3.2 The Nullspace of A: Solving $Ax = 0$

1 The **nullspace** $\mathbf{N}(A)$ in $\mathbf{R}^n$ contains all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. This includes $\boldsymbol{x} = \mathbf{0}$.

2 Elimination from A to U to R_0 does not change the nullspace: $\mathbf{N}(A) = \mathbf{N}(U) = \mathbf{N}(\boldsymbol{R_0})$.

3 **The reduced row echelon form $\boldsymbol{R_0} = \mathbf{rref}(\boldsymbol{A})$** has $\boldsymbol{I}$ in r columns and $\boldsymbol{F}$ in $n - r$ columns.

4 If column j of R_0 is free (no pivot), there is a "*special solution*" to $A\boldsymbol{x} = \mathbf{0}$ with $x_j = 1$.

5 Every short wide matrix with $m < n$ has nonzero solutions to $A\boldsymbol{x} = \mathbf{0}$ in its nullspace.

This section is about the nullspace containing all solutions to $A\boldsymbol{x} = \mathbf{0}$. The m by n matrix A can be square or rectangular. The right hand side is $\boldsymbol{b} = \mathbf{0}$. *One immediate solution is* $\boldsymbol{x} = \mathbf{0}$. For square invertible matrices this is the only solution. For other matrices, we find $n - r$ special solutions to $A\boldsymbol{x} = \mathbf{0}$. *Each solution $\boldsymbol{x}$ belongs to the nullspace of A.*

Elimination will find all solutions and identify this very important subspace.

***The nullspace* $\mathbf{N}(A)$ *consists of all solutions to* $Ax = \mathbf{0}$. *These vectors* x *are in* $\mathbf{R}^{\boldsymbol{n}}$.**

Check that those vectors form a subspace. Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the nullspace (this means $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$). The rules of matrix multiplication give $A(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{0} + \mathbf{0}$. The rules also give $A(c\boldsymbol{x}) = c\mathbf{0}$. The right sides are still zero. Therefore $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are also in the nullspace $\mathbf{N}(A)$, and the test for a subspace is passed.

To repeat: The solution vectors $\boldsymbol{x}$ have n components. They are vectors in $\mathbf{R}^n$, so *the nullspace is a subspace of* $\mathbf{R}^n$. The column space $\mathbf{C}(A)$ is a subspace of $\mathbf{R}^m$.

Example 1 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular!

Solution Apply elimination to change the linear equations $A\boldsymbol{x} = \mathbf{0}$ to $R_0\boldsymbol{x} = \mathbf{0}$:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 + 2x_2 &= 0 \\ \mathbf{0} &= \mathbf{0} \end{aligned} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \rightarrow R_0 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $\mathbf{N}(A)$. It contains all solutions $(x_1, x_2) = (-2c, c) = \boldsymbol{c}\,(\mathbf{-2}, \mathbf{1})$.

To describe the solutions to $A\boldsymbol{x} = \mathbf{0}$, here is an efficient way. Choose one "***special solution***". Then all solutions are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. **The special solution is $\boldsymbol{s} = (\mathbf{-2}, \mathbf{1})$.**

Special solution $A\boldsymbol{s} = \mathbf{0}$ The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $\boldsymbol{s} = \begin{bmatrix} -2 \\ \mathbf{1} \end{bmatrix}$.

This is the best way to describe the nullspace. **The solution $\boldsymbol{s}$ is special because the free variable is 1.** Simple formulas for $\boldsymbol{R}$ and $\boldsymbol{s}$ come at the end of this Section 3.2.

If $r < n$, $\mathbf{N}(A)$ consists of all combinations of the $n-r$ special solutions to $A\boldsymbol{x} = \mathbf{0}$.

Example 2 $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Then $A\boldsymbol{x} = \mathbf{0}$ produces a plane. All vectors on the plane are perpendicular to $(1, 2, 3)$. *The plane is the nullspace of A.* There are two free variables y and z: Set to 0 and 1.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has } \textbf{two special solutions } \boldsymbol{s}_1 = \begin{bmatrix} -2 \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} -3 \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

Those vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. In this example $A = R_0 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} I & F \end{bmatrix}$.

Notice what is special about $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. *The last two components are "free" and we choose them specially as $1, 0$ and $0, 1$.* Then the first components -2 and -3 are determined by the equation $x + 2y + 3z = 0$.

The solutions to $x + 2y + 3z = \mathbf{6}$ also lie on a plane, but that plane is not a subspace. The vector $\boldsymbol{x} = \mathbf{0}$ is only a solution if $\boldsymbol{b} = \mathbf{0}$. Section 3.3 will show how the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ (if there are any solutions) are shifted away from zero by one particular solution.

Two key steps	$(\mathbf{1})$ **reducing $\boldsymbol{A}$ to its row echelon forms $\boldsymbol{R_0}$ and $\boldsymbol{R}$**
in this section	$(\mathbf{2})$ **finding the $\boldsymbol{n-r}$ special solutions $\boldsymbol{s}$ to** $R\boldsymbol{x} = \mathbf{0}$
Section 3.3 has the final step	$(\mathbf{3})$ **finding a particular solution** to $A\boldsymbol{x} = \boldsymbol{b}$

R is connected to A by $A = CR$. As in Chapter 1, C contains r independent columns. Elimination (row operations) will now take us directly from A to R_0 to R, without C.

Example $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix}$. We can see that column 2 is 2 times column 1. Then columns 1 and 3 are independent and the rank is $r = 2$. But we don't want to use this information! We want a systematic way to find dependent columns for *any matrix A*. That systematic way is *a sequence of row operations on A that will lead directly to R.*

The row operations are like the elimination steps in Chapter 2, leading from A to U (upper triangular). But now **we don't stop at $\boldsymbol{U}$**. We will continue to R_0 and R. We are discovering R before C. **That matrix $\boldsymbol{R}$ will reveal the nullspace of $\boldsymbol{A}$.**

Step 1 Subtract multiples of row 1 to clear out column 1 below the first pivot. Column 2 is also cleared:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & 0 & \mathbf{3} \\ 0 & 0 & 6 \end{bmatrix}$$

Step 2 Divide row 2 by 3, to produce *second pivot* $= 1$. Use it to eliminate 6 and 1:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & \mathbf{2} & 1 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{0} \end{bmatrix} = \boldsymbol{R_0}$$

That matrix R_0 is the **reduced row echelon form**. It has the same rank as A (rank 2). The word *echelon* means that the 1's in R_0 go steadily down, left to right. R_0 has the same row space as A (all our row operations were invertible). R_0 has the same nullspace as A. The equations $R_0\boldsymbol{x} = \mathbf{0}$ are linear combinations of the equations $A\boldsymbol{x} = \mathbf{0}$.

Notice the zero row in R_0. We can and will remove it—no change in the row space or nullspace. **$\boldsymbol{R_0}$ becomes $\boldsymbol{R}$ with no zero rows**. *This is the R we wanted in Chapter* 1.

$$\boldsymbol{R_0} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \boldsymbol{A} = \boldsymbol{CR} \text{ is } \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

C contains the first r independent columns of A (columns 1 and 3)

R has the identity matrix in columns 1 and 3 and F in column 2 : rank $r = 2$

The special solution to $R\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{s} = (-2, 1, 0)$ with free variable $= 1$

The nullspace of A and R_0 and R contains all multiples of that solution $\boldsymbol{s}$

This is the same $\boldsymbol{A} = \boldsymbol{CR} = (\boldsymbol{m} \times \boldsymbol{r})\,(\boldsymbol{r} \times \boldsymbol{n})$ that Section 1.4 would produce by looking for independent columns in C. Now we have a good computational system: **Elimination steps from $\boldsymbol{A}$ to $\boldsymbol{R_0}$ and $\boldsymbol{R}$**, then look for the r by r identity matrix inside R.

By creating R, we know the correct columns of A that go into C. Those columns give the identity matrix I in R. Then $A = CR$ is the result of elimination on *any matrix*, going far beyond $A = LU$ to allow every matrix A.

Pivot Columns and Free Columns of $\boldsymbol{R}$ and $\boldsymbol{A}$

If $R\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = CR\boldsymbol{x} = \mathbf{0}$. There is a special solution $\boldsymbol{x} = \boldsymbol{s}$ for every column of A without a pivot. The r pivots are the 1's in I, leaving $n - r$ free columns of R. Here is the result of elimination on a 4 by 5 matrix when the rank is $r = 3$ and the independent columns of C are $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ and $\boldsymbol{a}_5$. The free columns $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$ are not in C.

$$\underset{4 \times 5}{\boldsymbol{A}} = \begin{bmatrix} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{a}_1 & \boldsymbol{a}_3 & \boldsymbol{a}_5 \\ \\ \end{bmatrix} \begin{bmatrix} \mathbf{1} & p & 0 & q & 0 \\ 0 & 0 & \mathbf{1} & r & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix} = \underset{4 \times 3}{\boldsymbol{C}} \;\; \underset{3 \times 5}{\boldsymbol{R}}$$

You see the 3 by 3 identity matrix in R. Elimination on the 4 by 5 matrix A led to a 4 by 5 matrix R_0. With rank $r = 3$, the fourth row of R_0 was all zeros. Removing that zero row from R_0 produced the perfect factorization $A = CR$.

Elimination on A is complete and it reached R. The remaining step is to read off the $5 - 3 = 2$ special solutions to $R\boldsymbol{x} = \mathbf{0}$.

What are the $n-r=5-3=2$ special solutions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$? Those vectors solve $R\boldsymbol{s}_1=\mathbf{0}$ and $R\boldsymbol{s}_2=\mathbf{0}$. Multiplying by C they also solve $A\boldsymbol{s}_1=\mathbf{0}$ and $A\boldsymbol{s}_2=\mathbf{0}$. **The combinations $c_1\boldsymbol{s}_1+c_2\boldsymbol{s}_2$ fill out the nullspace $\mathbf{N}(A)$.**

To find the special solutions, start with $\boldsymbol{s}_1=(\_,\mathbf{1},\_,\mathbf{0},\_)$ and $\boldsymbol{s}_2=(\_,\mathbf{0},\_,\mathbf{1},\_)$.

We are assigning the values $1,0$ and $0,1$ to the $n-r=5-3=2$ positions that *don't* correspond to the columns $1,3,5$ containing the identity matrix in R. The equations $R\boldsymbol{s}_1=\mathbf{0}$ and $R\boldsymbol{s}_2=\mathbf{0}$ tell us how to fill in the rest of those special solutions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$:

Special solutions to $R\boldsymbol{x}=\mathbf{0}$

$$\boldsymbol{s}_1=\begin{bmatrix}-p\\\mathbf{1}\\0\\\mathbf{0}\\0\end{bmatrix}\quad\text{and}\quad\boldsymbol{s}_2=\begin{bmatrix}-q\\\mathbf{0}\\-r\\\mathbf{1}\\0\end{bmatrix}$$

The nullspace $\mathbf{N}(A)=\mathbf{N}(R)$ contains all $\boldsymbol{x}=c_1\boldsymbol{s}_1+c_2\boldsymbol{s}_2$

Those three numbers $-p$ and $-q$ and $-r$ are just negatives of three numbers in R. Elimination has led systematically to $n-r=2$ independent vectors in the nullspace of R. Those are the two special solutions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ to $R\boldsymbol{x}=\mathbf{0}$ and $A\boldsymbol{x}=\mathbf{0}$.

The free components correspond to columns with no pivots. The special choice (one or zero) is only for the free variables in the special solutions.

Example 3 Find the nullspaces of A, B, M and the two special solutions to $M\boldsymbol{x}=\mathbf{0}$.

$$\boldsymbol{A}=\begin{bmatrix}1&2\\3&8\end{bmatrix}\qquad\boldsymbol{B}=\begin{bmatrix}A\\2A\end{bmatrix}=\begin{bmatrix}1&2\\3&8\\2&4\\6&16\end{bmatrix}\qquad\boldsymbol{M}=\begin{bmatrix}A&2A\end{bmatrix}=\begin{bmatrix}1&2&2&4\\3&8&6&16\end{bmatrix}.$$

Solution The equation $A\boldsymbol{x}=\mathbf{0}$ has only the zero solution $\boldsymbol{x}=\mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $\boldsymbol{x}=\mathbf{0}$ in $\mathbf{R}^2$. This fact comes from elimination:

$$A\boldsymbol{x}=\begin{bmatrix}1&2\\3&8\end{bmatrix}\to\begin{bmatrix}1&2\\0&2\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}=R=I\quad\textbf{No free variables}$$

A is invertible. There are no special solutions. Both columns of this matrix have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $B\boldsymbol{x}=\mathbf{0}$ again require $\boldsymbol{x}=\mathbf{0}$. The last two equations would also force $\boldsymbol{x}=\mathbf{0}$. When we add extra equations (giving extra rows), the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors $\boldsymbol{x}$ in the nullspace.

The rectangular matrix M is different. It has extra columns instead of extra rows. The solution vector $\boldsymbol{x}$ has *four* components. Elimination will produce pivots in the first two columns of M. **The last two columns of M are "free". They don't have pivots.**

$$\boldsymbol{M}=\begin{bmatrix}1&2&2&4\\3&8&6&16\end{bmatrix}\qquad\boldsymbol{U}=\begin{array}{c}\begin{bmatrix}\mathbf{1}&2&2&4\\0&\mathbf{2}&0&4\end{bmatrix}\\\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\end{array}\qquad\boldsymbol{R}=\begin{bmatrix}\mathbf{1}&\mathbf{0}&2&0\\\mathbf{0}&\mathbf{1}&0&2\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}&\boldsymbol{F}\end{bmatrix}$$

pivot columns **free columns**

For the free variables x_3 and x_4, we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are determined by the equation $U\boldsymbol{x} = \mathbf{0}$ (or $R\boldsymbol{x} = \mathbf{0}$). We get two special solutions in the nullspace of M. This is also the nullspace of U and R: *elimination doesn't change solutions.*

Special solutions

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{s_1} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s_2} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \textbf{2 pivot} \\ \leftarrow \quad \textbf{variables} \\ \leftarrow \textbf{2 free} \\ \leftarrow \quad \textbf{variables} \end{array}$$

$$R\boldsymbol{s}_1 = \mathbf{0} \quad R\boldsymbol{s}_2 = \mathbf{0}$$

The Reduced Row Echelon Form $\boldsymbol{R}$

Summary Elimination will not stop at the upper triangular U. We continue to R_0 and R.

1. ***Produce zeros above the pivots.*** **Use pivot rows to eliminate upward.**

2. ***Produce ones in the pivots.*** **Divide the whole pivot row by its pivot.**

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same: $\mathbf{N}(A) = \mathbf{N}(U) = \mathbf{N}(R)$. This nullspace becomes easiest to see when we reach the ***reduced row echelon form. The pivot columns of R contain I.***

$$\textbf{Reduced form } \boldsymbol{R} \qquad U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & 2 & 0 \\ \mathbf{0} & \mathbf{1} & 0 & 2 \end{bmatrix}.$$

I subtracted row 2 of U from row 1. Then I multiplied row 2 by $\frac{1}{2}$ to get pivot $= 1$.

Now (**free column 3**) $=$ **2** (**pivot column 1**), so -2 appears in $\boldsymbol{s}_1 = (-2, 0, 1, 0)$. The special solutions are much easier to find from the reduced system $R\boldsymbol{x} = \mathbf{0}$. In each pivot column of R, change all the signs to find $\boldsymbol{s}$. Second special solution $\boldsymbol{s}_2 = (0, -2, 0, 1)$.

Before moving to m by n matrices A and their nullspaces $\mathbf{N}(A)$ and special solutions, allow me to repeat one comment. For many matrices, the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. Their nullspaces $\mathbf{N}(A) = \mathbf{Z}$ contain only that zero vector: *no* special solutions. The only combination of the columns that produces $\boldsymbol{b} = \mathbf{0}$ is then the "zero combination".

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are **independent**. No combination of columns gives the zero vector (except $\boldsymbol{x} = \mathbf{0}$). But this can't happen if $n > m$. We can't have n independent columns in $\mathbf{R}^m$.

Important Suppose A has more columns than rows. With $n > m$ there is at least one free variable. The system $Ax = 0$ has at least one nonzero solution.

Suppose $A\boldsymbol{x} = \mathbf{0}$ has more unknowns than equations ($\boldsymbol{n} > \boldsymbol{m}$). There must be **at least** $\boldsymbol{n - m}$ **free columns**. $\boldsymbol{Ax = 0}$ **has nonzero solutions** in $\mathbf{N}(A)$.

The nullspace is a subspace. Its "dimension" is the number of free variables. This central idea—the ***dimension of a subspace***—is explained in Section 3.5 of this chapter.

Pivot Variables and Free Variables in the Echelon Matrix $\boldsymbol{R}$

$$\boldsymbol{A} = \begin{bmatrix} p & p & f & p & f \\ | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} \mathbf{1} & 0 & a & 0 & c \\ 0 & \mathbf{1} & b & 0 & d \\ 0 & 0 & 0 & \mathbf{1} & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{s}_1 = \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} \quad \boldsymbol{s}_2 = \begin{bmatrix} -c \\ -d \\ \mathbf{0} \\ -e \\ \mathbf{1} \end{bmatrix}$$

3 pivot columns p	I in pivot columns	special $R\boldsymbol{s}_1 = \mathbf{0}$ and $R\boldsymbol{s}_2 = \mathbf{0}$
2 free columns f	F in free columns	$-a$ to $-e$ come from R
to be revealed by R	3 pivots: rank $r = 3$	$R\boldsymbol{s} = \mathbf{0}$ means $A\boldsymbol{s} = \mathbf{0}$

R shows clearly: *column* $3 = \boldsymbol{a}\,($*column* $1) + \boldsymbol{b}\,($*column* $2)$. The same must be true for A. The special solution $\boldsymbol{s}_1$ repeats that combination so $(-\boldsymbol{a}, -\boldsymbol{b}, \mathbf{1}, 0, 0)$ has $R\boldsymbol{s}_1 = \mathbf{0}$. Nullspace of $A =$ Nullspace of $R =$ all combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

On the next page you will see simple formulas for the echelon matrix $\boldsymbol{R}$ and the $\boldsymbol{n - r}$ special solutions $\boldsymbol{x = s}$ to $\boldsymbol{Ax = 0}$ and $\boldsymbol{Rx = 0}$.

Example This 4 by 7 reduced row echelon matrix $\boldsymbol{R_0}$ has 3 pivots. Delete row 4 to find R.

$$\boldsymbol{R_0} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & x & x & x & \mathbf{0} & x \\ \mathbf{0} & \mathbf{1} & x & x & x & \mathbf{0} & x \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & \mathbf{1} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivot variables x_1, x_2, x_6
Four free variables x_3, x_4, x_5, x_7
Four special solutions $\boldsymbol{s}$ in $\mathbf{N}(R_0) = \mathbf{N}(R)$
The pivot rows and columns contain $\boldsymbol{I}$

$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{P}$ has row 4 removed. The permutation P puts column 3 of I into column 6.

Question What are the column space and the nullspace for this matrix $\boldsymbol{R}$?

Answer The columns of R_0 have four components so they lie in $\mathbf{R}^4$. (Not in $\mathbf{R}^3$!) The fourth component of every column is zero. *The column space of R_0 consists of all vectors of the form* $(b_1, b_2, b_3, 0)$. The nullspace $\mathbf{N}(R) = \mathbf{N}(R_0)$ is a subspace of $\mathbf{R}^7$. The solutions to $R_0\boldsymbol{x} = \mathbf{0}$ are combinations of the four special solutions—*one for each free variable*:

1. Columns $3, 4, 5, 7$ have no pivots. So the four free variables are x_3, x_4, x_5, x_7.
2. Set one free variable to 1 and set the other three free variables to zero.
3. To find each $\boldsymbol{s}$, solve $R\boldsymbol{s} = \mathbf{0}$ for the pivot variables x_1, x_2, x_6. Four special solutions.

To repeat: *A short wide matrix* $(n > m)$ *always has nonzero vectors in its nullspace.* There must be at least $n - m$ free variables, since the number of pivots cannot exceed m.

The Echelon Form and Special Solutions in Matrix Language

From the examples you see the steps to R_0 and R. Chapter 2 produced zeros below the pivots in U. Chapter 3 also has zeros *above* the pivots in R. All pivots are 1. We now have a systematic way to identify independent columns in A and to reach $A = CR$.

This row echelon form is famous, but its simple matrix formula is seldom given. This page will give formulas for R_0 and R, along with the special solutions to $A\boldsymbol{s} = \mathbf{0}$. Those $n - r$ special solutions combine to give the nullspace: all solutions to $A\boldsymbol{x} = \mathbf{0}$. R_0 comes from elimination (down and up) on A. Here are the basic formulas.

$$\boldsymbol{R_0} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{P} \qquad \boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{P} \qquad \boldsymbol{A} = \boldsymbol{CR} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{CF} \end{bmatrix} \boldsymbol{P} \tag{1}$$

That column permutation P puts the columns of I and F into their correct positions in R.

$\boldsymbol{F}$ tells how the independent columns in $\boldsymbol{A}$ combine into the dependent columns

Special solutions to $\boldsymbol{Ax} = \mathbf{0}$ Since A has rank r, we expect $n-r$ independent solutions.

$$\boldsymbol{Ax} = \mathbf{0} \quad \text{gives} \quad \boldsymbol{Rx} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{Px} = \mathbf{0}$$

Here I is r by r and F is r by $n - r$. Thanks to the simplicity of I, and the fact that $PP^{\mathrm{T}} = I$, we know immediately the matrix $\boldsymbol{S}$ of special solutions $\begin{bmatrix} \boldsymbol{s}_1 & \dots & \boldsymbol{s}_{n-r} \end{bmatrix}$.

$$\boxed{\boldsymbol{S} = \boldsymbol{P}^{\mathrm{T}} \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} \quad \text{and} \quad \boldsymbol{RS} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{PS} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \boldsymbol{PP}^{\mathrm{T}} \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \mathbf{0}}$$

S has n rows and $n-r$ columns (special solutions). *The identity matrix in S has size $n-r$.* Each column has a 1, as special solutions always do. The other nonzeros in that column come directly from F, with signs reversed to $-F$. The role of P^{T} is to move the 1's into the right positions (free positions) in these special solutions. If the r independent columns of A come first, then P is the identity matrix and S is truly simple: $RS = -F + F = 0$.

Here is a magic factorization that treats rows and columns of A in the same way. C contains the first r independent columns of A as always. **Suppose $\boldsymbol{R^*}$ contains the first $\boldsymbol{r}$ independent rows of $\boldsymbol{A}$.** (We know that row rank = column rank.) The rows of R^* will meet the columns of C in an r by r matrix W. Then $\boldsymbol{A}$ factors into $\boldsymbol{CW^{-1}R^*}$.

$$\boxed{\boldsymbol{A} = \begin{bmatrix} \\ \boldsymbol{C} \\ \\ \end{bmatrix} \begin{bmatrix} \boldsymbol{W^{-1}} \end{bmatrix} \begin{bmatrix} \quad \boldsymbol{R^*} \quad \end{bmatrix} \text{ as in } \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}}$$

The first columns of $W^{-1}R^*$ will be $W^{-1}W = I$. The last column will be the free part F. The permutation is just $P = I$, since the independent rows and columns came first in A.

$\boldsymbol{W^{-1}R^*}$ is the same matrix as $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$. The free part is $\boldsymbol{F} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{7} \\ \mathbf{8} \end{bmatrix}$.

Three Identical Factorizations $\boldsymbol{A}=\boldsymbol{C}\boldsymbol{R}=\boldsymbol{C}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{F}\end{bmatrix}\boldsymbol{P}=\boldsymbol{C}\boldsymbol{W}^{-1}\boldsymbol{R}^*$

This very optional page completes the presentation of $A = CR$ factorizations—all with the same C and R. C contains the first r independent columns of A, and R^* contains the first r independent rows of A. C and R^* meet in an r by r matrix W. Then $W^{-1} = M$ is the mixing matrix, and a small example from page 32 has grown into the "magic factorization" $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{W}^{-1}\boldsymbol{R}^*$.

$$A=\left[\begin{array}{cc|c}\mathbf{1} & \mathbf{4} & 7\\ \mathbf{2} & \mathbf{5} & 8\\ \hline 3 & 6 & 9\end{array}\right] \qquad R=W^{-1}R^*=\frac{1}{3}\begin{bmatrix}-5 & 4\\ 2 & -1\end{bmatrix}\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\end{bmatrix}=\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{-1}\\ \mathbf{0} & \mathbf{1} & \mathbf{2}\end{bmatrix}=\begin{bmatrix}I & F\end{bmatrix}$$

$\boldsymbol{A}$ = any matrix of rank r	$m \times n$
$\boldsymbol{C}$ = first r independent columns of A	$m \times r$
$\boldsymbol{R}^*$ = first r independent rows of A	$r \times n$
$\boldsymbol{W}$ = intersection of C and R^*	$r \times r$

$$\boldsymbol{A}=\begin{array}{c c}\begin{bmatrix}\boldsymbol{W} & \boldsymbol{H}\\ \boldsymbol{J} & \boldsymbol{K}\end{bmatrix} & \begin{array}{l} r\\ m-r\end{array}\\ \begin{array}{cc} r & n-r\end{array} & \end{array}$$

$$\boldsymbol{C}=\begin{bmatrix}\boldsymbol{W}\\ \boldsymbol{J}\end{bmatrix} \qquad \boldsymbol{R}^*=\begin{bmatrix}\boldsymbol{W} & \boldsymbol{H}\end{bmatrix}$$

Theorem **The $\boldsymbol{r}$ by $\boldsymbol{r}$ matrix $\boldsymbol{W}$ also has rank $\boldsymbol{r}$ and $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{W}^{-1}\boldsymbol{R}^*$.**

1. Combinations V of the rows of R^* must produce the dependent rows in $\begin{bmatrix}\boldsymbol{J} & \boldsymbol{K}\end{bmatrix}$

Then $\begin{bmatrix}\boldsymbol{J} & \boldsymbol{K}\end{bmatrix}=\boldsymbol{V}\boldsymbol{R}^*=\begin{bmatrix}\boldsymbol{V}\boldsymbol{W} & \boldsymbol{V}\boldsymbol{H}\end{bmatrix}$ for some matrix V and $\boldsymbol{C}=\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{V}\end{bmatrix}\boldsymbol{W}$

2. Combinations T of the columns of C must produce the dependent columns in $\begin{bmatrix}\boldsymbol{H}\\ \boldsymbol{K}\end{bmatrix}$

Then $\begin{bmatrix}\boldsymbol{H}\\ \boldsymbol{K}\end{bmatrix}=\boldsymbol{C}\boldsymbol{T}=\begin{bmatrix}\boldsymbol{W}\boldsymbol{T}\\ \boldsymbol{J}\boldsymbol{T}\end{bmatrix}$ for some matrix T and $\boldsymbol{R}^* = \boldsymbol{W}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{T}\end{bmatrix}$

3. $\boldsymbol{A}=\begin{bmatrix}W & H\\ VW & VH\end{bmatrix}=\begin{bmatrix}W & WT\\ VW & VWT\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{V}\end{bmatrix}\begin{bmatrix}\boldsymbol{W}\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{T}\end{bmatrix}=\boldsymbol{C}\boldsymbol{W}^{-1}\boldsymbol{R}^*$

Since A has rank r, its factors must have rank $\geq r$. From their shapes that means rank r. If C and R^* were not in the first r columns and rows of A, then permutations P_R of the rows and P_C of the columns will give $P_R\, A\, P_C = \begin{bmatrix}W & H\\ J & K\end{bmatrix}$ and the proof goes through.

I. Find C and R^* and W and W^{-1} and $R = W^{-1}R^*$ for the transpose of A above.

II. Explain these statements about the rank of augmented matrices $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix}$ and $\begin{bmatrix}C & D\end{bmatrix}$.

The rank of $\boldsymbol{A}$ equals the rank of $\begin{bmatrix}\boldsymbol{A} & \boldsymbol{b}\end{bmatrix}$ if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable.
The rank of $\boldsymbol{C}$ equals the rank of $\begin{bmatrix}\boldsymbol{C} & \boldsymbol{D}\end{bmatrix}$ if and only if $\boldsymbol{C}\boldsymbol{T} = \boldsymbol{D}$ is solvable.

III. If $A = C\,M\,R^*$ has sizes $(m\times r)\,(r\times r)\,(r\times n)$ and rank $A = r$, show that rank $M = r$.