DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 5.1, Page 258

Questions 1–10 are about the "subspace requirements": $\boldsymbol{v}+\boldsymbol{w}$ and $c\boldsymbol{v}$ (and then all linear combinations $c\boldsymbol{v}+d\boldsymbol{w}$) stay in the subspace.

1 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{v}+\boldsymbol{w}$ stays in the set but $\frac{1}{2}\boldsymbol{v}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{v}$ stays in the set but $\boldsymbol{v}+\boldsymbol{w}$ may be outside.

(a) The set of vectors with integer components (adding $v+w$ produces integers, multiplying by $\frac{1}{2}$ may not).

(b) One option for the set is to take two lines through $(0,0)$. Then $c\boldsymbol{v}$ stays on these lines but $\boldsymbol{v}+\boldsymbol{w}$ may not.

2 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1,4,0)$ and $\boldsymbol{w} = (2,2,2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

3 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

4 Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x+y-2z=4$. The origin $(0,0,0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

For the plane $v+y-2z=4$, the sum of $(4,0,0)$ and $(0,4,0)$ is not on the plane. (The key is that this plane does not go through $(0,0,0)$.)

5 Let $\mathbf{P}_0$ be the plane through $(0,0,0)$ parallel to the previous plane $\mathbf{P}$. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

The parallel plane $\mathbf{P}_0$ has the equation $v+y-2z=0$. Pick two points, for example $(2,0,1)$ and $(0,2,1)$, and their sum $(2,2,2)$ is in $\mathbf{P}_0$.

6 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

(a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0,0)$, and $(0,0)$ by itself (b) The subspaces of $\mathbf{R}^4$ are $\mathbf{R}^4$ itself, three-dimensional planes $\boldsymbol{n}\cdot\boldsymbol{v}=0$, two-dimensional subspaces ($\boldsymbol{n}_1\cdot\boldsymbol{v}=0$ and $\boldsymbol{n}_2\cdot\boldsymbol{v}=0$), one-dimensional lines through $(0,0,0,0)$, and $(0,0,0,0)$ by itself.

7 (a) The intersection of two planes through $(0,0,0)$ is probably a _____ but it could be a _____. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a _____ but it could be a _____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S}\cap\mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S}\cap\mathbf{T}$ consists of the vectors that lie in both subspaces. *Check the requirements on $\boldsymbol{v}+\boldsymbol{w}$ and $c\boldsymbol{v}$.*

(a) Two planes through $(0,0,0)$ probably intersect in a line through $(0,0,0)$
(b) The plane and line probably intersect in the point $(0,0,0)$
(c) If $\boldsymbol{v}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{v}+\boldsymbol{y}$ and $c\boldsymbol{v}$ are in both subspaces.

8 Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space $\mathbf{P}+\mathbf{L}$ containing both $\mathbf{P}$ and $\mathbf{L}$ is either _____ or _____.

The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

9 (a) Show that the set of *invertible* matrices in $\mathbf{M}$ is not a subspace.

(b) Show that the set of *singular* matrices in $\mathbf{M}$ is not a subspace.

(a) The invertible matrices do not include the zero matrix, so they are not a subspace
(b) The sum of singular matrices $\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}$ is not singular: not a subspace.

10 True or false (check addition in each case by an example):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}=A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}=-A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}\neq A$) form a subspace.

(a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}}=-A$ do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.

Questions 11–19 are about column spaces $\boldsymbol{C(A)}$ and the equation $\boldsymbol{Av=b}$.

11 Describe the column spaces (lines or planes) of these particular matrices:

$$A=\begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix}\qquad B=\begin{bmatrix}1&0\\0&2\\0&0\end{bmatrix}\qquad C=\begin{bmatrix}1&0\\2&0\\0&0\end{bmatrix}.$$

The column space of A is the x-axis $=$ all vectors $(x,0,0)$. The column space of B is the xy plane $=$ all vectors $(x,y,0)$. The column space of C is the line of vectors $(x,2x,0)$.

12 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + 2b_3$ in equation 3: Solution only if $b_3 = -b_1$.

13 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. Which matrices have the same column space? Which have the same *row space*?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space.

14 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

15 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{v} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger? Then it is the same for A and $[\,A \;\; \boldsymbol{b}\,]$.

The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$[\,A \;\; \boldsymbol{b}\,] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{v} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ ($\boldsymbol{b}$ is in column space) ($A\boldsymbol{v} = \boldsymbol{b}$ has a solution)

16 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B = 0$ and $A \neq 0$ is a case when $AB = 0$ has a smaller column space than A.

17 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, then $\boldsymbol{b} + \boldsymbol{b}^*$ is also in $\boldsymbol{C}(A)$.

The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

18 If A is any 5 by 5 invertible matrix, then its column space is ____ . Why ?

The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{v} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

19 True or false (with a counterexample if false) :

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

(a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$. (d) *False*: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

20 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

$A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $(1, 1, 1)$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line.

21 If the 9 by 12 system $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A)$ must be ____ .

When $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\mathbf{R}^9$.

Challenge Problems

22 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$. The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Then $\mathbf{S} + \mathbf{T}$ is a vector space.

If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and all vectors from $\mathbf{T}$. Explain this statement : *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

(a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

23 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M ? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \quad B\,]$.

24 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (this has extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$.

The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

25 An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an _____ matrix.

(Key question) The column space of an n by n matrix A is all of R^n exactly when $\boldsymbol{A}$ **is invertible**. In this invertible case, every vector $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$ because we can solve $A\boldsymbol{v} = \boldsymbol{b}$. And if A were not invertible, elimination would lead to a row of zeros—then $A\boldsymbol{v} = \boldsymbol{b}$ could not be solved for some (most !) vectors b.

Problem Set 5.2, Page 269

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

1 Reduce these matrices to their ordinary echelon forms U :

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables ?

(a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables v_2, v_4, v_5 Pivot variables v_1, v_3 (b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free v_3 Pivot v_1, v_2

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

(a) Free variables v_2, v_4, v_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$
(b) Free variable v_3: solution $(1, -1, 1)$. Special solution for each free variable.

3 By combining the special solutions in Problem 2, describe every solution to $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when there are no _____.

The complete solution to $A\boldsymbol{v} = \mathbf{0}$ is $(-2v_2, v_2, -2v_4 - 3v_5, v_4, v_5)$ with v_2, v_4, v_5 free. The complete solution to $B\boldsymbol{v} = \mathbf{0}$ is $(2v_3, -v_3, v_3)$. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when there are no free variables.

4 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false* : The nullspace of R equals the nullspace of U.

$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

5 By row operations reduce this new A and B to triangular echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}.$$

$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = LU$.

6 For the same A and B, find the special solutions to $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is _____.

(a) Special solutions $(3, 1, 0)$ and $(5, 0, 1)$ (b) $(3, 1, 0)$. Total of pivot and free is n.

7 In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors $\boldsymbol{v}$ that satisfy those equations as combinations of the special solutions.

(a) The nullspace of A in Problem 5 is the plane $-v + 3y + 5z = 0$; it contains all the vectors $(3y + 5z, y, z) = y(3, 1, 0) + z(5, 0, 1) =$ combination of special solutions. (b) The *line* through $(3, 1, 0)$ has equations $-v + 3y + 5z = 0$ and $-2v + 6y + 7z = 0$. The special solution for the free variable v_2 is $(3, 1, 0)$.

8 Reduce the echelon forms U in Problem 5 to R. For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

$R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Questions 9–17 are about free variables and pivot variables.

9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

(a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

10 Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

(a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

11 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) 2, 4, 5

(b) 1, 3, 6, 7

(c) 4 and 6.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

12 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) 2, 4, 5, 6

(b) 1, 3, 6, 7, 8.

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

13 Suppose column 4 of a 3 by 5 matrix is all zero. Then v_4 is certainly a _____ variable. The special solution for this variable is the vector $\boldsymbol{s} =$ _____.

If column 4 of a 3 by 5 matrix is all zero then v_4 is a *free* variable. Its special solution is $\boldsymbol{v} = (0,0,0,1,0)$, because 1 will multiply that zero column to give $A\boldsymbol{v} = \mathbf{0}$.

14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.

If column 1 = column 5 then v_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

15 Suppose an m by n matrix has r pivots. The number of special solutions is _____. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when $r =$ _____. The column space is all of $\mathbf{R}^m$ when $r =$ _____.

If a matrix has n columns and r pivots, there are $\boldsymbol{n} - \boldsymbol{r}$ special solutions. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when $\boldsymbol{r} = \boldsymbol{n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All important!

16 The nullspace of a 5 by 5 matrix contains only $\boldsymbol{v} = \mathbf{0}$ when the matrix has _____ pivots. The column space is $\mathbf{R}^5$ when there are _____ pivots. Explain why.

The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{v} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

17 The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are $(3,1,0)$ and _____.

$A = [1 \ \ -3 \ \ -1]$ gives the plane $v - 3y - z = 0$; y and z are free variables. The special solutions are $(3,1,0)$ and $(1,0,1)$.

18 (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Fill in **12** then **4** then **1** to get the complete solution to $v - 3y - z = 12$: $\begin{bmatrix} v \\ y \\ z \end{bmatrix} =$ $\begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{4} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{v}_{\text{nullspace}}$.

19 Prove that U and $A = LU$ have the same nullspace when L is invertible:

$$\text{If } U\boldsymbol{v} = \mathbf{0} \text{ then } LU\boldsymbol{v} = \mathbf{0}. \quad \text{If } LU\boldsymbol{v} = \mathbf{0}, \text{ how do you know } U\boldsymbol{v} = \mathbf{0}\,?$$

If $LU\boldsymbol{v} = \mathbf{0}$, multiply by L^{-1} to find $U\boldsymbol{v} = \mathbf{0}$. Then U and LU have the same nullspace.

20 Suppose column 1 + column 3 + column $5 = \mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

Questions 21–28 ask for matrices (if possible) with specific properties.

21 Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

For special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables v_3, v_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

22 Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through $(4, 3, 2, 1)$.

23 Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $(1, 1, 5)$ and $(0, 3, 1)$ in $\boldsymbol{C}(A)$ and $(1, 1, 2)$ in $\boldsymbol{N}(A)$. *Which other A's?*

24 Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

25 Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1,1,1)$ in $\boldsymbol{C}(A)$ and only the line (c,c,c,c) in $\boldsymbol{N}(A)$.

26 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

27 Why does no 3 by 3 matrix have a nullspace that equals its column space ?

If nullspace = column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

28 (Important) If $AB = 0$ then the column space of B is contained in the _____ of A. Give an example of A and B.

If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ *equals* $\boldsymbol{N}(A)$. (For $B = 0, \boldsymbol{C}(B)$ is smaller.)

29 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What reduced form R is virtually certain if the random A is 4 by 3 ?

For A = random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with fourth row of zeros. What about a random 3 by 4 matrix?

30 Show by example that these three statements are generally *false* :

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form of A then R^{T} is the reduced form of A^{T}.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31 If the nullspace of A consists of all multiples of $\boldsymbol{v} = (2,1,0,1)$, how many pivots appear in U ? What is R ?

If $\boldsymbol{N}(A)$ = line through $\boldsymbol{v} = (2,1,0,1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

32 If the special solutions to $R\boldsymbol{v} = \mathbf{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R :

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \text{ (empty 3 by 1).}$$

Any zero rows come after these rows: $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) What are the five 2 by 2 reduced echelon matrices R whose entries are all 0's and 1's?

(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R?

(a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's!

34 Explain why A and $-A$ always have the same reduced echelon form R.

One reason that R is the same for A and $-A$: They have the same nullspace. They also have the same column space, but that is not required for two matrices to share the same R. (R tells us the nullspace and row space.)

Challenge Problems

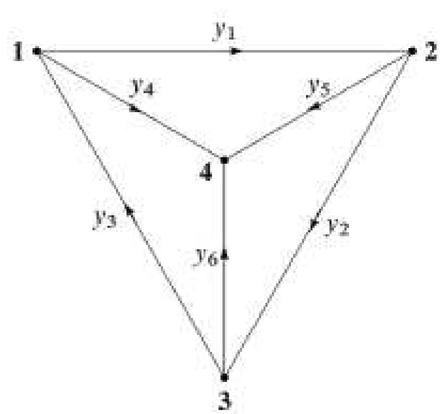
35 If A is 4 by 4 and invertible, describe all vectors in the nullspace of the 4 by 8 matrix $B = [A \;\; A]$.

The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{y}\\-\boldsymbol{y}\end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 How is the nullspace $\boldsymbol{N}(C)$ related to the spaces $\boldsymbol{N}(A)$ and $\boldsymbol{N}(B)$, if $C = \begin{bmatrix}A\\B\end{bmatrix}$?

If $C\boldsymbol{v} = \mathbf{0}$ then $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

37 Kirchhoff's Law says that *current in* $=$ *current out* at every node. This network has six currents $y_1, \ldots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $A\boldsymbol{y} = \mathbf{0}$ for Kirchhoff's Law at the four nodes. Reduce to $U\boldsymbol{y} = \mathbf{0}$. Find three special solutions in the nullspace of A.



Currents: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 5.3, Page 280

1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$$

$A\boldsymbol{v} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $s_1 = (-1, -1, 1, 0)$ and $s_2 = (2, -2, 0, 1)$; $v_{complete} = v_p + c_1 s_1 + c_2 s_2$;

$$[\,R \quad d\,] = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ gives the particular solution } v_p = (4, -1, 0, 0).$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $A\boldsymbol{v} = \boldsymbol{b}$ to be solvable. Together these two conditions put $\boldsymbol{b}$ into the _____ space.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} [\,2 \quad 1 \quad 3\,] = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix} \quad \text{Then } [\,R \quad \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$A\boldsymbol{v} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A)$ = line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{v}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{v}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

Questions 3–15 are about the solution of $A\boldsymbol{v} = \boldsymbol{b}$. Follow the steps in the text to $\boldsymbol{v_p}$ and $\boldsymbol{v_n}$. Start from the augmented matrix $[\,\boldsymbol{A} \quad \boldsymbol{b}\,]$.

3 Write the complete solution as $\boldsymbol{v}_p$ plus any multiple of $\boldsymbol{s}$ in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

$\boldsymbol{v}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. The matrix is singular but the equations are still solvable; $\boldsymbol{b}$ is in the column space. Our particular solution has free variable $y = 0$.

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$\boldsymbol{v}_{\text{complete}} = \boldsymbol{v}_p + \boldsymbol{v}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + v_2(-3, 1, 0, 0) + v_4(0, 0, -2, 1)$.

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include $\boldsymbol{b}$ as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$. Back-substitution gives the particular solution to $A\boldsymbol{v} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{v} = \boldsymbol{0}$: $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find $\boldsymbol{v}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{v}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination $y_1(\text{row } 1) + y_2(\text{row } 2) + y_3(\text{row } 3)$ gives the zero row?

$\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[\,0\ \ 0\ \ 0\ \ 0\,] =$ row 3 $-$ 2 (row 2) $+$ 4(row 1) **provided $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$.**

8 Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

(a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$.

(a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) We need $b_3 = 2b_2$, because $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$.

9 In Worked Example **5.3 A**, combine the pivot columns of A with the numbers -9 and 3 in the particular solution $\boldsymbol{v}_p$. What is that linear combination and why?

$$L[U \quad \boldsymbol{c}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix}$$

$= [A \quad b]$; particular $\boldsymbol{v}_p = (-9, 0, 3, 0)$ means $-9(1, 2, 3) + 3(3, 8, 7) = (0, 6, -6)$. This is $A\boldsymbol{v}_p = \boldsymbol{b}$.

10 Construct a 2 by 3 system $A\boldsymbol{v} = \boldsymbol{b}$ with particular solution $\boldsymbol{v}_p = (2, 4, 0)$ and null (homogeneous) solution $\boldsymbol{v}_n =$ any multiple of $(1, 1, 1)$.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$.

11 Why can't a 1 by 3 system have $\boldsymbol{v}_p = (2, 4, 0)$ and $\boldsymbol{v}_n =$ any multiple of $(1, 1, 1)$?

A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{v} = \boldsymbol{b}$ has two solutions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, find two solutions to $A\boldsymbol{v} = \mathbf{0}$.

(b) Then find another solution to $A\boldsymbol{v} = \boldsymbol{b}$.

(a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 Explain why these are all false:

(a) The complete solution is any linear combination of $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$.

(b) A system $A\boldsymbol{v} = \boldsymbol{b}$ has at most one particular solution.

(c) The solution $\boldsymbol{v}_p$ with all free variables zero is the shortest solution (minimum length $\|\boldsymbol{v}\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution $\boldsymbol{v}_n$ in the nullspace.

(a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 Suppose column 5 has no pivot. Then $\boldsymbol{v}_5$ is a _____ variable. The zero vector (is) (is not) the only solution to $A\boldsymbol{v} = \mathbf{0}$. If $A\boldsymbol{v} = \boldsymbol{b}$ has a solution, then it has _____ solutions.

If column 5 has no pivot, v_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 Suppose row 3 has no pivot. Then that row is _____. The equation $R\boldsymbol{v} = \boldsymbol{d}$ is only solvable provided _____. The equation $A\boldsymbol{v} = \boldsymbol{b}$ (*is*) (*is not*) (*might not be*) solvable.

If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $\boldsymbol{c}_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

Questions 16–21 are about matrices of "full rank" $r = m$ or $r = n$.

16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ (*always exists*) (*is unique*). The column space of A is _____. An example is $A =$ _____.

The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

17 The largest possible rank of a 6 by 4 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ (*always exists*) (*is unique*). The nullspace of A is _____. An example is $A =$ _____.

The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The solution is *unique*. The nullspace contains only the zero *vector*. An example is $A = R = [\,I \;\; F\,]$ for any 4 by 2 matrix F.

18 Find by elimination the rank of A and also the rank of A^{T}:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \text{ (rank depends on } q).$$

Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 Find the rank of A and also of $A^{\mathrm{T}}A$ and also of AA^{T}:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Both matrices A have rank 2. Always $A^{\mathrm{T}}A$ and AA^{T} have **the same rank** as A.

20 Reduce A to its echelon form U. Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

$$A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}; A = LU \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathbf{2} & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}.$$

21 Find the complete solution in the form $\boldsymbol{v}_p + \boldsymbol{v}_n$ to these full rank systems:

(a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution.

22 If $A\boldsymbol{v} = \boldsymbol{b}$ has infinitely many solutions, why is it impossible for $A\boldsymbol{v} = \boldsymbol{B}$ (new right side) to have only one solution? Could $A\boldsymbol{v} = \boldsymbol{B}$ have no solution?

If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

24 Give examples of matrices A for which the number of solutions to $A\boldsymbol{v} = \boldsymbol{b}$ is

(a) 0 or 1, depending on $\boldsymbol{b}$

(b) ∞, regardless of $\boldsymbol{b}$

(c) 0 or ∞, depending on $\boldsymbol{b}$

(d) 1, regardless of $\boldsymbol{b}$.

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\,x\,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $[1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\,b\,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

25 Write down all known relations between r and m and n if $A\boldsymbol{v} = \boldsymbol{b}$ has

(a) no solution for some $\boldsymbol{b}$

(b) infinitely many solutions for every $\boldsymbol{b}$

(c) exactly one solution for some $\boldsymbol{b}$, no solution for other $\boldsymbol{b}$

(d) exactly one solution for every $\boldsymbol{b}$.

(a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix $\boldsymbol{R}$.

26 Continue elimination from U to R. Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R:

$$U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R = I.$$

27 Suppose U is square with n pivots (an invertible matrix). *Explain why* $R = I$.

If U has n pivots, then R has n pivots **equal to 1**. Zeros above and below those pivots make $R = I$.

28 Apply Gauss-Jordan elimination to $U\boldsymbol{v} = \mathbf{0}$ and $U\boldsymbol{v} = \boldsymbol{c}$. Reach $R\boldsymbol{v} = \mathbf{0}$ and $R\boldsymbol{v} = \boldsymbol{d}$:

$$\begin{bmatrix} U & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $R\boldsymbol{v} = \mathbf{0}$ to find $\boldsymbol{v}_n$ (its free variable is $\boldsymbol{v}_2 = 1$). Solve $R\boldsymbol{v} = \boldsymbol{d}$ to find $\boldsymbol{v}_p$ (its free variable is $\boldsymbol{v}_2 = 0$).

$$\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix};\ \boldsymbol{v}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix};\ \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}.$$

Free $v_2 = 0$ gives $\boldsymbol{v}_p = (-1, 0, 2)$ because the pivot columns contain I.

29 Apply Gauss-Jordan elimination to reduce to $R\boldsymbol{v} = \mathbf{0}$ and $R\boldsymbol{v} = \boldsymbol{d}$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $U\boldsymbol{v} = \mathbf{0}$ or $R\boldsymbol{v} = \mathbf{0}$ to find $\boldsymbol{v}_n$ (free variable $= 1$). What are the solutions to $R\boldsymbol{v} = \boldsymbol{d}$?

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \text{ leads to } \boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}:$$

no solution because of the 3rd equation

30 Reduce to $U\boldsymbol{v} = \boldsymbol{c}$ (Gaussian elimination) and then $R\boldsymbol{v} = \boldsymbol{d}$ (Gauss-Jordan):

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \boldsymbol{b}.$$

Find a particular solution $\boldsymbol{v}_p$ and all homogeneous (null) solutions $\boldsymbol{v}_n$.

$$\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix};\ \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix};\ \boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

31 Find matrices A and B with the given property or explain why you can't:

(a) The only solution of $A\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$.

(b) The only solution of $B\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$.

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 Reduce $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ to $\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$ and find the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \boldsymbol{0}$ and $U\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since $\boldsymbol{b} = (1, 3, 6, 5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0, 0, 1)$ and the complete solution is $\boldsymbol{x} = (0, 0, 1) + c\boldsymbol{s}$. (Or use the particular solution $\boldsymbol{x}_p = (7, -2, 0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1, 0, 0, 0)$ elimination leads to $U\boldsymbol{x} = (1, -1, 0, 1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

33 The complete solution to $A\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + c \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$. Find A.

If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

Challenge Problems

34 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as the only special solution to $A\boldsymbol{v} = \boldsymbol{0}$.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{v} = \boldsymbol{0}$?

(b) What is the exact row reduced echelon form R of A? Good question.

(c) How do you know that $A\boldsymbol{v} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$?

(a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \boldsymbol{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (line of solution!). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solve for all $\boldsymbol{b}$, because A and R have *full row rank* $r = 3$.

35 If you have this information about the solutions to $A\boldsymbol{v} = \boldsymbol{b}$ for a specific $\boldsymbol{b}$, what does that tell you about the *shape* of A (m and n)? And possibly about $\boldsymbol{r}$ and $\boldsymbol{b}$.

1. There is exactly one solution.

2. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix} + c\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$.

3. There are no solutions.

4. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} + c\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

5. There are infinitely many solutions.

1. $r = n$ (no special solutions) and $\boldsymbol{b}$ is in the column space

2. $n - r = 1$ (one special solution)

3. $\boldsymbol{b}$ is not in the column space (so $r < m$)

4. Same conclusion as part **2**

5. $r < n$ (there are special solutions) and $\boldsymbol{b}$ is in the column space

36 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $C\boldsymbol{v} = \boldsymbol{b}$ have the same (complete) solutions for every $\boldsymbol{b}$. Is it true that $A = C$?

If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b}$ = column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 5.4, page 295

Questions 1–10 are about linear independence and linear dependence.

1 Show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are independent but $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are dependent:

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 + c_4\boldsymbol{u}_4 = \mathbf{0}$ or $A\boldsymbol{c} = \mathbf{0}$. The $\boldsymbol{u}$'s go in the columns of A. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}[\,\boldsymbol{c}\,] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{u}_1 + \boldsymbol{u}_2 - 4\boldsymbol{u}_3 + \boldsymbol{u}_4 = \mathbf{0}$ (dependent).

2 (Recommended) Find the largest possible number of independent vectors among

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \boldsymbol{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \boldsymbol{u}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{u} = 0$ so no four of these six vectors can be independent.

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $U\boldsymbol{v} = \mathbf{0}$ is $\boldsymbol{v} = \mathbf{0}$. Then the upper triangular U has independent columns.

$U\boldsymbol{v} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$. A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, columns add to $\mathbf{0}$.

6 Choose three independent columns of U and A. Then make two other choices.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

7 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero. Which singular matrix gives $[\, \boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3 \,] = [\, \boldsymbol{w}_1 \;\; \boldsymbol{w}_2 \;\; \boldsymbol{w}_3 \,]\, A$?

The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the difference are *dependent* and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

8 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the sums $\boldsymbol{v}_1 = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 + \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ are *independent*. (Write $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$ in terms of the $\boldsymbol{w}$'s. Find and solve equations for the c's, to show they are zero.)

If $c_1(\boldsymbol{w}_2+\boldsymbol{w}_3)+c_2(\boldsymbol{w}_1+\boldsymbol{w}_3)+c_3(\boldsymbol{w}_1+\boldsymbol{w}_2) = \mathbf{0}$ then $(c_2+c_3)\boldsymbol{w}_1+(c_1+c_3)\boldsymbol{w}_2+(c_1+c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

9 Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because _____ .
(b) The two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ will be dependent if _____ .
(c) The vectors $\boldsymbol{u}_1$ and $(0, 0, 0)$ are dependent because _____ .

(a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\,\boldsymbol{u}_1 \;\; \boldsymbol{u}_2\,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* "$\boldsymbol{u}_2$ is a multiple of $\boldsymbol{u}_1$" —since $\boldsymbol{u}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{u}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{u}_1 + 3(0, 0, 0) = \mathbf{0}$.

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

The plane is the nullspace of $A = [1 \;\; 2 - 3 - 1]$. Three free variables give three solutions $(x, y, z, t) = (2, -1 - 0 - 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

Questions 11–14 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors, to find the space they span.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$
(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$
(c) all vectors in $\mathbf{R}^3$ with whole number components
(d) all vectors with positive components.

(a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when _____ has a solution. The vector $\boldsymbol{c}$ is in the row space of A when _____ has a solution.

True or false: If the zero vector is in the row space, the rows are dependent.

$\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False*. The zero vector is always in the row space.

13 Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A (b) column space of U (c) row space of A (d) row space of U:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The column space and row space of A and U all have the same dimension = 2. *The row spaces of A and U are the same*, because the rows of U are combinations of the rows of A (and vice versa!).

14 $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ are combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Write $\boldsymbol{v}$ and $\boldsymbol{w}$ as combinations of $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$. The two pairs of vectors _____ the same space. When are they a basis for the same space?

$\boldsymbol{v}=\frac{1}{2}(\boldsymbol{v}+\boldsymbol{w})+\frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$ and $\boldsymbol{w}=\frac{1}{2}(\boldsymbol{v}+\boldsymbol{w})-\frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$. The two pairs *span* the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent*.

Questions 15–25 are about the requirements for a basis.

15 If $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are linearly independent, the space they span has dimension _____. These vectors are a _____ for that space. If the vectors are the columns of an m by n matrix, then m is _____ than n. If $m=n$, that matrix is _____.

The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \ge n)$.

16 Suppose $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do) (do not) (might not) span $\mathbf{R}^4$.

(b) Those vectors (are) (are not) (might be) linearly independent.

(c) Any four of those vectors (are) (are not) (might be) a basis for $\mathbf{R}^4$.

(a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent
(c) Any four *might be* a basis.

17 Find three different bases for the column space of $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$. Then find two different bases for the row space of U.

The column space of $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1+$ row 2) and (row 1 and $-$ row 2) are bases for the row spaces of U.

18 Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space and the nullspace of I $(4 \text{ by } 4)$.

These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\boldsymbol{N}(I)=\{\text{zero vector}\}$.

19 The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then? *Looking ahead*: The rank r counts the number of _____ columns.

n-independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m$ $\Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ *rank* $=m=n$. The rank counts the number of *independent* columns.

20 Find a basis for the plane $x-2y+3z=0$ in $\mathbf{R}^3$. Find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $A\boldsymbol{v}=\mathbf{0}$ has only the solution $\boldsymbol{v}=\mathbf{0}$ because ______.

(b) If $\boldsymbol{b}$ is in $\mathbf{R}^5$ then $A\boldsymbol{v}=\boldsymbol{b}$ is solvable because the basis vectors ______ $\mathbf{R}^5$.

Conclusion: A is invertible. Its rank is 5. Its rows are also a basis for $\mathbf{R}^5$.

(a) The only solution to $A\boldsymbol{v}=\mathbf{0}$ is $\boldsymbol{v}=\mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{v}=\boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 Suppose $\mathbf{S}$ is a 5-dimensional subspace of $\mathbf{R}^6$. True or false (example if false):

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

(a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

23 U comes from A by subtracting row 1 from row 3:

$$A=\begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix}\quad\text{and}\quad U=\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces.

24 True or false (give a good reason):

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

(a) *False* $A=[\,1\;\;1\,]$ has dependent columns, independent row (b) *False* column space $\neq$ row space for $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ (c) *True*: Both dimensions $=2$ if A is invertible, dimensions $=0$ if $A=0$, otherwise dimensions $=1$ (d) *False*, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

25 For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

Questions 26–28 are about spaces where the "vectors" are matrices.

26 Find a basis (and the dimension) for these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$

These are simple bases (among many others) for (a) diagonal matrices (b) skew-symmetric matrices. The dimensions are $3, 6, 3$.

27 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$. What space of 3 by 3 matrices do they span?

I, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; echelon matrices do *not* form a subspace; they ***span*** the upper triangular matrices (not every U is echelon).

The echelon matrices span all upper traingular matrices. (How could you produce the matrix with $a_{22} = 1$ as its only nanzero entry?)

28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$

Questions 29–32 are about spaces where the "vectors" are functions.

29 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

(a) $y(x) =$ constant C (b) $y(x) = 3x$ this is one basis for the 2 by 3 matrices with $(2,1,1)$ in their nullspace (4-dim subspace). (c) $y(x) = 3x + C = y_p + y_n$ solves $dy/dx = 3$.

30 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace $\boldsymbol{S}$ with $y(0) = 0$. What is the dimension of $\boldsymbol{S}$?

$y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

31 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$ (b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

(a) $y(x) = e^{2x}$ is a basis for, all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow 1$ basis function in solution space).

32 Suppose y_1, y_2, y_3 are three different functions of x. The space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

$y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

33 Find a basis for the space **S** of vectors (a,b,c,d) with $a + c + d = 0$ and also for the space **T** with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

Basis for **S**: $(1,0,-1,0), (0,1,0,0), (1,0,0,-1)$; basis for **T**: $(1,-1,0,0)$ and $(0,0,2,1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3,-3,2,1) =$ nullspace for 3 equation in $\mathbf{R}^4$ has dimension 1.

34 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1,2,0)$ and $(0,1,-1)$

(b) $(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)$

(c) $(1,2,2), (-1,2,1), (0,8,0)$

(d) $(1,2,2), (-1,2,1), (0,8,6)$

(a) No, 2 vectors don't span $\mathbf{R}^3$ (b) No, 4 vectors in $\mathbf{R}^3$ are dependent (c) Yes, a basis (d) No, these three vectors are dependent

35 Suppose A is 5 by 4 with rank 4. Show that $A\boldsymbol{v} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible. Show that $A\boldsymbol{v} = \boldsymbol{b}$ is solvable when $[\,A \;\; \boldsymbol{b}\,]$ is singular.

If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{v} = \boldsymbol{b}$ has a solution.

36 (a) Find a basis for all solutions to $d^4y/dx^4 = y(x)$.

(b) Find a particular solution to $d^4y/dx^4 = y(x) + 1$. Find the complete solution.

(a) The functions $y = \sin x$, $y = \cos x$, $y = e^x$, $y = e^{-x}$ are a basis for solutions to $d^4y/dx^4 = y(x)$.

(b) A particular solution to $d^4y/dx^4 = y(x) + 1$ is $y(x) = -\mathbf{1}$. The complete solution is $y(x) = -1 + c, \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$ (or use another basis for the nullspace of the 4th derivative).

Challenge Problems

37 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 =$ zero matrix, and prove that each $c_i = 0$.)

$$\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}. \quad \text{The six } P\text{'s are dependent}.$$

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

38 Intersections and sums have $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$. Start with a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ for the intersection $\mathbf{V} \cap \mathbf{W}$. Extend with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ to a basis for $\mathbf{V}$, and separately with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$ to a basis for $\mathbf{W}$. Prove that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s and $\boldsymbol{w}$'s together are ***independent***. The dimensions have $(r+s)+(r+t) = (r)+(r+s+t)$ as desired.

The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

39 Inside $\mathbf{R}^n$, suppose dimension $(\mathbf{V})$ $+$ dimension $(\mathbf{W}) > n$. Why is some nonzero vector in both $\mathbf{V}$ and $\mathbf{W}$? Start with bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_q$, $p + q > n$.

If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

40 Suppose A is 10 by 10 and $A^2 = 0$ (zero matrix): A times each column of A is $\mathbf{0}$. This means that the column space of A is contained in the _____. If A has rank r, those subspaces have dimension $r \leq 10 - r$. So the rank of A is $r \leq 5$, if $A^2 = 0$.

If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$, and we must have $r \leq 10 - r$ and $r \leq 5$.

Problem Set 5.5, page 308

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}}))$ $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space basis $=$ row $1 = (1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space basis $=$ column1 $= (1, 2)$; left nullspace $(-2, 1)$. B: Row space basis $=$ both rows $= (1, 2, 4)$ and $(2, 5, 8)$; column space basis $=$ two columns $= (1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$.

3 Row space basis = rows of $U = (0,1,2,3,4)$ and $(0,0,0,1,2)$; column space basis = pivot columns (of A not U) $=$ $(1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1)$ = last row of E^{-1}!

4 (a) $\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $\begin{bmatrix}1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}-9 & -3\\3 & 1\end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 0\end{bmatrix}$ has those rows spanning its row space $B = \begin{bmatrix}1 & -2 & 1\end{bmatrix}$ has the same rows spanning its nullspace and $BA^{\mathrm{T}} = 0$.

6 A: $\dim$ $\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: $\dim$ $\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix}A & A\end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix}I & 0\end{bmatrix}$ and $\begin{bmatrix}I & I; & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0\end{bmatrix} = 3$ by 2 have *row space dimensions* $= 3,3,0 =$ column space dimensions; *nullspace dimensions* $2,3,2$; left *nullspace dimensions* $0,2,3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank= 3, nullspace and left nullspace contain only $(0,0,0)$. For **rand** $(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here. (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix}1 & 1\\0 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\1 & 2 & 0\end{bmatrix} = \begin{bmatrix}2 & 2 & 1\\2 & 4 & 0\\1 & 0 & 1\end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space (same dimension!) (b) *True*: A and $-A$ have the same four subspaces (c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

17 Row space = yz plane; column space = xy plane; nullspace = x axis; left nullspace = z axis. For $I + A$: Row space = column space = $\mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row $3-2$ row $2+$ row $1=$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x}=\mathbf{0}$ leads to $0=b_3-b_2-b_1$ so $(-1,-1,1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3-2b_1=0$ and $b_4+b_2-4b_1=0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace. *Why*? Those vectors multiply the matrix to give *zero rows*. Section 4.1 will show another approach: $A\boldsymbol{x}=\boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ has 1 independent solution $=$ last row of E^{-1}. ($E^{-1}A=R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank <2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}+\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A=\begin{bmatrix}\boldsymbol{u} & \boldsymbol{w}\end{bmatrix}\begin{bmatrix}\boldsymbol{v}^{\mathrm{T}} & \boldsymbol{z}^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}1&2\\2&2\\4&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}3&2\\4&2\\5&1\end{bmatrix}$ has column space spanned by $\boldsymbol{u}$ and $\boldsymbol{w}$, row space spanned by $\boldsymbol{v}$ and $\boldsymbol{z}$.

23 As in Problem 22: Row space basis $(3,0,3),(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y}=\mathbf{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A=\begin{bmatrix}1 & 0\end{bmatrix}$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A)=\boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}}=A$ or $A^{\mathrm{T}}=-A$ they are also the same for A^{T})

26 The rows of $C=AB$ are combinations of the rows of B. So rank $C\leq$ rank B. Also rank $C\leq$ rank A, because the columns of C are combinations of the columns of A.

27 Choose $d=bc/a$ to make $\begin{bmatrix}\mathbf{a}&\mathbf{b}\\\mathbf{c}&\mathbf{d}\end{bmatrix}$ a rank-1 matrix. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$. Those two vectors are perpendicular !

28 B and C (checkers and chess) both have rank 2 if $p\neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero; $N(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $N(C)$ is a challenge.

29 $a_{11}=1,a_{12}=0,a_{13}=1,a_{22}=0,a_{32}=1,a_{31}=0,a_{23}=1,a_{33}=0,a_{21}=1.$

30 There are vectors along the floor and along a wall that are *not perpendicular*. In fact the vectors where the wall meets the floor are in both subspaces (and not perpendicular to themselves).

31 Every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ has $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$. Each row of A^{T} (= each column of A) has a *zero* dot product with $\boldsymbol{y}$—those dot products are the zeros on the right hand side of $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$.

32 The plane $\boldsymbol{P}$ is exactly the nullspace of the matrix $A=[1\ 1\ 1\ 1]$. Then $P^{\perp}$ is the row space of A, and the vector $v=(1,1,1,1)$ is a basic for P^{+}.

33 The vector $(1,4,5)$ in the row space of A would have to be orthogonal to $(4,5,1)$ in the nullspace—and it's not. So no matrix A.

34 The subspaces for $A=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B=cA$ with $c\neq 0$.

35 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

36 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

37 If a vector $\boldsymbol{v}$ is in the subspace S, then $\boldsymbol{v}$ is perpendicular to every vector in $S^\perp$. Therefore $\boldsymbol{v}$ belongs to $(S^\perp)^\perp$. Those lines show that S is **contained in** $(S^\perp)^\perp$. But if S has dimension d, $S^\perp$ will have dimension $n - d$ and $(S^\perp)^\perp$ will have dimension $n - (n - d) = d$.

If the d-dimensional space S is contained in the d-dimensional space $(S^\perp)^\perp$, the two spaces must be the same ! (Why is that true ?)

38 This problem shows that A and $A^{\mathrm{T}}A$ have the same nullspace (a very important fact, proved again on page 391). The proof here starts from $A^{\mathrm{T}}A\boldsymbol{v} = \mathbf{0}$, which puts $A\boldsymbol{v}$ in the nullspace of A^{T}. But $A\boldsymbol{v}$ is also in the column space of A ($A\boldsymbol{v}$ is always a combination of the columns, by matrix multiplication). So $A\boldsymbol{v}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A)$, perpendicular to itself and therefore $A\boldsymbol{v} = 0$.

Conclusion : $A^{\mathrm{T}}A\boldsymbol{v} = \mathbf{0}$ leads to $A\boldsymbol{v} = \mathbf{0}$. And certainly $A\boldsymbol{v} = \mathbf{0}$ leads to $A^{\mathrm{T}}A\boldsymbol{v} = \mathbf{0}$ (just multiply by A). So $\boldsymbol{N}(A^{\mathrm{T}}A) = \boldsymbol{N}(A)$.

Problem Set 5.6, page 319

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ for $\boldsymbol{y} = (1, -1, 1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination leads to

$$\begin{aligned} -v_1 + v_2 &= b_1 \\ -v_2 + v_3 &= b_2 - b_1 \\ -v_2 + v_3 &= b_3 \end{aligned} \qquad \text{and then} \qquad \begin{aligned} -v_1 + v_2 &= b_1 \\ -v_2 + v_3 &= b_2 - b_1 \\ \mathbf{0} &= \boldsymbol{b_3 - b_2 + b_1} \end{aligned}$$

The two nonzero rows of R are $1\ -1\ 0$ and $0\ 1\ -1$ (signs were reversed to make the pivot $= +1$). Row 3 of R is zero. The tree has edges from node 1 to 2 and node 2 to 3.

4 The equations in 5.6.3 can be solved when $b_3 - b_2 + b_1 = 0$ (this is actually Kirchhoff's Voltage Law). These are exactly all the vectors $\boldsymbol{b}$ that are orthogonal to $\boldsymbol{y} = (1, -1, 1)$. (If $\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{b} \neq 0$, then KVL fails and $A\boldsymbol{v} = \boldsymbol{b}$ has no solution.)

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1, -1, 0)$ and not solvable for $\boldsymbol{f} = (1, 0, 0)$; $\boldsymbol{f}$ must be orthogonal to $(1, 1, 1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{v} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{v} = 1, -1, 0$ and currents $-A\boldsymbol{v} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 The triangle graph has $A^{\mathrm{T}}A =$ graph Laplacian:

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

All vectors (c, c, c) are in nullspace of $A =$ nullspace of $A^{\mathrm{T}}A$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

9 Elimination on $A\boldsymbol{v} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

10 The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The nonzero rows of U keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 (a) The diagonal $2, 3, 3, 2$ counts edges that go in or out of nodes $1, 2, 3, 4$ on the graph. When A^{T} multiplies A, those diagonal entries are dot products (row i of A^{T}) $\cdot$ (column i of A) $= ||\text{column } i||^2 =$ number of -1's or 1's in column $i =$ degree of node i.

(b) Column i (from node i) overlays column j (from node j) only when an edge connects nodes i and j. Then the row of A for that edge has -1 and 1 in those columns—those numbers multiply to give -1.

12 The nullspace of $A^{\mathrm{T}}A$ contains $(1, 1, 1, 1)$ just like $\boldsymbol{N}(A)$. The rank is $4 - 1 = 3$. A vector $\boldsymbol{f}$ is in the column space of $A^{\mathrm{T}}A$ (= row space by symmetry) exactly when $\boldsymbol{f}$ is orthogonal to the nullspace—which means that $f_1 + f_2 + f_3 + f_4 = 0$. If you add up the 4 equations $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{f}$, you see this again.

13 The n by n *adjacency matrix* for the 4 node graph is

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad W^2 = \begin{bmatrix} 2 & \mathbf{1} & 1 & 2 \\ \mathbf{1} & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

You can check that the i, j entry of W^2 is the number of 2-*step paths* from i to j. When $i = j$ those paths go out and back. Only one 2-step path connects nodes 1 and 2, going through node 3.

14 The number of loops in this connected graph is $n - m + 1 = \mathbf{7} - \mathbf{7} + \mathbf{1} = \mathbf{1}$. What answer if the graph has two separate components (no edges between)?

15 Start from (4 nodes) $-$ (6 edges) $+$ (3 loops) $= 1$. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.

16 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

17 A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \cdots + (n-1) = n(n-1)/2$. Tree has 5 edges.

18 $\boldsymbol{N}(A)$ contains all multiples of $(1, 1, \ldots, 1)$ and *no other vectors*. The equations $A\boldsymbol{v} = \mathbf{0}$ tell you that $v_i = v_j$ when nodes i and j are connected by an edge. Then *every* $v_i = v_j$ whenever the graph is connected—just go from node i to node j using edges in the graph.

19 (a) With n nodes and all edges, $A^{\mathrm{T}}A$ will have $n-1$ along its diagonal (the degree of every edge). It will gave -1 in every off-diagonal entry (a complete graph has an edge between every pair of nodes i and j).

(b) If the edge connecting nodes 1 and 3 is removed, this reduces by 1 the degrees $(A^{\mathrm{T}}A)_{11}$ and $(A^{\mathrm{T}}A)_{33}$ on the diagonal: those degrees are now $n-2$. And $(A^{\mathrm{T}}A)_{13} = (A^{\mathrm{T}}A)_{31} = 0$ because that edge is gone.

20 With batteries b_1 to b_5 in the 5 edges of the square graph, the equation $A^{\mathrm{T}}(A\boldsymbol{v} - \boldsymbol{b}) = \mathbf{0}$ gives the voltages v_1, v_2, v_3, v_4 at the 4 nodes. Here $b = (1, 1, 1, 1, 1)$.

$$A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b} \text{ is } \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Notice that adding the 4 equations gives $0 = 0$: good. The solution $\boldsymbol{v}$ gives voltages

$$\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n = \begin{bmatrix} -2 \\ -5/4 \\ -3/4 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where the particular solution was chosen to have $v_4 = 0$.

Chapter 5 Notes, page 321

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

6 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1+c_2)\boldsymbol{f} = \boldsymbol{f}((c_1+c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.