; $A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \frac{1}{c^2}\begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

2 $(AB)^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = B^{\mathrm{T}}A^{\mathrm{T}}$. This answer is different from $A^{\mathrm{T}}B^{\mathrm{T}} = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ (except when $AB = BA$). $AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $A^{\mathrm{T}}A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

3 (a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.

(b) If U is upper triangular, so is U^{-1}: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. But the diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with *themselves*. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

5 (a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$

(b) This answer 5 is the row $\boldsymbol{x}^{\mathrm{T}}A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ times $\boldsymbol{y}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) This is also the row $\boldsymbol{x}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ times $A\boldsymbol{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

6 $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ *and* $\boldsymbol{B^{\mathrm{T}}} = \boldsymbol{C}$ *and* $D^{\mathrm{T}} = D$.

7 (a) False: $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$.

(b) False: The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$.

(c) True: Invertible symmetric matrices have symmetric inverses ! Easiest proof is to transpose $AA^{-1} = I$. So unsymmetric A has unsymmetric A^{-1}.

(d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices A, B, and $C)$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

9 $P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ but $P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

If P_3 and P_4 exchange *different* pairs of rows, then $P_3P_4 = P_4P_3 =$ both exchanges.

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ and $(4,3,2,1)$ exchange 2 pairs. $(1,2,3,4)$ makes 12 evens.

11 The "reverse identity" P takes $(1,\ldots,n)$ into $(n,\ldots,1)$. When rows and also columns are reversed, the $1,1$ and n,n entries of A change places in PAP. So do the $1,n$ and $n,1$ entries. In general $(PAP)_{ij}$ is $(A)_{n-i+1,\,n-j+1}$.

12 $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ since $P^{\mathrm{T}}P = I$. In general $P\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot P^{\mathrm{T}}\boldsymbol{y} \neq \boldsymbol{x}\cdot P\boldsymbol{y}$:

Non-equality where $P \neq P^{\mathrm{T}}$: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

13 $PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is upper triangular. Multiplying A *on the right* by a permutation matrix P_2 exchanges the *columns* of A. To make this A lower triangular, we also need P_1 to exchange rows 2 and 3:

$$P_1AP_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

14 A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$. The permutation $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same P has $\widehat{P}^4 = \widehat{P} \neq I$.

15 (a) If P sends row 1 to row 4, then P^{T} sends row 4 to row 1 (b) $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

16 $A^2 - B^2$ and also ABA are symmetric if A and B are symmetric. But $(A+B)(A-B)$ and $ABAB$ are generally *not* symmetric. Transposes $(A-B)\,(A+B)$ and $BABA$.

17 (a) $5+4+3+2+1 = 15$ independent entries if $S = S^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices. (d) The diagonal of $A^{\mathrm{T}}A$ contains $||$row 1$||$, $||$row 2$||, \ldots \Rightarrow$ never negative.

18 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{3} \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \boldsymbol{b} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{b} \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\mathbf{\frac{1}{2}} & 1 & \\ \mathbf{0} & -\mathbf{\frac{2}{3}} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{\frac{1}{2}} & \mathbf{0} \\ & 1 & -\mathbf{\frac{2}{3}} \\ & & 1 \end{bmatrix} = \boldsymbol{LDL}^{\mathrm{T}}.$$

19 $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$;

$$\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$$

20 $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$ and $L = U = I$. Elimination on this $A = P$ exchanges rows 1-2 then rows 2-3 then rows 3-4.

21 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

22 $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^{\mathrm{T}}$ has $0, 1, 2, 3$ in every row. I don't know any rules for a symmetric construction like this "Hankel matrix" with constant antidiagonals.

23 Reordering the rows and/or the columns of $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ will move the entry **a**. So the result cannot be the transpose (which doesn't move **a**).

24 (a) Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

(b) Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$. Six terms.

25 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^3 = I$ so three rotations for $360°$; P rotates every $\boldsymbol{v}$ around the $(1, 1, 1)$ line by $120°$.

26 $L(U^{\mathrm{T}})^{-1}$ is lower triangular times lower triangular, so *lower triangular*. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{T}\,\mathrm{T}} = U^{\mathrm{T}}DU$ again, so $U^{\mathrm{T}}DU$ is *symmetric*. The factorization multiplies lower triangular by symmetric to get LDU which is A.

27 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

28 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same permutation*. And if $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix} \text{ is 5 by 5 with } P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{P^6 = I}.$$

29 To split the matrix M into (symmetric S) + (anti-symmetric A), the only choice is $S = \frac{1}{2}(M + M^{\mathrm{T}})$ and $A = \frac{1}{2}(M - M^{\mathrm{T}})$.

30 Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for Q is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Problem Set 3.1, page 79

Note An interesting "max-plus" vector space comes from the real numbers $\mathbf{R}$ combined with $-\infty$. Change addition to give $x + y = \mathbf{max}(\boldsymbol{x}, \boldsymbol{y})$ and change multiplication to $xy = \textbf{usual } \boldsymbol{x} + \boldsymbol{y}$. Which y is the zero vector that gives $x + \mathbf{0} = \max(x, \mathbf{0}) = x$ for every x?

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

6 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1 + c_2)\boldsymbol{f} = \boldsymbol{f}((c_1 + c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.

8 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions: $(1, 1) + (-1, 1) = (0, 2)$ is removed.

9 The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

10 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

11 For the plane $x+y-2z=4$, the sum of $(4,0,0)$ and $(0,4,0)$ is not on the plane. (The key is that this plane does not go through $(0,0,0)$.)

12 The parallel plane $\mathbf{P}_0$ has the equation $x+y-2z=0$. Pick two points, for example $(2,0,1)$ and $(0,2,1)$, and their sum $(2,2,2)$ is in $\mathbf{P}_0$.

13 The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

14 (a) The invertible matrices do not include the zero matrix, so they are not a subspace (b) The sum of singular matrices $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ is not singular: not a subspace.

15 (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}}=-A$ do form a subspace (c) *True*: Any set of vectors from a vector space will span a subspace of that space.

16 The column space of A is the x-axis = all vectors $(x,0,0)$: a *line*. The column space of B is the xy plane = all vectors $(x,y,0)$. The column space of C is the line of vectors $(x,2x,0)$.

17 (a) Elimination leads to $0=b_2-2b_1$ and $0=b_1+b_3$ in equations 2 and 3: Solution only if $b_2=2b_1$ and $b_3=-b_1$ (b) Elimination leads to $0=b_1+b_3$ in equation 3: Solution only if $b_3=-b_1$.

18 A combination of the columns of C is also a combination of the columns of A. Then $C=\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$ and $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ have the same column space. $B=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$ has a different column space. The key word is "space".

19 (a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3=0$ (c) Solvable only if $b_3=b_2$.

20 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[\,A\ \ \boldsymbol{b}\,]=\begin{bmatrix}1 & 0 & \mathbf{1}\\ 0 & 0 & \mathbf{1}\end{bmatrix}\ \begin{matrix}\text{(larger column space)}\\ \text{(no solution to } A\boldsymbol{x}=\boldsymbol{b})\end{matrix}\quad \begin{bmatrix}1 & 0 & \mathbf{1}\\ 0 & 1 & \mathbf{1}\end{bmatrix}\ \begin{matrix}(\boldsymbol{b}\text{ is in column space})\\ (A\boldsymbol{x}=\boldsymbol{b}\text{ has a solution})\end{matrix}$$

21 The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B=$ zero matrix and $A\neq 0$ is a case when $AB=$ zero matrix has a smaller column space (it is just the zero space $\mathbf{Z}$) than A.

22 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\mathbf{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

23 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

24 (a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\mathbf{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\mathbf{C}(A) = \mathbf{C}(2A)$. (d) *False*: $\mathbf{C}(A - I) \neq \mathbf{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

25 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbf{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\mathbf{C}(A) =$ line in $\mathbf{R}^3$.

26 When $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\mathbf{C}(A) = \mathbf{R}^9$.

27 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\mathbf{S} + \mathbf{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\mathbf{S} + \mathbf{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: $\mathbf{S} + \mathbf{T} =$ *subspace*.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are different lines, then $\mathbf{S} \cup \mathbf{T}$ is just the two lines (*not a subspace*) but $\mathbf{S} + \mathbf{T}$ is the whole plane that they span.

28 If $\mathbf{S} = \mathbf{C}(A)$ and $\mathbf{T} = \mathbf{C}(B)$ then $\mathbf{S} + \mathbf{T}$ is the column space of $M = [\,A \quad B\,]$.

29 The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\mathbf{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

30 $y - e^{-x}$ and $y = e^x$ are independent solutions to $d^2y/dx^2 = y$. Also $y = \cos x$ and $y = \sin x$ are independent solutions to $d^2y/dx^2 = -y$. The solution space contains all combinations $A\cos x + B\sin x$.

31 If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the vector space $\mathbf{V} \cap \mathbf{W}$, then they are in both $\mathbf{V}$ and $\mathbf{W}$. So all combinations $c\boldsymbol{x} + d\boldsymbol{y}$ are in both $\mathbf{V}$ and $\mathbf{W}$. So all combinations are in $\mathbf{V} \cap \mathbf{W}$.

Problem Set 3.2, page 91

1 If $A\boldsymbol{x} = \mathbf{0}$ then $R\boldsymbol{x} = EA\boldsymbol{x} = \mathbf{0}$. And if $R\boldsymbol{x} = EA\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = E^{-1}R\boldsymbol{x} = \mathbf{0}$.

2 (a) If $c = 4$ then A has rank 1 and column 1 is its pivot column and $(-2, 1, 0)$ and $(-1, 0, 1)$ are special solutions to $A\boldsymbol{x} = \mathbf{0}$. If $c \neq 4$ then A has rank 2 and columns 1 and 3 are pivot columns and $(-2, 1, 0)$ is a special solution. If $c = 0$ then $B =$ zero matrix with rank 0 and $(1, 0)$ and $(0, 1)$ are special solutions to $B\boldsymbol{x} = \mathbf{0}$. If $c \neq 0$ then B has rank 1 and column 1 is its pivot column and $(-1, 1)$ is the special solution to $B\boldsymbol{x} = \mathbf{0}$.

3 $R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$. All matrices $A = CR$ with $C = 2$ by 2 invertible matrix have the same nullspace as R.

4 (a) $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3 (b) $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ Free x_3 Pivot x_1, x_2

5 Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$.

6 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

7 This question asks for U and not R. If it asked for R, pivot columns would have a single 1.

$$\begin{bmatrix} 0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8 $$R = \begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

9 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $\boldsymbol{x} = (0,0,0,1,0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

10 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can always solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

12 If a matrix has n columns and r pivots, there are $\boldsymbol{n - r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r} = \boldsymbol{n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

13 Fill in **12** then **3** then **1** to get the complete solution in $\mathbf{R}^3$ to $x - 3y - z = 12$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \text{one particular solution} + \text{all nullspace solutions.}$$

14 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1,0,1,0,1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (this nullspace is a line in $\mathbf{R}^5$).

15 To produce special solutions $(2,2,1,0)$ and $(3,1,0,1)$ with free variables x_3, x_4:

$$R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \text{ and } A \text{ can be any invertible 2 by 2 matrix times this } R.$$

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ The rank is 3 is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

17 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $\mathbf{C}(A)$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbf{N}(A)$. *Which other* A's?

18 $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

19 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\mathbf{N}(A) = \mathbf{C}(A)$. Notice that $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not A^{T}.

20 If nullspace = column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible. Only possible when n is even.

21 If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\mathbf{C}(B)$ *equals* $\mathbf{N}(A)$. For $B = 0, \mathbf{C}(B)$ is smaller than $\mathbf{N}(A)$.

22 For A = random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with a fourth row of zeros. What is R for a random 3 by 4 matrix?

23 If $\mathbf{N}(A)$ = line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

24 $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$. Any zero rows come after those rows.

25 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's !

26 The nullspace of $B = [\,A \;\; A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

27 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B) =$ *intersection*.

28 A has $R_0 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. B and C have $R_0 = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

And $R = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

29 $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

30 A and A^{T} have the same rank $r =$ number of pivots. But the pivot column is column 2 for this matrix A and column 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

31 The new entries keep rank 1 : $A = \begin{bmatrix} a & b & c \\ d & \frac{bd}{a} & \frac{cd}{a} \\ g & \frac{bg}{a} & \frac{cg}{a} \end{bmatrix}$ if $a \neq 0$, $\quad B = \begin{bmatrix} \mathbf{3} & 9 & \mathbf{-4.5} \\ 1 & \mathbf{3} & \mathbf{-1.5} \\ 2 & 6 & -3 \end{bmatrix}$,

$M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$ if $a \neq 0$.

32 With rank 1, the second row of R does not exist !

33 Invertible r by r submatrices
Use pivot rows and columns $\quad S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

34 (a) A and B will both have the same nullspace and row space as the R they share.

(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

35 CORRECTED : $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + y_5 = -y_2 + y_3 + y_6 = -y_4 - y_5 - y_6 = 0$.

These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1, 0, 0, 1, -1, 0)$ and $(0, 0, -1, -1, 0, 1)$ and $(0, -1, 0, 0, 1, -1)$. Those are flows around the 3 small loops.

36 $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix}$ C^{T} has pivot columns $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. The invertible S inside C is $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

37 The column space of AB contains all vectors $(AB)\boldsymbol{x}$. Those vectors are the same as $A(B\boldsymbol{x})$ so they are also in the column space of A.

38 By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns of B, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \leq$ rank (B). No new pivot columns !

39 We are given $AB = I$ which has rank n. Then rank$(AB) \leq$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

40 Certainly A and B have at most $\operatorname{rank} 2$. Then their product AB has at most $\operatorname{rank} 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$. *Example* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

41 $A = \begin{bmatrix} I & I \end{bmatrix}$ has $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$ has the same N; $C = \begin{bmatrix} I & I & I \end{bmatrix}$ has

$N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$.

42 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

43 $R_0 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; (b) $B = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (c) $C = \begin{bmatrix} I & 0 \end{bmatrix}$

$\mathbf{rref}(R_0^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\mathbf{rref}(R_0^{\mathrm{T}} R_0) = \text{same } R_0$

44 $R_0 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has $R_0^{\mathrm{T}} R_0 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and this matrix row reduces to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} =$ $\begin{bmatrix} R_0 \\ \text{zero row} \end{bmatrix}$. Always $R_0^{\mathrm{T}} R_0$ has the same nullspace as R_0, so its row reduced form must be R_0 with $n - m$ extra zero rows. R_0 is determined by its nullspace and shape !

45 The *row-column reduced echelon form* is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is r by r.

46 $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $W = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ is invertible if $W^{-1} = \dfrac{1}{3} \begin{bmatrix} -5 & 4 \\ 2 & -1 \end{bmatrix}$.

Then $W^{-1} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. This is the correct last column F of $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.

Problem Set 3.3, page 103

1 $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix} \begin{matrix} 4 \\ -1 \\ 0 \end{matrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{s}_2 = (2, -2, 0, 1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$$\begin{bmatrix} R_0 & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ gives the particular solution } x_p = (4, -1, 0, 0).$$

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix}$ Then $[\, R_0 \;\; \boldsymbol{d} \,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\mathbf{C}(A) =$ line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 $\boldsymbol{x}_{\text{complete}} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \end{bmatrix}; \quad \boldsymbol{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1).$

5 $\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.

Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \boldsymbol{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 2 & 0 & 2 & b_1 \\ 4 & 4 & 0 & b_2 \\ 8 & 8 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & b_1/2 \\ 0 & 1 & -1 & b_2/4 - b_1/2 \\ 0 & 0 & 0 & b_3 - 2b_2 \end{bmatrix}$

is solvable if $b_3 = 2b_2$. Then $\boldsymbol{x} = \begin{bmatrix} b_1/2 \\ b_2/4 - b_1/2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[\,0\ \ 0\ \ 0\ \ 0\,] =$ row 3 $-$ 2 (row 2) $+$ 4(row 1) **provided** $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$.

8 (a) Every $\boldsymbol{b}$ is in $\mathbf{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.

(b) We need $b_3 = 2b_2$, because $(\text{row}\,3) - 2(\text{row}\,2) = \mathbf{0}$.

9 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution from the nullspace.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$. Many possible A !

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and also $\boldsymbol{x} = \mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$

(b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}, A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1. $2\boldsymbol{x}_p$ would solve $A\boldsymbol{x} = 2\boldsymbol{b}$

(b) Any solution can be $\boldsymbol{x}_p$. If A has rank $= m$, the only $\boldsymbol{x}_p$ is $\boldsymbol{0}$.

(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)

(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \boldsymbol{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \boldsymbol{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $c_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F \,]$ for any 3 by 2 matrix F.

17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The columns are independent. The solution is *unique* (if there is a solution). The nullspace contains only the *zero vector*. Then $\mathbf{R_0} = \mathbf{rref}(A) = \begin{bmatrix} I \ \ (4 \text{ by } 4) \\ 0 \ \ (2 \text{ by } 4) \end{bmatrix}$.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{0}$ and we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

20 For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

21 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\, x \,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b \,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

22 (a) $r < m$, always $r \leq n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

23 $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R_0 = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R_0 = I = R$ and

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} \to R_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R.$$

24 $R_0 = I$ when A is square and invertible—so for a triangular matrix, all diagonal entries must be nonzero.

25 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ \mathbf{1} \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I. Note : $R_0 = R$.

26 $[\,R_0 \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ leads to $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $[\,R_0 \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$

leads to no solution because of the 3rd equation $0 = 5$.

27 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

28 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

29 $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \mathbf{0}$ and $U\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since $\boldsymbol{b} = (1, 3, 6, 5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0, 0, 1)$ and the complete solution is $\boldsymbol{x} = (0, 0, 1) + c\boldsymbol{s}$. (Another particular solution is $\boldsymbol{x}_p = (7, -2, 0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1, 0, 0, 0)$ elimination leads to $U\boldsymbol{x} = (1, -1, 0, 1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

30 If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

31 (a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \mathbf{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (a line of solutions). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R_0 must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$, because A and R_0 have *full row rank* $r = 3$.

32 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $\boldsymbol{A} = \boldsymbol{C}$!

33 The column space of R_0 (m by n with rank r) is spanned by its r pivot columns (the first r columns of an m by m identity matrix). The column space of R (after $m - r$ zero rows are removed from R_0) is r-dimensional space $\mathbf{R}^r$.

Problem Set 3.4, page 116

1 $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=0$ gives $c_3=c_2=c_1=0$. So those 3 column vectors are independent: no other combination gives **0**

$$\begin{bmatrix}1&1&1&2\\0&1&1&3\\0&0&1&4\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}0\\0\\0\end{bmatrix} \text{ is solved by } \boldsymbol{c}=\begin{bmatrix}1\\1\\-4\\1\end{bmatrix}. \text{ Then } \boldsymbol{v}_1+\boldsymbol{v}_2-4\boldsymbol{v}_3+\boldsymbol{v}_4=\mathbf{0} \text{ (dependent).}$$

2 $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors in $\mathbf{R}^4$ are on the plane $(1,1,1,1)\cdot\boldsymbol{v}=0$ so no four of these six vectors can be independent.

3 If $a=0$ then column $1=\mathbf{0}$; if $d=0$ then $b(\text{column }1)-a(\text{column }2)=\mathbf{0}$; if $f=0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x}=\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ gives $z=0$ then $y=0$ then $x=0$ (by back substitution). A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&-5&-7\\0&-1&-5\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&-5&-7\\0&0&-18/5\end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix}1&2&-3\\-3&1&2\\2&-3&1\end{bmatrix}\to\begin{bmatrix}1&2&-3\\0&7&-7\\0&-7&7\end{bmatrix}\to\begin{bmatrix}1&2&-3\\0&7&-7\\0&0&0\end{bmatrix}$; $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ columns add to **0**.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A. This is because $EA=U$ for the matrix E that subtracts 2 times row 1 from row 4. So A and U have the same nullspace (same dependencies of columns).

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the differences are *dependent* and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

(changing -1's to 1's for the matrix A in solution **7** above makes A invertible.)

9 (a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\,\boldsymbol{v}_1 \;\; \boldsymbol{v}_2\,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* "$\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$" —since $\boldsymbol{v}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{v}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{v}_1 + 3(0, 0, 0) = (0, 0, 0)$.

10 The plane is the nullspace of $A = [1 \;\; 2 \;\; -3 \;\; -1]$. Three free variables give three independent solutions $(x, y, z, t) = (-2, 1, 0, 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False* because the zero vector is always in the row space.

13 The column space and row space of A and U all have the same dimension = 2. *The row spaces of A and U are the same*, because the rows of U are combinations of the rows of A (and vice versa !).

14 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$. The two pairs *span* the same space. They are a basis for the same space when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent*.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$. *Invertible* if $m = n$.

16 These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\mathbf{N}(I) = \mathbf{Z} = \{\text{zero vector}\}$.

17 The column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1+$ row 2) or (row 1 and $-$ row 2) are bases for the row space of U.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

19 n independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ *rank* $= m = n$. The rank counts the number of *independent* columns.

20 One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 (a) The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$. Their combinations give every $\boldsymbol{b}$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

23 Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces. **Row spaces and nullspaces** stay fixed in elimination.

24 (a) *False* $A = [\,1 \;\; 1\,]$ has dependent columns, independent row (b) *False* Column space $\neq$ row space for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) *True*: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) *False*, columns may be dependent, in that case not a basis for $\mathbf{C}(A)$.

25 (a) Make $\boldsymbol{v}_1, \dots \boldsymbol{v}_k$ the columns of A. Then find the first n independent columns (we are told they span $\mathbf{R}^n$).

(b) Make $\boldsymbol{v}_1, \dots \boldsymbol{v}_j$ the rows of A and then include the n rows of the identity matrix. Row elimination will keep the first j independent rows and find $n - j$ more rows to form a basis for $\mathbf{R}^n$.

26 A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

27 (a) Basis for all diagonal matrices: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ = basis for symmetric matrices.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

Echelon matrices do *not* form a subspace; they ***span*** the upper triangular matrices (not every U is an echelon matrix).

29 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

30 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c) I by itself spans the space of all multiples cI.

31 $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. **Dimension** $= 4$.

32 (a) $y(x) =$ constant C (b) $y(x) = 3x$. (c) $y(x) = 3x + C = y_p + y_n$ solves $y' = 3$.

33 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 (a) $y(x) = e^{2x}$ is a basis for all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

35 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

36 Basis 1, x, x^2, x^3, for cubic polynomials; basis $x - 1$, $x^2 - 1$, $x^3 - 1$ for the subspace with $p(1) = 0$. (4-dimensional space and 3-dimensional subspace).

37 Basis for $\mathbf{S}$: $(1, 0, -1, 0), (0, 1, 0, 0), (1, 0, 0, -1)$; basis for $\mathbf{T}$: $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3, -3, 2, 1) =$ nullspace for 3 equations in $\mathbf{R}^4$ has dimension 1.

38 If the 5 by 5 matrix $[\, A \;\; \boldsymbol{b} \,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A: no solution to $A\boldsymbol{x} = \boldsymbol{b}$. If $[\, A \;\; \boldsymbol{b} \,]$ is singular, and the 4 columns of A are independent (rank 4), $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

39 One basis for $y'' = y$ is $y = e^x$ and $y = e^{-x}$. One basis for $y'' = -y$ is $y = \cos x$ and $y = \sin x$.

40 $\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 3rd cannot be (from $P_{22} = 1$) and also 1st ($P_{33} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

41 The dimension of $\boldsymbol{S}$ spanned by all rearrangements of $\boldsymbol{x}$ is (a) zero when $\boldsymbol{x} = \boldsymbol{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1, 1, 1, 1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim $\boldsymbol{S} = 2$. I owe this nice problem to Mike Artin—the answers are the same in higher dimensions: 0 or 1 or $n-1$ or n.

42 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination giving $\mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

43 If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

Here is a more basic approach : Put a basis for **V** and then a basis for **W** in the columns of a matrix A. Then A has more columns than rows and there is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$. That $\boldsymbol{x}$ gives a combination of the **V** columns $=$ a combination of the **W** columns.

44 If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$ by the Counting Theorem. So we must have $r \leq 10 - r$ and this leads to $r \leq 5$.

Problem Set 3.5, page 129

1 (a) Row and column space dimensions $9-5=5$, nullspace dimension $=4$, $\dim(\mathbf{N}(A^{\mathrm{T}}))$ $=9-7=2$ sum $5+5+4+2=16=m+n$

(b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$ (dimension zero).

2 A: Row space basis $=$ row $1=(1,2,4)$; nullspace $(-2,1,0)$ and $(-4,0,1)$; column space basis $=$ column $1=(1,2)$; left nullspace $(-2,1)$. B: Row space basis $=$ both rows $=(1,2,4)$ and $(2,5,8)$; column space basis $=$ two columns $=(1,2)$ and $(2,5)$; nullspace $(-4,0,1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y}=\mathbf{0}$: the rows of B are independent.

3 Row space basis $=$ first two rows of R; column space basis $=$ pivot columns (of A not R) $=(1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1)=$ last row of the elimination matrix $E^{-1}=L$.

4 (a) $\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $[1\ \ 1]$ (d) $\begin{bmatrix}9&-3\\3&-1\end{bmatrix}$

(e) *Impossible* Row space $=$ column space requires $m=n$. Then $m-r=n-r$; nullspaces have the same dimension. Section 4.1 will prove $\mathbf{N}(A)$ and $\mathbf{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A=\begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}$ has those rows spanning its row space. $B=\begin{bmatrix}1&-2&1\end{bmatrix}$ has the same vectors spanning its nullspace and $AB^{\mathrm{T}}=$ zero matrix (*not* AB).

6 A: $\dim$ $\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\mathbf{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: $\dim$ $\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\mathbf{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; \ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix}$ = 3 by 2 have *row space dimensions* $= 3, 3, 0 =$ column space dimensions; *nullspace dimensions* $2, 3, 2$; *left nullspace dimensions* $0, 2, 3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same

(b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand**(3), almost surely rank$= 3$, nullspace and left nullspace contain only $(0,0,0)$. For **rand**$(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here.

(b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space.

(b) *True*: A and $-A$ have the same four subspaces

(c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\mathbf{C}(A) = \mathbf{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$. So $\boldsymbol{v}$ is perpendicualr to $\boldsymbol{v}$: $\boldsymbol{v} = \mathbf{0}$.

17 Row space of $A = yz$ plane; column space of $A = xy$ plane; nullspace $= x$ axis; left nullspace $= z$ axis. For $I + A$: Row space $=$ column space $= \mathbf{R}^3$, both nullspaces contain only the zero vector.

18 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1.$ (Need to specify the five moves).

19 Row $3-2$ row $2+$ row $1 =$ zero row so the vectors $c(1, -2, 1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 (a) Special solutions $(-1, 2, 0, 0)$ and $(-\frac{1}{4}, 0, -3, 1)$ are perpendicular to the rows of R_0 (and rows of ER_0). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution $=$ last row of E^{-1}. ($E^{-1}A = R_0$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} \\ \boldsymbol{z}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$ $\boldsymbol{u}, \boldsymbol{w}$ span column space; $\boldsymbol{v}, \boldsymbol{z}$ span row space

23 As in Problem 22: Row space basis $(3, 0, 3), (1, 1, 2)$; column space basis $(1, 4, 2)$, $(2, 5, 7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A = [\,1 \quad 0\,]$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A) = \boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}} = A$ or $A^{\mathrm{T}} = -A$ they are also the same for A^{T})

26 Choose $d = bc/a$ to make $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ a rank-1 matrix. Then the row space has basis (a, b) and the nullspace has basis $(-b, a)$. Those two vectors are perpendicular !

27 B and C (checkers and chess) both have rank 2 if $p \neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero; $\mathbf{N}(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $\mathbf{N}(C)$ is a challenge : one vector in $\mathbf{N}(C)$ is $(1,0,\ldots,0,-1)$.

28 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

29 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

30 The key is equal row spaces. First row of A = combination of the rows of B : the only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

Problem Set 4.1, page 140

1 Both nullspace vectors will be orthogonal to the row space vector in $\mathbf{R}^3$. The column space of A and the nullspace of A^{T} are perpendicular lines in $\mathbf{R}^2$ because rank $= 1$.

2 The nullspace of a 3 by 2 matrix with rank 2 is $\mathbf{Z}$ (only the zero vector because the 2 columns are independent). So $\boldsymbol{x}_n = \mathbf{0}$, and row space $= \mathbf{R}^2$. Column space $=$ plane perpendicular to left nullspace $=$ line in $\mathbf{R}^3$ (because the rank is 2).

3 (a) One way is to use these two columns directly and make col 3 $= -$ col 1$-$ col 2. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$

(b) Impossible because $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathrm{T}})$ are orthogonal subspaces: $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ is not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbf{C}(A)$ and $\mathbf{N}(A^{\mathrm{T}})$ is impossible: not perpendicular

(d) Rows orthogonal to columns makes A times $A =$ zero matrix. An example is $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(e) $(1, 1, 1)$ in the nullspace (columns add to the zero vector) and also $(1, 1, 1)$ is in the row space: no such matrix.

4 If $AB = 0$, the columns of B are in the *nullspace* of A and the rows of A are in the *left nullspace* of B. If rank $= 2$, all those four subspaces have dimension at least 2 which is impossible for 3 by 3.

5 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$. $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = 0$. This says again that $\mathbf{C}(A)$ is orthogonal to $\mathbf{N}(A^{\mathrm{T}})$. (b) If $A^{\mathrm{T}}\boldsymbol{y} = (1, 1, 1)$ has a solution, $(1, 1, 1)$ is a combination of the rows of A. It is in the **row space** and is orthogonal to every $\boldsymbol{x}$ in the **nullspace**.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Now the equations add to $0 = 1$ so there is no solution. In subspace language, $\boldsymbol{y} = (1, 1, -1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}} A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ but here $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

7 Multiply the 3 equations by $\boldsymbol{y} = (1, 1, -1)$. Then $x_1 - x_2 = 1$ plus $x_2 - x_3 = 1$ *minus* $x_1 - x_3 = 1$ is $0 = 1$. Key point: This $\boldsymbol{y}$ in $\mathbf{N}(A^{\mathrm{T}})$ is not orthogonal to $\boldsymbol{b} = (1, 1, 1)$ so $\boldsymbol{b}$ is not in the column space and $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

8 Figure 4.3 has $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \mathbf{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. The example has $\boldsymbol{x} = (1, 0)$ and row space = line through $(1, 1)$ so the splitting is $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n = \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, -\frac{1}{2}\right)$. All $A\boldsymbol{x}$ are in $\mathbf{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the *nullspace* of A^{T}. Those subspaces are perpendicular. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column space and row space are the *same*. The nullspace is always perpendicular to the row space. (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" $\boldsymbol{x}$ and $\boldsymbol{z}$ have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

11 ***For A***: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $\mathbf{N}(A^{\mathrm{T}})$ is the perpendicular line through $(3, -1)$. ***For B***: The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for A.

12 $\boldsymbol{x} = (2, 0)$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1, -1) + (1, 1)$.

13 $V^{\mathrm{T}}W$ = zero matrix makes each column of $\mathbf{V}$ orthogonal to each column of W. This means: each basis vector for $\mathbf{V}$ is orthogonal to each basis vector for $\mathbf{W}$. Then *every* $\boldsymbol{v}$ in $\mathbf{V}$ (combinations of the basis vectors) is orthogonal to *every* $\boldsymbol{w}$ in $\mathbf{W}$.

14 $A\boldsymbol{x} = B\widehat{\boldsymbol{x}}$ means that $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ -\widehat{\boldsymbol{x}} \end{bmatrix} = \mathbf{0}$. Three homogeneous equations (zero right hand sides) in four unknowns always have a nonzero solution. Here $\boldsymbol{x} = (3, 1)$ and $\widehat{\boldsymbol{x}} = (1, 0)$ and $A\boldsymbol{x} = B\widehat{\boldsymbol{x}} = (5, 6, 5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

15 A p-dimensional and a q-dimensional subspace of $\mathbf{R}^n$ share at least a line if $\boldsymbol{p} + \boldsymbol{q} > \boldsymbol{n}$. (The $p + q$ basis vectors of **V** and **W** cannot be independent, so some combination of the basis vectors of **V** is also a combination of the basis vectors of $\boldsymbol{W}$.)

16 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ leads to $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = 0$. Then $\boldsymbol{y} \perp A\boldsymbol{x}$ and $\mathbf{N}(A^{\mathrm{T}}) \perp \mathbf{C}(A)$.

17 If **S** is the subspace of $\mathbf{R}^3$ containing only the zero vector, then $\mathbf{S}^\perp$ is all of $\mathbf{R}^3$. If **S** is spanned by $(1, 1, 1)$, then $\mathbf{S}^\perp$ is the plane spanned by $(1, -1, 0)$ and $(1, 0, -1)$. If **S** is spanned by $(1, 1, 1)$ and $(1, 1, -1)$, then $\mathbf{S}^\perp$ is the line spanned by $(1, -1, 0)$.

18 $\mathbf{S}^\perp$ contains all vectors perpendicular to those two given vectors. So $\mathbf{S}^\perp$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $\mathbf{S}^\perp$ is a *subspace* even if **S** is not.

19 $\mathbf{L}^\perp$ is the 2-*dimensional subspace* (*a plane*) in $\mathbf{R}^3$ perpendicular to **L**. Then $(\mathbf{L}^\perp)^\perp$ is a 1-*dimensional subspace* (*a line*) perpendicular to $\mathbf{L}^\perp$. In fact $(\mathbf{L}^\perp)^\perp$ is **L**.

20 If **V** is the whole space $\mathbf{R}^4$, then $\mathbf{V}^\perp$ contains only the *zero vector*. Then $(\mathbf{V}^\perp)^\perp =$ all vectors perpendicular to the zero vector $= \mathbf{R}^4 = \mathbf{V}$.

21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $\mathbf{S}^\perp$ =nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

22 $(1, 1, 1, 1)$ is a basis for the line $\mathbf{P}^\perp$ orthogonal to the hyperplane **P**.

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has **P** as its nullspace and $\mathbf{P}^\perp$ as its row space.

23 $\boldsymbol{x}$ in $\mathbf{V}^\perp$ is perpendicular to every vector in **V**. Since **V** contains all the vectors in **S**, $\boldsymbol{x}$ is perpendicular to every vector in **S**. So every $\boldsymbol{x}$ in $\mathbf{V}^\perp$ is also in $\mathbf{S}^\perp$.

24 $AA^{-1} = I$: Column 1 of A^{-1} is orthogonal to rows $2, 3, \ldots, n$ of A and therefore it is orthogonal to the space spanned by those rows.

25 If the columns of A are unit vectors, all mutually perpendicular, then $A^{\mathrm{T}}A = I$. Simple but important! We write Q for such a matrix.

26 $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, This example shows a matrix with perpendicular columns. $A^{\mathrm{T}}A = 9I$ is *diagonal*: $(A^{\mathrm{T}}A)_{ij} =$ (column i of A) $\cdot$ (column j of A). When the columns are *unit vectors*, then $A^{\mathrm{T}}A = I$.

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are **parallel**. They are the same line if $b_2 = 2b_1$. In that case (b_1, b_2) is perpendicular to $\boldsymbol{(-2, 1)}$. The nullspace of the 2 by 2 matrix is the line $3x + y = \mathbf{0}$. One particular vector in the nullspace is $\boldsymbol{(-1, 3)}$.

28 (a) $(1, -1, 0)$ is in both planes. Normal vectors are perpendicular to each other, but planes can still intersect! Two planes in $\mathbf{R}^3$ can't be orthogonal.

(b) Need *three* orthogonal vectors to span the whole orthogonal complement in $\mathbf{R}^5$.

(c) Lines in $\mathbf{R}^3$ can meet at the zero vector without being orthogonal.

29 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$; A has $\boldsymbol{v} = (1, 2, 3)$ in row and column spaces; B has $\boldsymbol{v}$ in its column space and nullspace. $\boldsymbol{v}$ **can not** be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} \neq 0$.

30 When $AB = 0$, every column of B is multiplied by A to give zero. So the column space of B is contained in the nullspace of A. Therefore the dimension of $\mathbf{C}(B) \leq$ dimension of $\mathbf{N}(A)$. This means rank$(B) \leq 4 -$ rank(A).

31 null(N') produces a basis for the *row space* of A (perpendicular to $\mathbf{N}(A)$).

32 We need $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{n} = 0$ and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\ell} = 0$. All possible examples have the form $A = a\boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ with $a \neq 0$.

33 Both $\boldsymbol{r}$'s must be orthogonal to both $\boldsymbol{n}$'s, both $\boldsymbol{c}$'s must be orthogonal to both $\boldsymbol{\ell}$'s, each pair ($\boldsymbol{r}$'s, $\boldsymbol{n}$'s, $\boldsymbol{c}$'s, and $\boldsymbol{\ell}$'s) must be independent. Fact: All A's with these subspaces have the form $[\boldsymbol{c}_1\ \boldsymbol{c}_2]M[\boldsymbol{r}_1\ \boldsymbol{r}_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 150

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; projection $\boldsymbol{p} = 5\boldsymbol{a}/3 = (5/3, 5/3, 5/3)$; $\boldsymbol{e} = (-2, 1, 1)/3$

(b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; projection $\boldsymbol{p} = -\boldsymbol{a}$; $\boldsymbol{e} = \boldsymbol{0}$.

2 (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$

(b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

The picture for part (a) has the vector $\boldsymbol{b}$ at an angle θ with the horizontal $\boldsymbol{a}$. The picture for part (b) has vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ at a 90° angle.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1\boldsymbol{b} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

4 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. P_1 projects onto $(1, 0)$, P_2 projects onto $(1, -1)$. $P_1P_2 \neq 0$ and $P_1 + P_2$ is not a projection matrix. $(P_1 + P_2)^2$ is different from $P_1 + P_2$.

5 $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$ and $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$.

P_1 and P_2 are the projection matrices onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. $P_1P_2 =$ *zero matrix because* $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

6 $\boldsymbol{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = \boldsymbol{b}$.

7 $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I$.

We *can* add projections onto *orthogonal vectors* to get the projection matrix onto the larger space. This is important.

8 The projections of $(1, 1)$ onto $(1, 0)$ and $(1, 2)$ are $\boldsymbol{p}_1 = (1, 0)$ and $\boldsymbol{p}_2 = \frac{3}{5}(1, 2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$. The sum of projections is not a projection onto the space spanned by $(1, 0)$ and $(1, 2)$ because those vectors are *not orthogonal*.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ separates into $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$. And I is the projection matrix onto all of $\mathbf{R}^2$.

10 $P_2 = \dfrac{\boldsymbol{a}_2\boldsymbol{a}_2^{\mathrm{T}}}{\boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{a}_2} = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}, P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, P_1 = \dfrac{\boldsymbol{a}_1\boldsymbol{a}_1^{\mathrm{T}}}{\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P_1P_2\boldsymbol{a}_1 =$ $\begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$. This is not $\boldsymbol{a}_1 = (1, 0)$
No, $\boldsymbol{P_1P_2} \neq (P_1P_2)^2$.

11 (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2, 3, 0)$, $\boldsymbol{e} = (0, 0, 4)$, $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$

(b) $\boldsymbol{p} = (4, 4, 6)$ and $\boldsymbol{e} = \mathbf{0}$ because $\boldsymbol{b}$ is in the column space of A.

12 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = Projection matrix $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ onto the second column space. Certainly $(P_2)^2 = P_2$. A true projection matrix.

13 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, P = square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

14 The projection of this $\boldsymbol{b}$ onto the column space of A is $\boldsymbol{b}$ itself because $\boldsymbol{b}$ is in that column space. But P is not necessarily I. Here $\boldsymbol{b} = 2$(column 1 of A) :

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} \text{ gives } P = \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix} \text{ and } \boldsymbol{b} = P\boldsymbol{b} = \boldsymbol{p} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

15 $2A$ has the same column space as A. Then P is the same for A and $2A$, but $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. So $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

17 If $P^2 = P$ then $\boldsymbol{(I - P)^2} = (I - P)(I - P) = I - PI - IP + P^2 = \boldsymbol{I - P}$. When P projects onto the column space, $I - P$ projects onto the *left nullspace*.

18 (a) $I - P$ is the projection matrix onto $(1, -1)$ in the perpendicular direction to $(1, 1)$

(b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1, 1, 1)$.

19 For any basis vectors in the plane $x - y - 2z = 0$, say $(1, 1, 0)$ and $(2, 0, 1)$, the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

20 $\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \frac{\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}}{\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is also $P\boldsymbol{b}$.

22 $P^{\mathrm{T}} = (A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}})^{\mathrm{T}} = A((A^{\mathrm{T}}A)^{-1})^{\mathrm{T}}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = P$. ($A^{\mathrm{T}}A$ is symmetric!)

23 If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \mathbf{0}$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\mathbf{0}$.

25 **The column space of P is the space that P projects onto**. The column space of A always contains all outputs $A\boldsymbol{x}$ and here the outputs $P\boldsymbol{x}$ fill the subspace S. Then rank of P = dimension of $S = n$.

26 A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.

27 If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is in the **nullspace of A^{T}**. But $A\boldsymbol{x}$ is always in the **column space of A**. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace*: $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ exactly when $A\boldsymbol{x} = \mathbf{0}$.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2, 2)$ entry of P equals the $(2, 2)$ entry of $P^{\mathrm{T}}P$. But the $(2, 2)$ entry of $P^{\mathrm{T}}P$ is the length squared of column 2.

29 $A = B^{\mathrm{T}}$ has independent columns, so $A^{\mathrm{T}}A$ (which is BB^{T}) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

The formula $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ needs independent columns—this A has dependent columns. The update formula is correct. The column space of A is a line.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ (columns of A project to themselves) and $AP_R = A$. Then $\boldsymbol{P_C A P_R = A}$.

31 *Test*: The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 $\dfrac{1}{999}\left(b_1+\cdots+b_{999}\right)+\left(1-\dfrac{1}{1000}\right)+\dfrac{b_{1000}}{1000} = \left(b_1+\cdots+b_{999}\right)\left(\dfrac{1}{1000}\right)+\dfrac{b_{1000}}{1000} = \widehat{x}_{1000}$.

33 If $P_1P_2 = P_2P_1$ then $P_1P_2P_1P_2 = P_1P_1P_2P_2 = P_1P_2$. Also $(P_1P_2)^{\mathrm{T}} = P_2^{\mathrm{T}}P_1^{\mathrm{T}} = P_2P_1 = P_1P_2$.

Suppose $P_1P_2 \neq P_2P_1$. Then the previous equation fails: $(P_1P_2)^{\mathrm{T}} \neq P_1P_2$ and $P_1P_2 \neq$ projection.

Problem Set 4.3, page 161

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

$E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

2 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. This $A\boldsymbol{x} = \boldsymbol{b}$ is unsolvable. Project $\boldsymbol{b}$ to $\boldsymbol{p} = P\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$; When $\boldsymbol{p}$ replaces $\boldsymbol{b}$,

$\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ exactly solves $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$.

3 In Problem 2, $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (1, 5, 13, 17)$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1, 3, -5, 3)$. This $\boldsymbol{e}$ is perpendicular to both columns of A. This shortest distance $\|\boldsymbol{e}\|$ is $\sqrt{44}$.

4 $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$. These two normal equations are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [\,1 \;\; 1 \;\; 1 \;\; 1\,]$ and $A^{\mathrm{T}}A = [\,4\,]$. $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = \mathbf{9}$ = best height C for the horizontal line. Errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-9, -1, -1, 11)$ still add to zero.

6 $\boldsymbol{a} = (1,1,1,1)$ and $\boldsymbol{b} = (0,8,8,20)$ give $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$ and the projection is $\widehat{x}\boldsymbol{a} = \boldsymbol{p} = (9,9,9,9)$. Then $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = (-9,-1,-1,11)^{\mathrm{T}}(1,1,1,1) = 0$ and the shortest distance from $\boldsymbol{b}$ to the line through $\boldsymbol{a}$ is $\|\boldsymbol{e}\| = \sqrt{204}$.

7 Now the 4 by 1 matrix in $A\boldsymbol{x} = \boldsymbol{b}$ is $A = [\,0 \;\; 1 \;\; 3 \;\; 4\,]^{\mathrm{T}}$. Then $A^{\mathrm{T}}A = [\,26\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,112\,]$. Best $D = 112/26 = 56/13$.

8 $\widehat{\boldsymbol{x}} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 56/13$ and $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$ from Problems 1-4. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

Figure 4.9 (a) is fitting 4 points and 4.9 (b) is a projection in $\mathbf{R}^4$: same problem!

10 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Then $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$. **Exact cubic so $\boldsymbol{p} = \boldsymbol{b}$, $\boldsymbol{e} = \mathbf{0}$.** This Vandermonde matrix gives exact interpolation by a cubic at $0, 1, 3, 4$

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{b} = 9$ at center time, $\widehat{t} = 2$.

(b) The first equation $Cm + D\sum\, t_i = \sum\, b_i$ divided by m gives $C + D\widehat{t} = \widehat{\boldsymbol{b}}$. This shows: The best line goes through $\widehat{\boldsymbol{b}}$ at time $\widehat{t}$.

12 (a) $\boldsymbol{a} = (1,\ldots,1)$ has $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/m$ is the **mean** of the b's (their average value)

(b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$ and $\|\boldsymbol{e}\|^2 = (b_1 - \text{ mean })^2 + \cdots + (b_m - \text{ mean })^2 =$ **variance** (denoted by $\boldsymbol{\sigma^2}$).

(c) $\boldsymbol{p} = (3,3,3)$ and $\boldsymbol{e} = (-2,-1,3)$ $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. Projection matrix $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

13 $(A^{\mathrm T}A)^{-1}A^{\mathrm T}(\boldsymbol{b}-A\boldsymbol{x})=\widehat{\boldsymbol{x}}-\boldsymbol{x}$. This tells us: When the components of $A\boldsymbol{x}-\boldsymbol{b}$ add to zero, so do the components of $\widehat{\boldsymbol{x}}-\boldsymbol{x}$: Unbiased.

14 The matrix $(\widehat{\boldsymbol{x}}-\boldsymbol{x})(\widehat{\boldsymbol{x}}-\boldsymbol{x})^{\mathrm T}$ is $(A^{\mathrm T}A)^{-1}A^{\mathrm T}(\boldsymbol{b}-A\boldsymbol{x})(\boldsymbol{b}-A\boldsymbol{x})^{\mathrm T}A(A^{\mathrm T}A)^{-1}$. When the average of $(\boldsymbol{b}-A\boldsymbol{x})(\boldsymbol{b}-A\boldsymbol{x})^{\mathrm T}$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}}-\boldsymbol{x})(\widehat{\boldsymbol{x}}-\boldsymbol{x})^{\mathrm T}$ will be the *output covariance matrix* $(A^{\mathrm T}A)^{-1}A^{\mathrm T}\sigma^2A(A^{\mathrm T}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm T}A)^{-1}$. That gives the average of the squared output errors $\widehat{\boldsymbol{x}}-\boldsymbol{x}$.

15 When A has 1 column of 4 ones, Problem 14 gives the expected error $(\widehat{x}-x)^2$ as $\sigma^2(A^{\mathrm T}A)^{-1}=\sigma^2/4$. By taking m measurements, the variance drops from σ^2 to σ^2/m.

16 $\dfrac{1}{10}b_{10}+\dfrac{\mathbf{9}}{\mathbf{10}}\widehat{x}_9=\dfrac{1}{10}(b_1+\cdots+b_{10})$. Knowing $\widehat{x}_9$ avoids adding all ten b's.

17 $\begin{bmatrix}1 & -1\\ 1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}7\\ 7\\ 21\end{bmatrix}$. The solution $\widehat{\boldsymbol{x}}=\begin{bmatrix}\mathbf{9}\\ \mathbf{4}\end{bmatrix}$ comes from $\begin{bmatrix}3 & 2\\ 2 & 6\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}35\\ 42\end{bmatrix}$.

18 $\boldsymbol{p}=A\widehat{\boldsymbol{x}}=(5,13,17)$ gives the heights of the closest line. The vertical errors are $\boldsymbol{b}-\boldsymbol{p}=(2,-6,4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e}=P\boldsymbol{b}-P\boldsymbol{p}=\boldsymbol{p}-\boldsymbol{p}=\mathbf{0}$.

19 If $\boldsymbol{b}=$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p}=\mathbf{0}$.

20 The matrix A has columns $1,1,1$ and $-1,1,2$. If $\boldsymbol{b}=A\widehat{\boldsymbol{x}}=(5,13,17)$ then $\widehat{\boldsymbol{x}}=(9,4)$ and $\boldsymbol{e}=\mathbf{0}$ since $\boldsymbol{b}=9$ (column 1) $+\,4$ (column 2) is *in the column space of* A.

21 $\boldsymbol{e}$ is in $\mathbf{N}(A^{\mathrm T})$; $\boldsymbol{p}$ is in $\mathbf{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\mathbf{C}(A^{\mathrm T})$; $\mathbf{N}(A)=\{\mathbf{0}\}=$ zero vector only.

22 The least squares equation is $\begin{bmatrix}5 & \mathbf{0}\\ \mathbf{0} & 10\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}5\\ -10\end{bmatrix}$. Solution: $C=1$, $D=-1$. The best line is $b=1-t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm T}A\Rightarrow$ easy solution.

23 $\boldsymbol{e}$ is orthogonal to $\boldsymbol{p}$ in $\mathbf{R}^m$; then $\|\boldsymbol{e}\|^2=e^{\mathrm T}(\boldsymbol{b}-\boldsymbol{p})=\boldsymbol{e}^{\mathrm T}\boldsymbol{b}=\boldsymbol{b}^{\mathrm T}\boldsymbol{b}-\boldsymbol{b}^{\mathrm T}\boldsymbol{p}$.

24 The derivatives of $\|A\boldsymbol{x}-\boldsymbol{b}\|^2=\boldsymbol{x}^{\mathrm T}A^{\mathrm T}A\boldsymbol{x}-2\boldsymbol{b}^{\mathrm T}A\boldsymbol{x}+\boldsymbol{b}^{\mathrm T}\boldsymbol{b}$ (this last term is constant) are zero when $2A^{\mathrm T}A\boldsymbol{x}=2A^{\mathrm T}\boldsymbol{b}$, or $\boldsymbol{x}=(A^{\mathrm T}A)^{-1}A^{\mathrm T}\boldsymbol{b}$.

25 3 points on a line will give **equal slopes** $(b_2-b_1)/(t_2-t_1)=(b_3-b_2)/(t_3-t_2)$. Linear algebra: Orthogonal to the columns $(1,1,1)$ and (t_1,t_2,t_3) is $\boldsymbol{y}=(t_2-t_3,t_3-t_1,t_1-t_2)$ in the left nullspace of A. $\boldsymbol{b}$ is in the column space ! Then $\boldsymbol{y}^{\mathrm T}\boldsymbol{b}=0$ is the same equal slopes condition written as $(b_2-b_1)(t_3-t_2)=(b_3-b_2)(t_2-t_1)$.

26 The unsolvable equations for $C + Dx + Ey = (0,1,3,4)$ at the 4 corners are

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}. \; A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}; \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}.$$

At $x, y = 0, 0$ the best plane $2 - x - \frac{3}{2}y$ has height $C = \mathbf{2}$ = average of $0, 1, 3, 4$.

27 The shortest link connecting two lines in space is *perpendicular to those lines*.

28 If A has dependent columns, then $A^{\mathrm{T}}A$ is not invertable and the usual formula $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ will fail. Replace A in that formula by the matrix B that keeps *only the pivot columns of* A.

29 Only 1 plane contains $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$. If they are dependent, there is a vector $\boldsymbol{v}$ perpendicular to all the $\boldsymbol{a}$'s. Then they all (including $\mathbf{0}$) lie on the plane $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x} = 0$ going through $\boldsymbol{x} = (0, 0, \ldots, 0)$.

30 When A has orthogonal columns $(1, \ldots, 1)$ and $(T_1, \ldots, T_m)$, the matrix $A^{\mathrm{T}}A$ is **diagonal** with entries m and $T_1^2 + \cdots + T_m^2$. Also $A^{\mathrm{T}}\boldsymbol{b}$ has entries $b_1 + \cdots + b_m$ and $T_1b_1 + \cdots + T_mb_m$. The solution with that diagonal $A^{\mathrm{T}}A$ is just the given $\widehat{\boldsymbol{x}} = (C, D)$.

Problem Set 4.4, page 174

1 (a) *Independent* (b) *Independent* and *orthogonal* (c) *Independent* and *orthonormal*.
For orthonormal vectors, (a) becomes $(1,0)$, $(0,1)$ and (b) is $(.6,.8)$, $(.8,-.6)$.

2 Divide by length 3 to get $\boldsymbol{q}_1 = (\frac{2}{3},\frac{2}{3},-\frac{1}{3})$. $\boldsymbol{q}_2 = (-\frac{1}{3},\frac{2}{3},\frac{2}{3})$. $Q^{\mathrm{T}}Q = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ but $QQ^{\mathrm{T}} = \begin{bmatrix}5/9 & 2/9 & -4/9\\ 2/9 & 8/9 & 2/9\\ -4/9 & 2/9 & 5/9\end{bmatrix}$.

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries $1^2, 2^2, 3^2 = 1, 4, 9$.

4 (a) $Q = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$, $QQ^{\mathrm{T}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \neq I$. Any Q with $n < m$ has $QQ^{\mathrm{T}} \neq I$.
(b) $(1,0)$ and $(0,0)$ are *orthogonal*, not *independent*. Nonzero orthogonal vectors *are* independent. (c) From $\boldsymbol{q}_1 = (1,1,1)/\sqrt{3}$ my favorite is $\boldsymbol{q}_2 = (1,-1,0)/\sqrt{2}$ and $\boldsymbol{q}_3 = (1,1,-2)/\sqrt{6}$.

5 *Orthogonal* vectors are $(1,-1,0)$ and $(1,1,-1)$. *Orthonormal* after dividing by their lengths: $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)$ and $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I$. Another approach is to see that $(Q_1Q_1)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}} = (Q_1Q_2)^{\mathrm{T}}$.

7 When Gram-Schmidt gives Q with orthonormal columns, $Q^{\mathrm{T}}Q\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$ becomes $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$. No cost to solve the normal equations !

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $\boldsymbol{p} = (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$. The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

9 (a) $Q = \begin{bmatrix}.8 & -.6\\ .6 & .8\\ 0 & 0\end{bmatrix}$ has $P = QQ^{\mathrm{T}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ = projection on the xy plane.

(b) $(QQ^{\mathrm{T}})(QQ^{\mathrm{T}}) = Q(Q^{\mathrm{T}}Q)Q^{\mathrm{T}} = QQ^{\mathrm{T}}$.

10 (a) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are *orthonormal* then the dot product of $\boldsymbol{q}_1$ with $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3 = \boldsymbol{0}$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. This proves: *Independent* $\boldsymbol{q}$'s

(b) $Q\boldsymbol{x} = \boldsymbol{0}$ leads to $Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{0}$ which says $\boldsymbol{x} = \boldsymbol{0}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1 = \frac{1}{10}(1, 3, 4, 5, 7)$ and $\boldsymbol{q}_2 = \frac{1}{10}(-7, 3, 4, -5, 1)$

(b) Closest vector = *projection* $QQ^{\mathrm{T}}(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$.

12 (a) Orthonormal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1) = x_1$

(b) Orthogonal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1)$. Therefore $x_1 = \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1$

(c) x_1 is the first component of A^{-1} times $\boldsymbol{b}$ (A is 3 by 3 and invertible).

13 The multiple to subtract is $\dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$. Then $\boldsymbol{B} = \boldsymbol{b} - \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

14 $$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$$

15 (a) Gram-Schmidt chooses $\boldsymbol{q}_1 = \boldsymbol{a}/||\boldsymbol{a}|| = \frac{\mathbf{1}}{\mathbf{3}}(1, 2, -2)$ and $\boldsymbol{q}_2 = \frac{1}{3}(2, 1, 2)$. Then $\boldsymbol{q}_3 = \frac{1}{3}(2, -2, -1)$.

(b) The nullspace of A^{T} contains $\boldsymbol{q}_3$

(c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1, 2, 7) = (1, 2)$.

16 $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$. $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4, 5, 2, 2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1, 4, -4, -4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

17 $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = (3, 3, 3)$ and $\boldsymbol{e} = (-2, 0, 2)$. Then Gram-Schmidt will choose $\boldsymbol{q}_1 = (1, 1, 1)/\sqrt{3}$ and $\boldsymbol{q}_2 = (-1, 0, 1)/\sqrt{2}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1, -1, 0, 0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

Gram-Schmidt would go on to normalize $\boldsymbol{q}_1 = \boldsymbol{A}/||\boldsymbol{A}||, \boldsymbol{q}_2 = \boldsymbol{B}/||\boldsymbol{B}||, \boldsymbol{q}_3 = \boldsymbol{C}/||\boldsymbol{C}||$.

19 If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}Q^{\mathrm{T}}QR = R^{\mathrm{T}}R =$ *lower* triangular times *upper* triangular (this Cholesky factorization of $A^{\mathrm{T}}A$ uses the same R as Gram-Schmidt!). The example has $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$ and the same R appears in
$A^{\mathrm{T}}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^{\mathrm{T}}R.$

20 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2 = (-7,-3,-1,3)/2$. And $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

21 $A = (1,1,2),\ B = (1,-1,0),\ C = (-1,-1,1)$. These are not yet unit vectors. Gram-Schmidt will divide by $||\boldsymbol{A}|| = \sqrt{6}$ and $||\boldsymbol{B}|| = \sqrt{2}$ and $||\boldsymbol{C}|| = \sqrt{3}$.

22 You can see why $\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} =$ QR. This Q is just a permutation matrix—certainly orthogonal.

23 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

24 When $\boldsymbol{a}$ and $\boldsymbol{b}$ are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. We must use the orthogonal $\boldsymbol{A}$ and $\boldsymbol{B}$ (or orthonormal $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$) to be allowed to add projections on those lines.

25 There are $\frac{1}{2}m^2n$ multiplications to find the numbers r_{kj} and the same for v_{ij}.

26 $\boldsymbol{q}_1 = \frac{1}{3}(2,2,-1), \boldsymbol{q}_2 = \frac{1}{3}(2,-1,2), \boldsymbol{q}_3 = \frac{1}{3}(1,-2,-2)$.

27 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

28 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

29 (a) $Q\boldsymbol{u} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$. This is $-\boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ equals 1 (b) $Q\boldsymbol{v} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{v} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{v}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$.

30 Starting from $\boldsymbol{A} = (1,-1,0,0)$, the orthogonal (not orthonormal) vectors $\boldsymbol{B} = (1,1,-2,0)$ and $\boldsymbol{C} = (1,1,1,-3)$ and $\boldsymbol{D} = (1,1,1,1)$ are in the directions of $\boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$. The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal Q!) are

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix}$$

31 $[Q,R] = \boldsymbol{qr}(A)$ produces from A (m by n of rank n) a "*full-size*" square $Q = [\, Q_1 \;\; Q_2 \,]$ and $\begin{bmatrix} R \\ 0 \end{bmatrix}$. The columns of Q_1 are the orthonormal basis from Gram-Schmidt of the *column space* of A. The $m-n$ columns of Q_2 are an orthonormal basis for the *left nullspace* of A. Together the columns of $Q = [\, Q_1 \;\; Q_2 \,]$ are an orthonormal basis for $\mathbf{R}^m$.

32 This question describes the next $\boldsymbol{q}_{n+1}$ in Gram-Schmidt using the matrix Q with the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ (instead of using those $\boldsymbol{q}$'s separately). Start from $\boldsymbol{a}$, *subtract its projection* $\boldsymbol{p} = QQ^{\mathrm{T}}\boldsymbol{a}$ onto the earlier $\boldsymbol{q}$'s, *divide by the length* of $\boldsymbol{e} = \boldsymbol{a} - QQ^{\mathrm{T}}\boldsymbol{a}$ to get the next $\boldsymbol{q}_{n+1} = \boldsymbol{e}/\|\boldsymbol{e}\|$.

Problem Set 5.1, page 181

1 $\det(2A) = 2^4 \det A = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$. If $\det A = 0$ then $\det A/2 = \det(-A) = \det A^2 = 0$; no A^{-1}.

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (except when $n = 1$) (b) *True*: The product rule extends to ABC (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$

(d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

4 Exchange rows 1 and 3 to show $\det J_3 = -1$. Exchange rows 1 and 4, then rows 2 and 3 to show $\det J_4 = 1$. Two exchanges = even permutation.

5 $|J_5| = 1$ by exchanging row 1 with 5 and row 2 with 4. $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat in cycles of length 4 so the determinant of J_{101} is $+1$.

6 $\det A = 4, \det B = 0, \det C = 0$.

7 The 6 terms become $a(q+b)z - b(p+a)z + \cdots$ (4 more). The approach in the display (using linearity to split up row 2) is better. Result: $\det$ does not change if row 2 is added to row 1.

8 $\det A^{\mathrm{T}} = \begin{bmatrix} a & p & x \\ b & q & y \\ c & r & z \end{bmatrix} = \begin{matrix} aqz + cpy + brx \\ -\, ary - bpz - cqx \end{matrix} =$ **same six terms as det A**

Key point: $\det P^{\mathrm{T}} = \det P$ for every permutation, because the number of row exchanges is the same (just done in reverse order). Then P is even when P^{T} is even.

9 $\det A = 1$ from two row exchanges. $\det B = 2$ (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). $\det C = 0$ and $\det D = 0$ (equal rows).

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 If P_1 needs n exchanges to reach I and P_2 needs N exchanges then P_1P_2 reaches I after those $n{+}N$ exchanges. So $\det(P_1P_2) = (-1)^{n+N} = (-1)^n(-1)^N = (\det P_1)\,(\det P_2)$.

12 We can pair off even permutations with odd permutations: odd $=$ even followed by exchanging 1 and 2. Number of even permutations$=\frac{1}{2}n!=$number of odd permutations.

13 Pivots $1, 1, 1$ give determinant $= \mathbf{1}$; pivots $1, -2, -3/2$ give determinant $= \mathbf{3}$.

14 $\det(A) = \mathbf{36}$ and the 4 by 4 second difference matrix has det $= \mathbf{5}$.

15 The first determinant is $\mathbf{0}$, the second is $1{+}t^4{+}t^4{-}t^4{-}t^2{-}t^2 = 1{-}2t^2{+}t^4 = (1{-}t^2)^2$.

16 A singular rank one matrix has determinant $= 0$. The skew-symmetric A also has $\det A = 0$. A skew-symmetric matrix A of odd order 3: Changing every sign will multiply $\det A$ by $(-1)^3$ but also keep the same $\det A = \det A^{\mathrm{T}}$. So $\det A = 0$.

17 When the i, j entry is i times j, row $2 = 2$ times row 1 so $\det A = 0$.

When the ij entry is $i + j$, row $3 -$ row $2 =$ row2 $-$ row 1 so A is singular: $\det A = 0$.

18 $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \text{ to reach 2 by 2.}$$

We eliminated a and a^2 in row 1 by subtracting a and a^2 times column 1 from columns 2 and 3. Factor out $b - a$ and $c - a$ from the 2 by 2:

$$(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$$

19 Fill a row (or column) by 4 zeros to guarantee $\det = 0$. Leave only the main diagonal (12 zeros) to allow $\det A \neq 0$.

20 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms will be zeros and 2 terms can be nonzero. (b) 15 terms must be zero. Effectively we are counting the permutations that make everyone move; $2, 3, 1$ and $3, 1, 2$ for $n = 3$ mean that the other 4 permutations take a term from the main diagonal of A; so those terms are 0 when the diagonal is all zeros.

21 The cofactor formula $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$ gives $\det = 0$ if all cofactors are zero. The 2 by 2 matrix of 1's has $\det = 0$ even though no cofactors are zero.

22 A tridiagonal matrix has $\det = a_{11}C_{11} + a_{12}C_{12}$ since $a_{13}, \ldots, a_{1n}$ are zero. If $n = 5$ then C_{11} has 5 nonzeros. C_{12} has $a_{21} =$ only nonzero in column 1. That a_{21} multiplies a 3 by 3 tridiagonal determinant (with 3 nonzeros). So $\det A$ has $5 + 3 = 8$ nonzeros.

23 Two equal rows imply $\mathbf{det = 0}$. Proof for 3×3 if row $1 =$ row 2. Then $a = p, b = q,$ $c = r$. Then $aqz+brx+cpy-ary-bpz-cqx = abz+bcx+cay-acy-baz-cbx = 0$.

24 If A has two equal rows then A^{T} has two equal columns (say columns j and k). Then the columns are not independent. So $\det A^{\mathrm{T}} = 0$ and $\det A = 0$. Other proofs also reach this conclusion.

25 (a) a_{14} multiplies $3! = 6$ terms.

(b) Only 2 terms include both a_{13} and a_{22}. Those terms are $a_{13}a_{22}(-a_{31}a_{44}+a_{34}a_{41})$.

(c) If the main diagonal is all zero, we are counting only the permutations like $a_{12}a_{23}a_{34}a_{41}$ that involve no diagonal entries. Those are called **derangements**: How many ways can $n = 4$ students grade each other's tests ? Wikipedia says there are **9 ways** and uses the symbol $!n$.

Problem Set 5.2, page 190

1 If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$, $\det = A^n = 2^n$, and $\det A^{\mathrm{T}} = 2$.

2 $\det A = -2$, independent columns; $\det B = 0$, dependent columns; $\det C = -1$, independent columns but $\det D = 0$ because its submatrix B has dependent rows (and dependent columns).

3 The problem suggests 3 ways to see that $\det A = 0$: All cofactors of row 1 are zero. A has rank ≤ 2. Each of the 6 terms in $\det A$ is zero. Notice also that column 2 has no pivot.

4 (a) $A = \begin{bmatrix} 0.9 & -0.9 \\ 0.9 & 0.9 \end{bmatrix}$ has $\det A = 1.62$ and $\det A^n = (1.62)^n \to \infty$.

(b) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has $\det A = 0$ and $\det A^n = 0$ even if $A_{ij} = 2$.

5 (a) $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$, $|B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -6$, $|B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4, |B_1| = 3, |B_2| = -2, |B_3| = 1$. Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.

6 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$. That is because B_2 with $(1, 0, 0)$ in column 2 has $\det B_2 = fg - id$.

7 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0}/\mathbf{0}$: *undetermined*.

8 (a) $x_1 = \det\bigl([\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,]\bigr) / \det A$, if $\det A \neq 0$. This is $|B_1|/|A|$.

(b) The determinant is linear in its first column so $|x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + x_3\,\boldsymbol{a}_3\;\boldsymbol{a}_2\;\boldsymbol{a}_3|$ splits into $x_1|\boldsymbol{a}_1\;\boldsymbol{a}_2\;\boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\;\boldsymbol{a}_2\;\boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\;\boldsymbol{a}_2\;\boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\;\boldsymbol{a}_2\;\boldsymbol{a}_3|$ which is $x_1 \det A$.

9 If the first column in A is also the right side b then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.

10 (a) Area $\begin{vmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{4} \end{vmatrix} = 10$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

11 (a) Area $\frac{1}{2}\begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{vmatrix} = 5 + 7 = 12.$

12 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = \mathbf{16}$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$ in 4 dimensions.)

13 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients come from $(2+x)^n$. Cube from $2I$ has volume 2^n.

14 The pyramid has volume $\dfrac{1}{6}$. The 4-dimensional pyramid has volume $\dfrac{1}{24}$

15 The pattern $\det = 1, 0, -1, -1, 0, 1$ repeats. So $E_{100} = E_4$ after 16 repeats of length 6. And $E_4 = \mathbf{-1}$.

16 Take the determinant of $AC^{\mathrm{T}} = (\det A)I$. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

17 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

18 If the entries are 1 to 9, the maximum determinant **may be** $412 = \begin{vmatrix} 9 & 3 & 5 \\ 4 & 8 & 1 \\ 2 & 6 & 7 \end{vmatrix}$.

19 Orthogonal matrix $\Rightarrow$ box volume $= 1$, singular matrix $\Rightarrow$ box volume $= 0$, box for $2E$ has volume $2^n V$,

20 Geometric proof that $\det A = \det A^{\mathrm{T}}$: A student showed me how to slide the edges of the parallelogram for $\det A$ along themselves to get the parallelogram for $\det A^{\mathrm{T}}$. *No change in area.*

21 The matrix $\begin{bmatrix} u & 0 \\ v & w \end{bmatrix}$ still gives area uw for this parallelogram: Rotate the usual parallelogram by $90°$ to see base $= w$ and height $= u$ and area $= uw$.

Problem Set 5.3, page 199

1 With $\boldsymbol{w} = \mathbf{0}$ linearity gives $T(\boldsymbol{v} + \mathbf{0}) = T(\boldsymbol{v}) + T(\mathbf{0})$. Cancel $T(\boldsymbol{v})$: $T(\mathbf{0}) = \mathbf{0}$. With $c = -1$ linearity gives $T(-\mathbf{0}) = -T(\mathbf{0})$. This is a second proof that $T(\mathbf{0}) = \mathbf{0}$.

2 $T(\boldsymbol{v}) = (4,4)$ and $(2,2)$ and $(2,2)$; if $\boldsymbol{v} = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v}) = b(2,2) + (0,0)$.

3 (d) $T(\boldsymbol{v}) = (0,1) =$ constant and (f) $T(\boldsymbol{v}) = v_1 v_2$ are not linear.

4 (a) $T(S(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(S(\boldsymbol{v}_1) + S(\boldsymbol{v}_2)) = T(S(\boldsymbol{v}_1)) + T(S(\boldsymbol{v}_2))$: linear.

5 Choose $\boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (-1,0)$. Then $T(\boldsymbol{v}) + T(\boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w})$ but $T(\boldsymbol{v} + \boldsymbol{w}) = T(0,1) = (0,0)$.

6 False, unless those n vectors are independent and thus a basis for $\mathbf{R}^n$.

7 (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ does not satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ or $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ (b) and (c) are linear (d) satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ but T is not linear.

8 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

9 $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles the width of the house. $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$ *projects* the house (since $A^2 = A$ from trace $= 1$ and $\lambda = 0, 1$). The projection is onto the column space of $A =$ line through $(.7, .3)$. $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ will *shear* the house horizontally: The point at (x,y) moves over to $(x+y, y)$.

10 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ leaves the house AH sitting straight up (b) $A = 3I$ expands the house by 3 (c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the house.

11 $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ compresses vertical distances by 10 to 1. $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects onto the 45° line. $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ rotates by 45° clockwise and contracts by a factor of $\sqrt{2}$ (the columns have length $1/\sqrt{2}$). $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant -1 so the house is "flipped and sheared." One way to see this is to factor the matrix as LDL^{T}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \text{(shear) (flip left-right) (shear).}$$

12 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0, 0)$.)

13 Circles $||x|| = 1$ are always transformed by A to ellipses (see figure in Section 7.1).

14 (a) $T(\boldsymbol{v}_1) = \boldsymbol{v}_2, T(\boldsymbol{v}_2) = \boldsymbol{v}_1$ is its own inverse (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1, T(\boldsymbol{v}_2) = 0$ has $T^2 = T$ (c) If $T^2 = I$ for part (a) and $T^2 = T$ for part (b), then T must be I.

15 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} r \\ t \end{bmatrix}$ and $\begin{bmatrix} s \\ u \end{bmatrix}$; this is the "easy" direction. (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ transforms in the inverse direction, back to the standard basis vectors. (c) $ad = bc$ will make the forward matrix singular and the inverse impossible.

17 $MW = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 Reordering basis vectors is done by a *permutation matrix*. Changing lengths is done by a *positive diagonal matrix*.

19 Differentiation has no inverse because $\dfrac{d}{dx}(1)$ = derivative of a constant $= 0$.

Problem Set 6.1, page 211

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to $\mathbf{0}$ and $\mathbf{6}$. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 $\det(A - \lambda I) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. Then A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $\boldsymbol{x}_1 = (3, -2)$ and $\boldsymbol{x}_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3 (their diagonal entries : triangular matrices). $A + B$ has $\lambda^2 + 8\lambda + 15 = 0$ and $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda^2 - 4\lambda + 1 = 0$ and the quadratic formula gives $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved at the end of Section 6.2).

7 The eigenvalues of U (on its diagonal) are the *pivots* of A. The eigenvalues of L (on its diagonal) are all 1's. The eigenvalues of A *are not* the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply $A\boldsymbol{x} = \lambda\boldsymbol{x}$ by A : $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$

(b) Multiply by A^{-1} : $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$

(c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 $\det(A-\lambda I)=\lambda^2-1.4\lambda+0.4$ so A has $\lambda_1=1$ and $\lambda_2=0.4$ with $\boldsymbol{x}_1=(1,2)$ and $\boldsymbol{x}_2=(1,-1)$. A^∞ has $\lambda_1=1$ and $\lambda_2=0$ (same eigenvectors as A). A^{100} has $\lambda_1=1$ and $\lambda_2=(0.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Proof 1. $A-\lambda_1 I$ is singular so its two columns are in the same direction. Also $(A-\lambda_1 I)\boldsymbol{x}_2=(\lambda_2-\lambda_1)\boldsymbol{x}_2$. So $\boldsymbol{x}_2$ is in the column space and both columns must be multiples of $\boldsymbol{x}_2$. Here is also a **second proof**: Columns of $A-\lambda_1 I$ are in the nullspace of $A-\lambda_2 I$ because $M=(A-\lambda_2 I)(A-\lambda_1 I)$ is the zero matrix **[**this is the *Cayley-Hamilton Theorem* in Problem 6.2.30**]**. Notice that M has *zero eigenvalues* $(\lambda_1-\lambda_2)(\lambda_1-\lambda_1)=0$ and $(\lambda_2-\lambda_2)(\lambda_2-\lambda_1)=0$. So those columns solve $(A-\lambda_2 I)\,\boldsymbol{x}=\boldsymbol{0}$, they are eigenvectors.

12 The projection matrix P has $\lambda=1,0,1$ with eigenvectors $(1,2,0)$, $(2,-1,0)$, $(0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda=1$. The whole column space of P contains eigenvectors with $\lambda=1$! *Note* $P^2=P$ leads to $\lambda^2=\lambda$ so $\lambda=0$ or 1.

13 (a) $P\boldsymbol{u}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u})=\boldsymbol{u}$ so $P^{100}\boldsymbol{u}=\boldsymbol{u}$ (b) $P\boldsymbol{v}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})=\boldsymbol{0}$ (c) $\boldsymbol{x}_1=(-1,1,0,0)$, $\boldsymbol{x}_2=(-3,0,1,0)$, $\boldsymbol{x}_3=(-5,0,0,1)$ all have $P\boldsymbol{x}=0\boldsymbol{x}=\boldsymbol{0}$.

14 $\det(Q-\lambda I)=\lambda^2-2\lambda\cos\theta+1=0$ when $\lambda=\cos\theta\pm i\sin\theta=e^{i\theta}$ and $e^{-i\theta}$. Check $\lambda_1\lambda_2=\cos^2\theta+\sin^2\theta=1$ and $\lambda_1+\lambda_2=2\cos\theta$. Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1=(1,i)$ and $\boldsymbol{x}_2=(1,-i)$ (or $c\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ with $cd\neq 0$).

15 The other two eigenvalues are $\lambda=\frac{1}{2}(-1\pm i\sqrt{3})$. Those three eigenvalues add to $0=$ trace of P. The three eigenvalues of the second P are $1,1,-1$.

16 Set $\lambda=0$ in $\det(A-\lambda I)=(\lambda_1-\lambda)\ldots(\lambda_n-\lambda)$ to find $\det A=(\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 These 3 matrices have $\lambda=4$ and 5, trace 9, det 20: $\begin{bmatrix}4&0\\0&5\end{bmatrix}$, $\begin{bmatrix}3&2\\-1&6\end{bmatrix}$, $\begin{bmatrix}2&2\\-3&7\end{bmatrix}$.

18 (a) rank $=2$ (b) $\det(B^{\mathrm{T}}B)=0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1,\frac{1}{2},\frac{1}{5}$.

19 $A = \begin{bmatrix} 0 & 1 \\ \mathbf{-28} & \mathbf{11} \end{bmatrix}$ has trace 11 and determinant 28, so $\lambda = 4$ and 7. Moving to a 3 by 3 companion matrix, for eigenvalues $1, 2, 3$ we want $\det(C - \lambda I) = (1 - \lambda)(2 - \lambda)(3 - \lambda)$. Multiply out to get $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$. To get those numbers $6, -11, 6$ from a companion matrix you just put them into the last row:

$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{6} & \mathbf{-11} & \mathbf{6} \end{bmatrix}$ Notice the trace $6 = 1 + 2 + 3$ and determinant $6 = (1)(2)(3)$.

20 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. Pick $M = A - \lambda I$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ have } \textit{different eigenvectors } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

21 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.30.

22 $\lambda = \mathbf{0, 0, 6}$ (*notice rank* 1 *and trace* 6). Two eigenvectors of $\boldsymbol{uv}^{\mathrm{T}}$ are perpendicular to $\boldsymbol{v}$ and the third eigenvector is $\boldsymbol{u}$: $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

23 When A and B have the same n λ's and $\boldsymbol{x}$'s, look at any combination $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. Multiply by A and B: $A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **equals** $B\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ **for all vectors** $\boldsymbol{v}$. So $A = B$.

24 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

25 A is triangular: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; C has rank one: $\lambda(C) = 0, 0, 6$.

26 (a) $\boldsymbol{u}$ is a basis for the nullspace (we know $A\boldsymbol{u} = 0\boldsymbol{u}$); $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space (we know $A\boldsymbol{v}$ and $A\boldsymbol{w}$ are in the column space).

(b) $A(\boldsymbol{v}/3 + \boldsymbol{w}/5) = 3\boldsymbol{v}/3 + 5\boldsymbol{w}/5 = \boldsymbol{v} + \boldsymbol{w}$. So $\boldsymbol{x} = \boldsymbol{v}/3 + \boldsymbol{w}/5$ is a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Add any $c\boldsymbol{u}$ from the nullspace to find all solutions.

(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space : wrong dimension 3.

27 Always $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})\,\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})$ so $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ with $\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$. (Watch numbers $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$, vectors $\boldsymbol{u}$, matrices $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$!!) If $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$ then $A^2 = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{v}^{\mathrm{T}}$ is the zero matrix and $\lambda^2 = 0, 0$ and $\lambda = 0, 0$ and trace $(A) = 0$. This zero trace also comes from adding the diagonal entries of $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{bmatrix} \quad \text{has trace } \; u_1 v_1 + u_2 v_2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$$

28 The six 3 by 3 permutation matrices include $P = I$ and three single row exchange matrices P_{12}, P_{13}, P_{23} and two double exchange matrices like $P_{12}P_{13}$. Since $P^{\mathrm{T}}P = I$ gives $(\det P)^2 = 1$, the determinant of P is 1 or -1. The pivots are always 1 (but there may be row exchanges). The trace of P can be 3 (for $P = I$) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and -1 and $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

29 $AB - BA = I$ can happen only for infinite matrices. If $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = -B$ then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\,(AB - BA)\,\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\,(A^{\mathrm{T}}B + B^{\mathrm{T}}A)\,\boldsymbol{x} \le ||A\boldsymbol{x}||\,||B\boldsymbol{x}|| + ||B\boldsymbol{x}||\,||A\boldsymbol{x}||.$$

Therefore $||A\boldsymbol{x}||\,||B\boldsymbol{x}|| \ge \frac{1}{2}||\boldsymbol{x}||^2$ and $(||A\boldsymbol{x}||/||\boldsymbol{x}||)\,(||B\boldsymbol{x}||/||\boldsymbol{x}||) \ge \frac{1}{2}$.

30 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and $\det$. So does every $M^{-1}AM$!

Problem Set 6.2, page 223

1 Eigenvectors in $\boldsymbol{X}$ and eigenvalues 1 and 3 in Λ. Then $A = X\Lambda X^{-1}$ is

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$ The second matrix has $\lambda = 0$ (rank 1) and

$\lambda = 4$ (trace $= 4$). Then $A = X\Lambda X^{-1}$ is $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

$A^3 = X\Lambda^3 X^{-1}$ and $A^{-1} = X\Lambda^{-1}X^{-1}$.

2 Put the eigenvectors in X and eigenvalues $2, 5$ in Λ. $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 If $A = X\Lambda X^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still X. So $A + 2I = X(\Lambda + 2I)X^{-1} = X\Lambda X^{-1} + X(2I)X^{-1} = A + 2I$.

4 (a) False: We are not given the λ's (b) True (c) True since X has independent columns. (d) False: For this we would need the eigenvectors of X.

5 With $X = I, A = X\Lambda X^{-1} = \Lambda$ is a diagonal matrix. If X is triangular, then X^{-1} is triangular, so $X\Lambda X^{-1}$ is also triangular.

6 The columns of X are nonzero multiples of $(2,1)$ and $(0,1)$: either order. The same eigenvector matrices diagonalize A and A^{-1}.

7 $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

$$X\Lambda^k X^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The second component is $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

8 (a) The equations are $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ with $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$. This matrix has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1,1),\ \boldsymbol{x}_2 = (1,-2)$

(b) $A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

9 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

10 $A = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} \lambda_1+\lambda_2 & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix} /2.$

These are the matrices $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, their eigenvectors are $(1, 1)$ and $(1, -1)$.

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know if $\lambda = 0$ or not.

(b) True: an eigenvector is missing, which can only happen for a repeated eigenvalue.

(c) True: We know there is only one line of eigenvectors.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (the new A will have two different eigenvalues)

15 $A^k = X\Lambda^k X^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; A_1 is a Markov matrix so $\lambda_{\text{max}} = 1$ and $A_1^k \to A_1^\infty$, A_2 has $\lambda = .6 \pm .3$ so $A_2^k \to 0$.

16 $\begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} = X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $A_1^k = X\Lambda^k X^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 A_2 is $X\Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$ and $X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Then $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = X\Lambda X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and

$$A^k = X\Lambda^k X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Multiply those last three matrices to get $A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 $B^k = X\Lambda^k X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 $\det A = (\det X)(\det \Lambda)(\det X^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof ($\det =$ product of λ's) works when A is *diagonalizable*. The formula is always true.

21 trace $XY = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace YX. Diagonalizable case: the trace of $X\Lambda X^{-1} =$ trace of $(\Lambda X^{-1})X =$ trace of $\Lambda = \Sigma\lambda_i$.

22 $A = B\Lambda B^{-1}$ when A has n independent eigenvectors. They go into the columns of $B =$ eigenvector matrix.

23 If $A = X\Lambda X^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}\begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix}$. So B has the original λ's from A and the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$ from $2A$.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same X. When $X = I$ the A's with those eigenvectors give the subspace of **diagonal matrices**. The dimension of that matrix space is 4 since the matrices are 4 by 4.

25 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces $\mathbf{C}(A)$ and $\mathbf{N}(A)$ add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 $R = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\boldsymbol{R^2 = A}$.

$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, the trace (their sum) is not real so $\sqrt{B}$ cannot be real. Note that the square root of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has *two* imaginary eigenvalues $\sqrt{-1} = i$ and $-i$, real trace 0, real square root $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 The factorizations of A and B into $X\Lambda X^{-1}$ are the same. So $A = B$.

29 $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from

$X\Lambda_1 X^{-1} X\Lambda_2 X^{-1} = X\boldsymbol{\Lambda_1\Lambda_2}X^{-1} = X\boldsymbol{\Lambda_2\Lambda_1}X^{-1} = X\Lambda_2 X^{-1} X\Lambda_1 X^{-1} = BA.$

30 (a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\det(A - \lambda I) = \lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

31 When $A = X\Lambda X^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = X(\Lambda - \lambda_j I)X^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. The product $p(A)$ becomes

$$p(A) = (A - \lambda_1 I) \cdots (A - \lambda_n I) = X(\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I)X^{-1}.$$

That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen the following Cayley-Hamilton proof but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.1 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)C^{\mathrm{T}}(\lambda) = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix A for λ. If other matrices B are substituted for λ, does the identity remain true ? If $AB \neq BA$, even the order of multiplication seems unclear . . .

32 If $AB = BA$, then B has the same eigenvectors $(1,0)$ and $(0,1)$ as A. So B is also diagonal $b = c = 0$. The nullspace for the following equation is 2-dimensional:

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{b} \\ \boldsymbol{c} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Those 4 equations $0 = 0, -b = 0, c = 0, 0 = 0$ have a 4 by 4 coefficient matrix with rank $= 4 - 2 = 2$.

33 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1. Then $B^4 = I$ and $B^{1024} = I$.
C has $\lambda = (1 \pm \sqrt{3}i)/2$. This λ is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ which leads to $C^{1024} = (-I)^{341}C = -C$.

34 The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and determinant $\lambda_1\lambda_2 = 1$). Their eigenvectors are $(1, -i)$ and $(1, i)$:

$$A^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} /2i$$

$$= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \cdots \\ (e^{in\theta} - e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

35 Columns of X times rows of ΛX^{-1} gives a sum of r rank-1 matrices ($r =$ rank of A). Those matrices are $\lambda_1 \boldsymbol{x}_1 \boldsymbol{y}_1^{\mathrm{T}}$ to $\lambda_r \boldsymbol{x}_r \boldsymbol{y}_r^{\mathrm{T}}$.

36 Multiply $\mathsf{ones}(n) * \mathsf{ones}(n) = n * \mathsf{ones}(n)$. Then

$$AA^{-1} = (\mathsf{eye}(n) + \mathsf{ones}(n)) * (\mathsf{eye}(n) + C * \mathsf{ones}(n))$$

$$= \mathsf{eye}(n) + (1 + C + Cn) * \mathsf{ones}(n) = \mathsf{eye}(n) \text{ for } C = \boldsymbol{-1/(n+1)}.$$

37 $B = A_1^{-1}$ leads to $A_2A_1 = B(A_1A_2)B^{-1}$. Then A_2A_1 is similar to A_1A_2 : they have the same eigenvectors (not zero because A_1 and A_2 are invertible).

38 This matrix has column $1 = 2$ (column 2) so $\boldsymbol{x}_1 = (1, -2, 0)$ is an eigenvector with $\lambda_1 = 0$. Also $A(1,1,1) = (1,1,1)$ and $\lambda_2 = 1$. Trace = zero so $\lambda_3 = -1$. Then $1^{2020} = 1$ and $(-1)^{2020} = 1$ and $(0)^{2020} = 0$. So A^{2019} has the same eigenvalues and eigenvectors as A: $A^{2019} = A$ and $A^{2020} = A^2$.

Problem Set 6.3, page 238

1 (a) ASB stays symmetric like S when $B = A^{\mathrm{T}}$

(b) ASB is similar to S when $B = A^{-1}$

To have both (a) and (b) we need $B = A^{\mathrm{T}} = A^{-1}$ to be an **orthogonal matrix** Q. Then QSQ^{T} is similar to S and also symmetric like S.

2 $\lambda = 0, 4, -2$; unit vectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$. Those are for S. The eigenvalues of T are $\lambda = 0, \sqrt{5}, -\sqrt{5}$ in Λ (trace $= 0$). The eigenvectors of T are $\frac{1}{3}(2, 2, -1)$ and $(1+\sqrt{5}, 1-\sqrt{5}, 2)$ and $(1-\sqrt{5}, 1+\sqrt{5}, 4)$.

3 $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has $\lambda = 0$ and 25 so the columns of Q are the two eigenvectors: $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or we can exchange columns or reverse the signs of any column.

4 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots $1, 1 - b^2$ have the same signs as the λ's

(c) The trace is $\lambda_1 + \lambda_2 = 2$, so S can't have two negative eigenvalues.

5 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C will be 6 by 6 and the triple product $A^{\mathrm{T}}CA$ is 3 by 3.

6 $\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. Then $S = Q\Lambda Q^{\mathrm{T}}$.

If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is *symmetric* then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ requires $\Lambda = 0$. The only *symmetric* A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

7 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

8 $\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix}$ is an orthogonal matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} =$

$\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = QQ^{\mathrm{T}} = I$; also $P_1P_2 = \boldsymbol{x}_1(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2)\boldsymbol{x}_2^{\mathrm{T}} =$ zero matrix.

Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

9 $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ has $\lambda = ib$ and $-ib$. The block matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are also skew-symmetric with $\lambda = ib$ (twice) and $\lambda = -ib$ (twice).

10 M is skew-symmetric and **orthogonal**; every λ is imaginary with $|\lambda| = 1$. So λ's must be $i, i, -i, -i$ to have trace zero.

11 $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$. The good property for complex matrices is not $A^{\mathrm{T}} = A$ (symmetric) but $\overline{A}^{\mathrm{T}} = A$ (Hermitian with real eigenvalues and orthogonal eigenvectors).

12 S has $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular in Q. Not perpendicular in X since $S^{\mathrm{T}} = S$ but $B^{\mathrm{T}} \neq B$

13 $S = \begin{bmatrix} 1 & 3+4i \\ 3-4i & 1 \end{bmatrix}$ is a *Hermitian matrix* ($\overline{S}^{\mathrm{T}} = S$). Its eigenvalues 6 and -4 are *real*. Here is the proof that λ is always real when $\overline{S}^{\mathrm{T}} = S$:

$S\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $\overline{S}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. Transpose to $\overline{x}^{\mathrm{T}}S = \overline{x}^{\mathrm{T}}\overline{\lambda}$ using $\overline{S}^{\mathrm{T}} = S$.

Then $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x}$ and also $\overline{\boldsymbol{x}}^{\mathrm{T}}S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda}\boldsymbol{x}$. So $\lambda = \overline{\lambda}$ is real.

14 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$ (c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

15 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ is first for A^{T}.

16 A is invertible, orthogonal, permutation, diagonalizable; B is projection, diagonalizable. A allows $QR, X\Lambda X^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $X\Lambda X^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

17 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b=1$; repeated λ and no X if $b=-1$; singular if $b=0$.

18 Orthogonal and symmetric requires $|\lambda|=1$ and λ real, so $\lambda=\pm 1$. Then $S=\pm I$ or
$$\pm S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}.$$

19 Eigenvectors $(1,0)$ and $(\mathbf{1},\mathbf{1})$ give a 45° angle even with A^{T} very close to A.

20 a_{11} is $\begin{bmatrix}q_{11} \ldots q_{1n}\end{bmatrix}\begin{bmatrix}\lambda_1\overline{q}_{11} \ldots \lambda_n\overline{q}_{1n}\end{bmatrix}^{\mathrm{T}} \leq \lambda_{\max}\left(|q_{11}|^2+\cdots+|q_{1n}|^2\right)=\lambda_{\max}$.

21 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x})=(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x}=-\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ so $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}=0$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y}=0+0$ (c) $\det A=\lambda_1\ldots\lambda_n\geq 0$: because pairs of λ's $=ib,-ib$ multiply to give $+b^2$.

22 Since S is diagonalizable with eigenvalue matrix $\Lambda=2I$, the matrix S itself has to be $X\Lambda X^{-1}=X(2I)X^{-1}=2I$. The unsymmetric matrix [2 1 ; 0 2] also has $\lambda=2,2$ but this matrix can't be diagonalized.

23 (a) $S^{\mathrm{T}}=S$ and $S^{\mathrm{T}}S=I$ lead to $S^2=I$.

(b) The only possible eigenvalues of S are 1 and -1.

(c) $\Lambda=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$ so $\boldsymbol{S}=\begin{bmatrix}Q_1 & Q_2\end{bmatrix}\Lambda\begin{bmatrix}Q_1^{\mathrm{T}}\\ Q_2^{\mathrm{T}}\end{bmatrix}=\boldsymbol{Q_1Q_1^{\mathrm{T}}-Q_2Q_2^{\mathrm{T}}}$ with $Q_1^{\mathrm{T}}Q_2=0$.

24 Suppose $a>0$ and $ac>b^2$ so that also $c>b^2/a>0$.

(i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2=\det=ac-b^2>0$.

(ii) That sign is *positive* because $\lambda_1+\lambda_2>0$ (it equals the trace $a+c>0$).

25 Only $S_4=\begin{bmatrix}1 & 10\\ 10 & 101\end{bmatrix}$ has two positive eigenvalues since $101>10^2$.

$\boldsymbol{x}^{\mathrm{T}}S_1\boldsymbol{x}=5x_1^2+12x_1x_2+7x_2^2$ is negative for example when $x_1=4$ and $x_2=-3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has $\det=0$.

26 Positive definite for $-3 < b < 3$
$$\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$$

Positive definite for $c > 8$
$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}.$$

Positive definite for $c > |b|$
$$L = \begin{bmatrix}1 & 0\\ -b/c & 1\end{bmatrix} \quad D = \begin{bmatrix}c & 0\\ 0 & c-b^2/c\end{bmatrix} \quad S = LDL^{\mathrm{T}}.$$

27 $x^2+4xy+3y^2 = (x+2y)^2 - y^2 =$ *difference of squares* is negative at $x = 2, y = -1$, where the first square is zero.

28 $S = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ produces $f(x,y) = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \boldsymbol{2xy}$. S has $\lambda = 1$ and $\lambda = -1$. Then S is an *indefinite matrix* and $f(x,y) = 2xy$ has a *saddle point*.

29 $A^{\mathrm{T}}A = \begin{bmatrix}1 & 2\\ 2 & 13\end{bmatrix}$ and $A^{\mathrm{T}}A = \begin{bmatrix}6 & 5\\ 5 & 6\end{bmatrix}$ are positive definite; $A^{\mathrm{T}}A = \begin{bmatrix}2 & 3 & 3\\ 3 & 5 & 4\\ 3 & 4 & 5\end{bmatrix}$ is singular (and positive semidefinite). The first two A's have independent columns. The 2 by 3 A cannot have full column rank 3, with only 2 rows; third $A^{\mathrm{T}}A$ is singular.

30 $S = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $T = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $T\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

31 Corner determinants $|S_1| = 2$, $|S_2| = 6$, $|S_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

32 S is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$. T is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

33 $S = \begin{bmatrix}1 & 5\\ 5 & 10\end{bmatrix}$ is an example with $a + c > 2b$ but $ac < b^2$, so not positive definite.

34 The eigenvalues of S^{-1} are positive because they are $1/\lambda(S)$. Also the energy is $\boldsymbol{x}^{\mathrm{T}}S^{-1}\boldsymbol{x} = (S^{-1}\boldsymbol{x})^{\mathrm{T}}S(S^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$.

35 $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

36 If a_{jj} were smaller than all λ's, $S - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $S - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 35.

37 (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues

(d) $S = -I$ has $\det = +1$ when n is even, but this S is *negative* definite.

38 S is positive definite when $s > 8$; T is positive definite when $t > 5$ by determinants.

39 $A = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $A = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

40 The ellipse $x^2 + xy + y^2 = 1$ comes from $S = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ with $\lambda = \dfrac{1}{2}$ and $\dfrac{3}{2}$.

The axes have half-lengths $\sqrt{2}$ and $\sqrt{2/3}$.

41 $\begin{matrix} S = C^{\mathrm{T}}C \\ \boldsymbol{S} \text{ not } \boldsymbol{A} \end{matrix} = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

42 The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = S$.

43 (a) $\det S = (1)(10)(1) = 10$; (b) $\lambda = 2$ and 5; (c) $\boldsymbol{x}_1 = (\cos\theta \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; (d) The λ's are positive, so S is positive definite.

44 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$) because the determinant $ac - b^2$ is *negative*.

45 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero along the line $2x + 3y = 0$.

46 A product ST of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $ST = M^{-1}K$. (Often we use **eig**(K, M) without actually inverting M.) All eigenvalues λ of ST are positive:

$$ST\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (T\boldsymbol{x})^{\mathrm{T}}ST\boldsymbol{x} = (T\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}T^{\mathrm{T}}ST\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0.$$

47 Put parentheses in $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \mathbf{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \mathbf{0}$ only happens when $\boldsymbol{x} = \mathbf{0}$. Thus $A^{\mathrm{T}}CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}}CA$ in a wide range of applications. I believe positive definiteness of $A^{\mathrm{T}}CA$ is a unifying concept from linear algebra.

48 (a) The eigenvalues of $\lambda_1 I - S$ are $\lambda_1 - \lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$. Those are ≥ 0; $\lambda_1 I - S$ is semidefinite.

(b) Semidefinite matrices have energy $\boldsymbol{x}^{\mathrm{T}}\,(\lambda_1 I - S)\,\boldsymbol{x}_2 \geq 0$. Then $\lambda_1 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$.

(c) Part (b) says $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq \lambda_1$ for all $\boldsymbol{x}$. Equality at the eigenvector with $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$. So the maximum value of $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is λ_1.

49 Energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = a\,(x_1 + x_2 + x_3)^2 + c\,(x_2 - x_3)^2 \geq 0$ if $a \geq 0$ and $c \geq 0$: semidefinite. S has rank ≤ 2 and determinant $= 0$; cannot be positive definite for any a and c.

Problem Set 6.4, page 253

1 Eigenvalues 4 and 1 with eigenvectors $(1,0)$ and $(1,-1)$ give solutions $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{u}_2 = e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = \begin{bmatrix}5\\-2\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix}$, then use those coefficients 3 and 2 : $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$.

2 $z(t) = 2e^t$ solves $dz/dt = z$ with $z(0) = 2$. Then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 (a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix}-2 & 3\\2 & -3\end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3,2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1,-1)$. Then the usual 3 steps:

1. Write $u(0) = \begin{bmatrix}4\\1\end{bmatrix}$ as $\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2 =$ combination of eigenvectors
2. The solutions follow those eigenvectors: $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{u}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3,2)$ since $e^{-5t} \to 0$.

4 $d(v+w)/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant.
$A = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\\lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

$$\begin{bmatrix}v(0)\\w(0)\end{bmatrix} = \begin{bmatrix}30\\10\end{bmatrix} = 20\begin{bmatrix}1\\1\end{bmatrix} + 10\begin{bmatrix}1\\-1\end{bmatrix} \text{ leads to } \begin{matrix}v(1) = 20 + 10e^{-2} & v(\infty) = 20\\ w(1) = 20 - 10e^{-2} & w(\infty) = 20\end{matrix}$$

5 $\dfrac{d}{dt}\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ has $\lambda = 0$ and $\lambda = \mathbf{+2}$: $v(t) = \mathbf{20 + 10e^{2t}} \to -\infty$ as $t \to \infty$.

6 $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$. In this case the solutions of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ approach zero.

$B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$. In this case all solutions of $d\boldsymbol{v}/dt = B\boldsymbol{v}$ approach zero.

7 A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (c, c)$. Eigenvectors with $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (c, -c)$. For the solution to $d\boldsymbol{u}/dt = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; (somewhat against nature) e^{5t} dominates.

9 (a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ i \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $u(t) = 2e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

10 $\dfrac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$. This correctly gives $y' = y'$ and $y'' = 4y + 5y'$.

$A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

11 The series for e^{At} is $e^{At} = I + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} +$ zeros $= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

Then $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$. This $y(t) = y(0) + y'(0)t$ solves the equation—the factor t tells us that A had only one eigenvector: not diagonalizable.

12 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$. Substitute $y = te^{3t}$ to show that this gives the needed second solution ($y = e^{3t}$ is the first solution).

13 (a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with eigenvectors $x = \begin{bmatrix} 1 \\ 3i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$. Then $\boldsymbol{u}(t) = \frac{3}{2}e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} \mathbf{3\cos 3t} \\ \mathbf{-9\sin 3t} \end{bmatrix}$.

14 When A is skew-symmetric, the derivative of $||u(t)||^2$ is *zero*. Then $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ stays at $\|\boldsymbol{u}(0)\|$. So the matrix e^{At} is *orthogonal* when A is skew-symmetric ($A^{\mathrm{T}} = -A$).

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$. For the matrix equation, the particular solution $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1 e^t \begin{bmatrix} 1 \\ t \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

16 $d/dt(e^{At}) = A + A^2 t + \frac{1}{2}A^3 t^2 + \frac{1}{6}A^4 t^3 + \cdots = A(I + At + \frac{1}{2}A^2 t^2 + \frac{1}{6}A^3 t^3 + \cdots)$. This is exactly Ae^{At}, the derivative we expect from e^{At}.

17 $e^{Bt} = I + Bt$ (short series with $B^2 = 0$) $= \begin{bmatrix} \mathbf{1} & \mathbf{-4t} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. Derivative $= \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = Be^{Bt} = B$ in this example.

18 The solution at time $t + T$ is $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

19 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \frac{1}{6}\boldsymbol{At^3} + \cdots = I + (e^t - 1)A$.

20 $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$ from **21** and $e^B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ from **19**. By direct multiplication $e^A e^B \neq e^B e^A \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$.

21 The matrix has $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$. *Then all* $A^n = A$. So $e^{At} = I + (t + t^2/2! + \cdots)A = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 0 \end{bmatrix}$ as in Problem 19.

22 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$. To see $e^{At}\boldsymbol{x}$, write $(I + At + \frac{1}{2}A^2t^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$.

23 I intend to give this example: $\begin{matrix} dx/dt = & 0x - 4y \\ dy/dt = & -2x + 2y \end{matrix}$ becomes $\begin{matrix} dy/dt = & -2x + 2y \\ dx/dt = & 0x - 4y \end{matrix}$ $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. **The correct matrix for the exchanged** $\boldsymbol{u} = \begin{bmatrix} y \\ x \end{bmatrix}$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

24 Invert $\begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}$ to produce $\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \boldsymbol{U}_n = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix} \boldsymbol{U}_n$.

At $\Delta t = 1$, $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$. Both eigenvalues have $\lambda^6 = 1$ so $\boldsymbol{A^6 = I}$. Therefore $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

25 First A has $\lambda = \pm i$ and $A^4 = I$.
Second A has $\lambda = -1, -1$ and $A^n = (-1)^n \begin{bmatrix} 1 - 2n & -2n \\ 2n & 2n + 1 \end{bmatrix}$ Linear growth.

26 With $a = \Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1} = \dfrac{1}{1 + a^2} \begin{bmatrix} 1 - a^2 & 2a \\ -2a & 1 - a^2 \end{bmatrix} \boldsymbol{U}_n$.

That matrix has orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\| = \|\boldsymbol{U}_n\|$

27 For proof 2, square the start of the series to see $(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3)^2 = I + 2A + \frac{1}{2}(2A)^2 + \frac{1}{6}(2A)^3 + \cdots$. The diagonalizing proof is easiest when it works (but it needs a diagonalizable A).

Problem Set 7.1, page 267

1 $A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{bmatrix} \quad AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ give $\sigma_1 = 8$ and $\sigma_2 = 1$.
$\boldsymbol{v}_1 = (0,0,1)$, $\boldsymbol{v}_2 = (0,1,0)$, $\boldsymbol{u}_1 = (0,1,0)$, $\boldsymbol{u}_2 = (1,0,4)$. After removing row 3 of A and column 3 of A^{T}, $\begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix}$ still has $\sigma_1^2 = 64$ and $\sigma_2^2 = 1$.

2 $\det(B - \lambda I) = -\lambda^3 + \frac{1}{125} = 0$ gives $\lambda = \frac{1}{5}$ times 1 and $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The singular values are $\sigma = 8$ and 1 and $1/1000$. So λ changed by $1/5$ and σ only changed by $1/1000$.

3 A^{T} has the same singular values as A, and the singular vectors change from $A\boldsymbol{v} = \sigma\boldsymbol{u}$ to $A\boldsymbol{u} = \sigma\boldsymbol{v}$.

4 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{v}_k \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v}_k \\ A^{\mathrm{T}}\boldsymbol{u}_k \end{bmatrix} = \sigma_k \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{v}_k \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} -\boldsymbol{u}_k \\ \boldsymbol{v}_k \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v}_k \\ -A^{\mathrm{T}}\boldsymbol{u}_k \end{bmatrix} = -\sigma_k \begin{bmatrix} -\boldsymbol{u}_k \\ \boldsymbol{v}_k \end{bmatrix}$
So the symmetric matrix S reveals the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s and σ's in the SVD of A.

5 $A^{\mathrm{T}}A$ is symmetric with $\sigma_1 = 25$ and $\lambda_2 = 0$ so A has $\sigma_1 = 5$. Its eigenvectors are $\boldsymbol{v}_1 = (2,1)$ and $\boldsymbol{v}_2 = (-1,2)$: *orthogonal*. They are the $\boldsymbol{v}$'s in $A = U\Sigma V^{\mathrm{T}}$.

6 The singular values σ_1, σ_2 of A are the square roots of the eigenvalues λ_1, λ_2 of $A^{\mathrm{T}}A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\lambda^2 - 3\lambda + 1 = 0$ gives $\sigma + 1 = 3\lambda = 3\sigma^2$.

7 There are 20 singular values because a random 20 by 40 matrix almost surely has rank 20.

8 $\dfrac{\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} = \dfrac{\lambda_1 c_1^2 + \cdots + \lambda_n c_n^2}{c_1^2 + \cdots + c_n^2}$ has its maximum λ_1 when $c_1 = 1$ and the other c_k are zero. Then $\boldsymbol{x} = \boldsymbol{v}_1 =$ first singular vector of A. The ratio is a minimum λ_n when $c_n = 1$ and the other c_k are zero. That means $\boldsymbol{x} = \boldsymbol{v}_n =$ last singular vector of A.

9 Requiring $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{v}_1 = 0$ means $c_1 = 0$. Then the maximum ratio comes when $c_2 = 1$ and the other c's are zero. In that case $\boldsymbol{x} = \boldsymbol{v}_2 =$ second singular vector of $A =$ second eigenvector of S.

10 The first matrix has $A^{\mathrm{T}}A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ with $\lambda = 8$ and $\lambda = 2$. The eigenvectors of $A^{\mathrm{T}}A =$ right singular vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A = (1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$. The left singular vectors come from $\boldsymbol{u} = A\boldsymbol{v}/\sigma = (4,0)/\sqrt{2}\sqrt{8} = (1,0)$ and $\boldsymbol{u} = (0,2)/\sqrt{2}\sqrt{2} = (0,1)$.

The second matrix has $A^{\mathrm{T}}A = \begin{bmatrix} 25 & 25 \\ 25 & 25 \end{bmatrix}$ so $\lambda = 50$ and $\lambda = 0$. The right singular vectors of A are again $\boldsymbol{v}_1 = (1,1)/\sqrt{2}$ with $\sigma_1 = \sqrt{50}$ and $\boldsymbol{v}_2 = (1,-1)/\sqrt{2}$ with no σ_2 (or you could say $\sigma_2 = 0$ but our convention is no σ_2). Then $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sqrt{50} = (3,4)/5$.

11 This matrix has $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 3, 1, 0$ and $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ and no σ_3. The eigenvectors of $A^{\mathrm{T}}A$ are $\boldsymbol{v}_1 = (1,2,1)/\sqrt{6}$ and $\boldsymbol{v}_2 = (1,0,-1)/\sqrt{2}$ and $\boldsymbol{v}_3 = (1,-1,1)/\sqrt{3}$. Then $A\boldsymbol{v} = \sigma\boldsymbol{u}$ gives $\boldsymbol{u} = (1,1)/\sqrt{2}$ and $\boldsymbol{u}_2 = (1,-1)/\sqrt{2}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} /\sqrt{6} \\ /\sqrt{2} \end{matrix}$$

12 This small question is a key to everything. It is based on the associative law $(AA^{\mathrm{T}})A = A(A^{\mathrm{T}}A)$. Here we are applying both sides to an eigenvector $\boldsymbol{v}$ of $A^{\mathrm{T}}A$:

$$(AA^{\mathrm{T}})A\boldsymbol{v} = A(A^{\mathrm{T}}A)\boldsymbol{v} = A\lambda\boldsymbol{v} = \lambda A\boldsymbol{v}.$$

So $\boldsymbol{A\boldsymbol{v}}$ **is an eigenvector of** $\boldsymbol{AA^{\mathrm{T}}}$ with the same eigenvalue λ.

13 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

This $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a 2 by 2 matrix of rank 1. Its row space has basis $\boldsymbol{v}_1$, its nullspace has basis $\boldsymbol{v}_2$, its column space has basis $\boldsymbol{u}_1$, its left nullspace has basis $\boldsymbol{u}_2$:

$$\text{Row space} \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Nullspace} \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{Column space} \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{N}(A^{\mathrm{T}}) \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

14 (a) The main diagonal of $A^{\mathrm{T}}A$ contains the squared lengths $||\text{row } 1||^2, \cdots, ||\text{row } m||^2$. So the trace of $A^{\mathrm{T}}A$ is the sum of all a_{ij}^2.

(b) If A has rank 1, then $A^{\mathrm{T}}A$ has rank 1. So the only singular value of A is $\sigma_1 = (\text{trace } A^{\mathrm{T}}A)^{1/2}$.

15 The number $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$ is the same as $\sigma_{\max}(A)/\sigma_{\min}(A)$. This is ≥ 1. It equals 1 if all σ's are equal, and $A = U\Sigma V^{\mathrm{T}}$ is a multiple of an orthogonal matrix. The ratio $\sigma_{\max}/\sigma_{\min}$ is the important **condition number** of A.

16 The smallest change in A is to set its smallest singular value σ_2 to zero.

17 The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A + I)^{\mathrm{T}}(A + I)$. Test the diagonal matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

Problem Set 7.2, page 273

1 All the singular values of I are $\boldsymbol{\sigma} = \mathbf{1}$. We cannot leave out any of the terms $\boldsymbol{u}_i \cdot \boldsymbol{v}_i^{\mathrm{T}}$ without making an error of size 1. And the matrix $A = I$ starts with size 1 ! None of the SVD pieces can be left out.

Notice that the SVD is $I = (U)(I)(U^{\mathrm{T}})$ so that $U = V$. The natural choice for the SVD is just $U\Sigma V^{\mathrm{T}} = III$. But we could actually choose any orthogonal matrix U. (The eigenvectors of I are very far from unique—many choices ! Any orthogonal matrix U holds orthonormal eigenvectors of I.)

One possible rank 3 flag with a cross of zeros is $A = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix}$.

2 $$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \text{pivot} \\ \text{columns} \end{bmatrix} \begin{bmatrix} \text{rows} \\ \text{of } R \end{bmatrix} \text{ is } A = CR$$

3 $$BB^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 13 & 19 \end{bmatrix}. \text{ Trace } \mathbf{28}, \quad \text{Determinant } \mathbf{2}.$$

$$B^{\mathrm{T}}B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 13 & 13 \\ 5 & 13 & 13 \end{bmatrix}. \text{ Trace } \mathbf{28}, \quad \text{Determinant } \mathbf{0}.$$

With a small singular value $\sigma_2 \approx \frac{1}{\sqrt{14}}$, B is compressible. But we don't just keep the first row and column of B. The good row $\boldsymbol{v}_1$ and column $\boldsymbol{u}_1$ are eigenvectors of $B^{\mathrm{T}}B$ and BB^{T}.

4 My hand calculation produced $A^{\mathrm{T}}A = \begin{bmatrix} 7 & 10 & 7 \\ 10 & 16 & 10 \\ 7 & 10 & 7 \end{bmatrix}$ and $\det(A^{\mathrm{T}}A - \lambda I) = -\lambda^3 + 30\lambda^2 - 24\lambda$. This gives $\lambda = 0$ as one eigenvalue of $A^{\mathrm{T}}A$ (correct). For the others,

$$\lambda^2 - 30\lambda + 24 = 0 \quad \text{gives} \quad \lambda = 15 \pm \sqrt{15^2 - 24} \approx 15 \pm 14 = 29 \text{ and } 1.$$

So $\sigma_1 \approx \sqrt{29}$ and $\sigma_2 \approx 1$. The **svd** (A) command in MATLAB will give accurate σ's and U and V.

5 A circle full of 1's contains a square full of 1's. The square touches the circle at angles $\pm 45°$ and $\pm 135°$ measured from angle $0°$ at the x axis. *The square of all* 1*'s only has rank* 1. Now look at the 1's outside that square.

The shapes above and below the square each have about $N - \sqrt{2}N/2$ rows. The shapes right and left of the square each have that same number of columns. (The rows below copy the rows above, and the left columns copy the right columns.) Adding those two equal numbers gives a rank of about $(2 - \sqrt{2})N$ which can be confirmed numerically.

6 For $F_1 = xy$ the matrix entries A_{ij} are ij/N^2. This matrix has column $j = j$ times column 1, so the rank is 1. For $F_2 = x + y$ the matrix has $A_{ij} = (i + j)/N$. This is the sum of two rank -1 matrices and its rank is **2**. For $F_3 = x^2 + y^2$ the matrix has $A_{ij} = (i^2 + j^2)/N^2$, again with rank 2. So A is positive semidefinite with σ's $= \lambda$'s. The trace of A is $2(1^2 + \cdots + N^2)/N^2 \approx 2(N^3/3)/N^2 = 2N/3$. That trace is $\lambda_1 + \lambda_2$ and numerical linear algebra would estimate λ_1 and λ_2.

7 When $A_{ij} = F(i/N, j/N)$, this matrix is symmetric when $F(x, y) = F(y, x)$. The matrix A is antisymmetric when $F(x, y) = -F(y, x)$. The matrix is singular when $F(x, y) = F(x)\,F(y)$. The matrix has rank 2 when $F(x, y) = F_1(x)\,F_1(y) + F_2(x)\,F_2(y)$.

Problem Set 7.3, page 279

1 The row averages of A_0 are 3 and 0. Therefore

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & -2 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad S = \frac{AA^{\mathrm{T}}}{4} = \frac{1}{4}\begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}$$

The eigenvalues of S are $\lambda_1 = \dfrac{10}{4}$ and $\lambda_2 = \dfrac{4}{4} = 1$. The top eigenvector of S is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. I think this means that a **horizontal line** (the x axis) is closer to the five points $(2,-1),\ldots,(-2,-1)$ in the columns of A than any other line through the origin $(0,0)$.

2 Now the row averages of A_0 are $\frac{1}{2}$ and 2. Therefore

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad S = \frac{AA^{\mathrm{T}}}{5} = \frac{1}{5}\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 4 \end{bmatrix}.$$

Again the rows of A are accidentally orthogonal (because of the special patterns of those rows). This time the top eigenvector of S is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So a **horizontal line** is closer to the six points $\left(\frac{1}{2},-1\right),\ldots,\left(-\frac{1}{2},-1\right)$ from the columns of A than any other line through the center point $(0,0)$.

3 $A_0 = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 2 \end{bmatrix}$ has row averages 2 and 3 so $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & -1 \end{bmatrix}$.

Then $S = \dfrac{1}{2}AA^{\mathrm{T}} = \dfrac{1}{2}\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$.

Then trace $(S) = \frac{1}{2}(8)$ and $\det(S) = \left(\frac{1}{2}\right)^2(3)$. The eigenvalues $\lambda(S)$ are $\frac{1}{2}$ times the roots of $\lambda^2 - 8\lambda + 3 = 0$. Those roots are $4 \pm \sqrt{16-3}$. Then the σ's are $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.

4 This matrix A with orthogonal rows has $S = \dfrac{AA^{\mathrm{T}}}{n-1} = \dfrac{1}{3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

With λ's in descending order $\lambda_1 > \lambda_2 > \lambda_3$, the eigenvectors are $(0,1,0)$ and $(0,0,1)$ and $(1,0,0)$. The first eigenvector shows the $\boldsymbol{u}_1$ direction $= y$ axis. Combined with the second eigenvector $\boldsymbol{u}_2$ in the z direction, the best plane is the yz plane.

These problems are examples where the sample **correlation matrix** (rescaling S so all its diagonal entries are 1) would be the identity matrix. If we think the original scaling is not meaningful and the rows should have the same length, then there is no reason to choose $\boldsymbol{u}_1 = (0,1,0)$ from the 8 in row 2.

5 Ordinary least squares is different from PCA = perpendicular least squares.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}b$ is $\begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix}\widehat{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ leads to $\widehat{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 5/14 \end{bmatrix}$. Best line is $y = \dfrac{5}{14}t$.

PCA finds the line through $(0,0)$ whose perpendicular distances to the points $(-3,-1)$, $(1,0)$, $(2,1)$ is smallest. The computation finds the top eigenvector of $A^{\mathrm{T}}A$, where A is now the 2 by 3 matrix of data points:

$$AA^{\mathrm{T}} = \begin{bmatrix} -3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 2 \end{bmatrix} \text{ has } \lambda^2 - 16\lambda + 3 = 0.$$

Then $\lambda = 8 \pm \sqrt{61}$ and the top eigenvector of AA^{T} is in the direction of $(5, \sqrt{61} - 6)$ $\approx \mathbf{(5, 1.8)}$. That is the direction of the line $y = \dfrac{1.8}{5}t$.

6 See **eigenfaces** on Wikipedia.

7 The closest matrix A_3 of rank 3 has the 3 top singular values $5, 4, 3$. Then $A - A_3$ has singular values **2** and **1**.

8 If A has $\sigma_1 = 9$ and B has $\sigma_1 = 4$, then $A + B$ has $\sigma_1 \le 13$ because $||A + B|| \le ||A|| + ||B||$. Also $\sigma_1 \ge 5$ for $A + B$ because $||A + B|| + || - B|| \ge ||A||$.

Problem Set 7.4, page 285

1 $\boldsymbol{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ has $\overline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $||\boldsymbol{v}||^2 = 2$ and $\overline{\boldsymbol{v}}^{\mathrm{T}} = \begin{bmatrix} 1 & -i \end{bmatrix}$ and $||\overline{\boldsymbol{v}}||^2 = 2$.

2 $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos\theta - i\sin\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$ and its conjugate

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos\theta + i\sin\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has eigenvalues 1 and -1.

3 Every permutation matrix has columns of length 1 and those columns are orthogonal. Then $P^{\mathrm{T}}P = I$ and $P^{-1} = P^{\mathrm{T}}$. The inverse is also a permutation matrix.

4 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. Then the four λ's are $1, i, i^2, i^3$. The eigenvectors of P are given in the problem and they are orthogonal (as they must be for an orthogonal matrix like P with no repeated eigenvalues). In this case the eigenvector matrix for P is the Fourier matrix !

5 The eight Haar wavelets are the columns of this matrix W_8.

$$W = W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Then $W^{\mathrm{T}}W$ is a diagonal matrix D with diagonal entries $(8, 8, 4, 4, 2, 2, 2, 2)$. Also $W^{\mathrm{T}}W = D$ gives $W^{-1} = D^{-1}W^{\mathrm{T}}$. The 8 Harr wavelets are the 4 wavelets shown and 4 more wavelets $(1, -1, 0, 0, 0, 0, 0, 0)$, $(0, 0, 1, -1, 0, 0, 0, 0)$, $(0, 0, 0, 0, 1, -1, 0, 0)$ and $(0, 0, 0, 0, 0, 0, 1, -1)$.

Problem Set 8.3, page 319

1 $P^2 = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a}^{\mathrm{T}}/(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})^2 = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = P.$

$P\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x})$ is a multiple of $\boldsymbol{a}$.

$P\boldsymbol{a} = \boldsymbol{a}$ so $\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{x} - P\boldsymbol{x}) = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} - (P\boldsymbol{a})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0}$.

2 Equation (15) in Section 8.3 page 322 includes the term $\dfrac{b_i\boldsymbol{a}_i}{||\boldsymbol{a}_i||^2} = \dfrac{\boldsymbol{a}_i\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}^*}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{a}_i}$ because $A\boldsymbol{x}^* = \boldsymbol{b}$.

3 We have $e_2/e_1 = \cos\theta$ and $e_3/e_2 = \cos\theta$. So $e_3 = e_2\cos\theta = e_1\cos^2\theta$. *Every step reduces the error by* $\cos\theta$.

4 The gradient vector is $\nabla F = (\partial F/\partial x, \partial F/\partial y) = 2(x - y, 2y - x) = (0, 2)$ at $x = 1, y = 1$. So the step starts at $(1, 1)$ and moves in the direction of $(0, 2)$ to the point $(1, 1 - 25) = (1, 0) = (x_1, y_1)$.

5 With $B = 2$ samples x_i and x_k in a step, we minimize the loss $F(x_i) + F(x_k) = ||\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i||^2 + ||\boldsymbol{a}_k^{\mathrm{T}}\boldsymbol{x} - b_k||^2$. The gradient of this sum is $2(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{a}_i + \boldsymbol{a}_k^{\mathrm{T}}\boldsymbol{a}_k)$.

6 Numerical experiment.

7 Numerical experiment.

Problem Set 8.4, page 333

1 $\mu = 20$ and $S^2 = 0$; $\mu = 20.5$ and $S^2 = \dfrac{24}{23}\left(\dfrac{1}{2}\right)^2$.

10 $\mathrm{E}[x^2] = \mathrm{E}[(x-m)^2] - 2m\mathrm{E}[x] + m^2 = \sigma^2 - 2m^2 + m^2 = \sigma^2 - m^2$.