DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

Gilbert Strang

Massachusetts Institute of Technology

Book Website	**math.mit.edu/dela**
Email address	**diffeqla@gmail.com**
Course page	web.mit.edu/18.06
OpenCourseWare	video lectures: ocw.mit.edu
Home page	math.mit.edu/$\sim$gs
Publisher	www.wellesleycambridge.com
Direct email	gs@math.mit.edu

Wellesley - Cambridge Press

Box 812060

Wellesley, Massachusetts 02482

Problem Set 3.1, page 160

1 (a) Why do two isoclines $f(t,y)=s_1$ and $f(t,y)=s_2$ never meet ?

(b) Along the isocline $f(t,y)=s$, what is the slope of all the arrows ?

(c) Then all solution curves go only one way across an _____ .

Solution (a) Isoclines can't meet because $f(t,y)$ has one fixed value along an isocline.

(b) The slope of the arrows is fixed at s along the isocline $f(t,y)=s$.

(c) All solution curves go one way (with slope s) across the isocline $f(t,y)=s$.

2 (a) Are isoclines $f(t,y)=s_1$ and $f(t,y)=s_2$ always parallel ? Always straight ?

(b) An isocline $f(t,y)=s$ is a solution curve when its slope equals _____ .

(c) The zerocline $f(t,y)=0$ is a solution curve only when y is _____ : slope 0.

Solution (a) In case $f(t,y)$ does not depend on t (autonomous equation) the isoclines are horizontal lines. In general isoclines need to be parallel or straight.

(b) If the slope of the isoclines $f(t,y)=s$ happens to be s (slope of arrows equals slope of curve, so the arrows go along the isocline) then the isocline is actually a solution curve. Example: A steady state where $f(y)=0$ has arrows of slope zero. That horizontal isocline is also the graph of the constant solution $y(t)=Y$.

(c) The zerocline is a solution curve when the slope is zero and y is **constant**.

3 If $y_1(0)<y_2(0)$, what continuity of $f(t,y)$ assures that $y_1(t)<y_2(t)$ for all t ?

Solution Two solution curves $y_1(t)$ and $y_2(t)$ can't meet or cross if they are continuous curves : this will be true if f and $\partial f/\partial y$ are continuous.

4 The equation $dy/dt=t/y$ is completely safe if $y(0)\neq 0$. Write the equation as $y\,dy=t\,dt$ and find its unique solution starting from $y(0)=-1$. The solution curves are hyperbolas—can you draw two on the same graph ?

Solution $dy/dt=t/y$ leads to $\int y\,dy=\int t\,dt$ and $y^2=t^2+C$. If $y(0)=-1$ then $y(t)=-\sqrt{t^2+1}$. The hyperbolas $y^2=t^2+C$ are asymptotic to the 45° and -45° lines $y=t$ and $y=-t$.

5 The equation $dy/dt=y/t$ has many solutions $y=Ct$ in case $y(0)=0$. It has no solution if $y(0)\neq 0$. When you look at all solution curves $y=Ct$, which points in the t, y plane have no curve passing through ?

Solution The solution curves $y=Ct$ (allowing all numbers C) go through all points (t,y) with suitable $C=y/t$—**except** the points on the vertical line $t=0$ (other than the origin $(0,0)$ that all the lines $y=Ct$ will pass through). You cannot solve $dy/dt=y/t$ with an initial value like $y(0)=1$, because the right side y/t would be $1/0$.

6 For $y'=ty$ draw the isoclines $ty=1$ and $ty=2$ (those will be hyperbolas). On each isocline draw four arrows (they have slopes 1 and 2). Sketch pieces of solution curves that fit your picture between the isoclines.

Solution The solution curves $dy/dt=ty$ have $dy/y=t\,dt$ and $\ln y=\frac{1}{2}t^2+c$ and $y=\exp\left(\frac{1}{2}t^2+c\right)=C\exp\left(\frac{1}{2}t^2\right)$. Solution curves cross isoclines $f(t,y)=s$ with that slope s! **The arrows with that slope are tangent to the curves as they cross the isocline**.

7 The solutions to $y' = y$ are $y = Ce^t$. Changing C gives a higher or lower curve. But $y' = y$ is autonomous, its solution curves should be shifting right and left!

Draw $y = 2e^t$ and $y = -2e^t$ to show that they really are *right-left shifts* of $y = e^t$ and $y = -e^t$. The shifted solutions to $y' = y$ are e^{t+C} and $-e^{t+C}$.

Solution For all autonomous equations $dy/dt = f(y)$, the solution curves are horizontal shifts of each other. In particular for $f(y) = y$, the curves $y = Ce^t$ shift right-left as C increases-decreases.

8 For $y' = 1 - y^2$ the flat lines $y =$ constant are isoclines $1 - y^2 = s$. Draw the lines $y = 0$ and $y = 1$ and $y = -1$. On each line draw arrows with slope $1 - y^2$. The picture says that $y =$ _____ and $y =$ _____ are steady state solutions. From the arrows on $y = 0$, guess a shape for the solution curve $y = (e^t - e^{-t})/(e^t + e^{-t})$.

Solution The picture will show the horizontal lines $y = 1$ and $y = -1$ as "zeroclines" where $f(t, y) = s = 1 - y^2 = 0$. So those are steady state solution curves $y(t) = Y = 1$ or -1.

The isocline $y = 0$ is the x-axis, along with $f(t, y) = 1 - y^2 = 1 = s$. (The arrows cross the x-axis at 45°, with slope $s = 1$.) So the solution curves are S-curves going up from the line $y = -1$ to the line $y = 1$, rising at 45° along the x-axis halfway between those two lines.

9 The parabola $y = t^2/4$ and the line $y = 0$ are both solution curves for $\boldsymbol{y' = \sqrt{|y|}}$. Those curves meet at the point $t = 0$, $y = 0$. What continuity requirement is failed by $f(y) = \sqrt{|y|}$, to allow more than one solution through that point?

Solution The function $f(y) = \sqrt{|y|}$ is continuous at $y = 0$ but its derivative $df/dy = 1/2\sqrt{|y|}$ blows up (because of $1/0$.) So two solutions can start from the same initial value $y(0) = 0$, and they do.

10 Suppose $y = 0$ up to time T is followed by the curve $y = (t - T)^2/4$. Does this solve $\boldsymbol{y' = \sqrt{|y|}}$? Draw this $y(t)$ going through flat isoclines $\sqrt{|y|} = 1$ and 2.

Solution Yes, $y' = \sqrt{|y|}$ is solved by the constant $y(t) = 0$. It is also solved by the curve $y(t) = (t - T)^2/4$ because $dy/dt = (t - T)/2$ equals the square root of $|y(t)|$. So solution curves can lift off the x-axis $y = 0$ anywhere they want, and start upwards on a parabola.

11 The equation $\boldsymbol{y' = y^2 - t}$ is often a favorite in MIT's course 18.03: not too easy. Why do solutions $y(t)$ rise to their maximum on $y^2 = t$ and then descend?

Solution Below the parabola $y^2 = t$ (which opens to the right instead of opening upwards) the right side of $dy/dt = y^2 - t$ will be negative. The solution curves have negative slope and they can't cross the rising parabola.

12 Construct $f(t, y)$ with two isoclines so solution curves go *up* through the higher isocline and other solution curves go *down* through the lower isocline. *True or false*: Some solution curve will stay between those isoclines: **A continental divide**.

Solution We want the isocline $f(t, y) = s = 1$ to be *above* the isocline $f(t, y) = s = -1$. A simple example would be $\boldsymbol{f(t, y) = y}$. Then the equation $dy/dt = y$ has solution curves $y = Ce^t, C > 0$ going *up* through the isocline $f(t, y) = 1$ (which is the flat line $y = 1$). The curves $y = Ce^t$ with $C < 0$ go down through $y = -1$. The **continental divide** is the solution curve $y(t) = 0$ with $C = 0$. Certainly $y(t) = 0$ does solve $dy/dt = y$.

There is always a "continental divide" where solution curves (like water in the Rockies) can't choose between the Atlantic and the Pacific.

Problem Set 3.2, page 168

1 Draw Figure 3.6 for a sink (the missing middle figure) with $y = c_1e^{-2t} + c_2e^{-t}$. Which term dominates as $t \to \infty$? The paths approach the dominating line as they go in toward zero. **The slopes of the lines are -2 and -1** (the numbers s_1 and s_2).

Solution The c_2e^{-t} term dominates at $t \to \infty$ since it decays at a slower rate.

$$\text{Then } y(t) = \frac{\sin \omega t}{\omega(a^2 - \omega^2)} - \frac{\sin at}{a(a^2 - \omega^2)}.$$

2 Draw Figure 3.7 for a spiral sink (the missing middle figure) with roots $s = -1 \pm i$. The solutions are $y = C_1e^{-t}\cos t + C_2e^{-t}\sin t$. They approach zero because of the factor e^{-t}. They spiral around the origin because of $\cos t$ and $\sin t$.

Solution The spiral goes clockwise in toward $(0, 0)$. Not easy to draw to scale, by hand!

3 Which path does the solution take in Figure 3.6 if $y = e^t + e^{t/2}$? Draw the curve $(y(t), y'(t))$ more carefully starting at $t = 0$ where $(y, y') = (2, 1.5)$.

Solution As $t \to \infty$, the path of the point $(y(t), y'(t))$ comes closer and closer to the path for $y = e^t$—**because e^t dominates the other term $e^{t/2}$**. The path for $y = e^t$ has points $(y, y') = (e^t, e^t)$ so it is a straight 45° line in the (y, y') plane.

4 Which path does the solution take around the saddle in Figure3.6 if $y = e^{t/2} + e^{-t}$? Draw the curve more carefully starting at $t = 0$ where $(y, y') = (2, -\frac{1}{2})$.

Solution The function $y = e^{t/2} + e^{-t}$ comes from exponents $\frac{1}{2}$ and -1 (positive and negative will give a **saddle point**. The graph shows the spiral is unwinding clockwise as it leaves the tight spiral and goes outward. For large t the dominant part of (y, y') will be $(e^{t/2}, \frac{1}{2}e^{t/2})$ from the growing term $e^{t/2}$ in y.

5 Redraw the first part of Figure 3.6 when the roots are equal: $s_1 = s_2 = 1$ and $y = c_1e^t + c_2te^t$. *There is no s_2-line*. Sketch the path for $y = e^t + te^t$.

Solution $y = e^t + te^t$ has $y' = 2e^t + te^t$. The larger term te^t gives $(y, y') \approx (te^t, te^t)$ on the 45° line in the y, y' plane. At $t = 0$ it starts from $(y(0), y'(0)) = (1, 2)$.

6 The solution $y = e^{2t} - 4e^t$ gives a source (Figure 3.6), with $y' = 2e^{2t} - 4e^t$. Starting at $t = 0$ with $(y, y') = (-3, -2)$, where is (y, y') when $e^t = 1.1$ and $e^t = .25$ and $e^t = 2$?

Solution Substituting the values $t = \ln 1.1$ and $\ln 0.25$ and $\ln 2$, we get:

1. For $e^t = 1.1$ we have $(y, y') = (-3.19, -1.98)$
2. For $e^t = .25$ we have $(y, y') = (-0.9375, -0.875)$
3. For $e^t = 2$ we have $(y, y') = (-4, 0)$

Those early times don't show the situation for large t, when the dominant term e^{2t} gives $(y, y') = (e^{2t}, 2e^{2t})$ and the path approaches a straight line **with slope 2**.

7 The solution $y = e^t(\cos t + \sin t)$ has $y' = 2e^t \cos t$. This spirals out because of e^t. Plot the points (y, y') at $t = 0$ and $t = \pi/2$ and $t = \pi$, and try to connect them with a spiral. Note that $e^{\pi/2} \approx 4.8$ and $e^\pi \approx 23$.

Solution

1. For $t = 0, (y, y') = (1, 2)$

2. For $t = \frac{\pi}{2}, (y, y') = (e^{\pi/2}, 0) \approx (4.8, 0)$

3. For $t = \pi, (y, y') = (-e^\pi, -2e^\pi) \approx (-23.1, -46.2)$

Maybe we can see the path better by writing $(y, y') = e^t(\cos t, \cos t) + e^t(\sin t, \cos t)$. The first term goes forward and back on the 45° line. the second term circles around and spirals out because of e^t. So we have a big circle around a moving slider.

8 The roots s_1 and s_2 are $\pm 2i$ when the differential equation is _____. Starting from $y(0) = 1$ and $y'(0) = 0$, draw the path of $(y(t), y'(t))$ around the center. Mark the points when $t = \pi/2, \pi, 3\pi/2, 2\pi$. Does the path go clockwise?

Solution The differential equation is $\boldsymbol{y'' + 4y = 0}$. The solution starting at $(y, y') = (1, 0)$ is $(y(t), y'(t)) = \mathbf{(\cos 2t, -2\sin 2t)}$. This is an ellipse in the equation

$$y^2 + \frac{1}{4}(y')^2 = \cos^2 2t + sin^2 2t = 1.$$

The path is clockwise around that elliptical center.

9 The equation $y'' + By' + y = 0$ leads to $s^2 + Bs + 1 = 0$. For $B = -3, -2, -1, 0, 1, 2, 3$ decide which of the six figures is involved. For $B = -2$ and 2, why do we not have a perfect match with the source and sink figures?

Solution To determine which figure is involved, we solve the quadratic equation:

$$s_1 \text{ and } s_2 \text{ are } \frac{-B \pm \sqrt{B^2 - 4}}{2}$$

$\boldsymbol{B = -3}$ has $s_1 = \frac{3-\sqrt{5}}{2} \approx 0.38$ and $s_2 = \frac{3+\sqrt{5}}{2} \approx 2.6$. **Source** with $\boldsymbol{0 < s_1 < s_2}$

$\boldsymbol{B = -2}$ has $s_1 = 1$ and $s_2 = 1$. Since $0 < s_1 = s_2$ we have a **source**

$\boldsymbol{B = -1}$ has $s_1 = \frac{1+\sqrt{3i}}{2}$ and $s_2 = \frac{1+\sqrt{3i}}{2}$. **Spiral Source (outward)** $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) > 0$

$\boldsymbol{B = \ \ 0}$ has $s_1 = i$ and $s_2 = -i$. Since $0 = \mathrm{Re}(s_1) = \mathrm{Re}(s_2)$ we have a **center**

$\boldsymbol{B = \ \ 1}$ has $s_1 = \frac{-1+\sqrt{3i}}{2}$ and $s_2 = \frac{-1+\sqrt{3i}}{2}$. **Spiral Sink (inward)** $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) < 0$

$\boldsymbol{B = \ \ 2}$ has $s_1 = -1$ and $s_2 = -1$. Since $s_1 = s_2 < 0$ we have a **sink**

$\boldsymbol{B = \ \ 3}$ has $s_1 = \frac{-3-\sqrt{5}}{2} \approx -2.6$ and $s_2 = \frac{-3+\sqrt{5}}{2} \approx -0.38$. $s_1 < s_2 < 0$. This is a **sink**

The special case $B = 2$ and $B = -2$ gave **equal roots** $\boldsymbol{s_1 = s_2}$. Then there will be a factor "t" in the null solution. The path won't close on itself like a circle or ellipse. As it turns, it will go slowly outward from that factor t.

10 For $y'' + y' + Cy = 0$ with damping $B = 1$, the characteristic equation will be $s^2 + s + C = 0$. Which C gives the changeover from a *sink* (overdamping) to a spiral *sink* (underdamping)? Which figure has $C < 0$?

Solution The solutions to the quadratic equation $s^2 + s + C = 0$ are

$$s_1 \text{ and } s_2 \text{ are } \frac{-1 \pm \sqrt{1-4C}}{2}$$

The change from a sink to a spiral sink occurs at $C = \frac{1}{4}$. Those are sinks because the real part of s is negative. When C is less than zero, we change to one positive root and one negative root. Then the path becomes a **saddle**.

Problems 11–18 are about $dy/dt = Ay$ with companion matrices $\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$.

11 The eigenvalue equation is $\boldsymbol{\lambda^2 + B\lambda + C = 0}$. Which values of B and C give complex eigenvalues? Which values of B and C give $\lambda_1 = \lambda_2$?

Solution Look at the solution to the quadratic equation $\lambda^2 + B\lambda + C = 0$:

$$\lambda_1 \text{ and } \lambda_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

Therefore when $B^2 < 4C$ we get complex eigenvalues.

On the other hand, when $B^2 = 4C$ we get $\lambda_1 = \lambda_2 = -B/2$ (the square root is 0).

12 Find λ_1 and λ_2 if $B = 8$ and $C = 7$. Which eigenvalue is more important as $t \to \infty$? Is this a sink or a saddle?

Solution We solve the quadratic eigenvalue equation for λ_1 and λ_2:

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-8 \pm \sqrt{64-28}}{2} \text{ gives } \boldsymbol{\lambda_1 = -7} \text{ and } \boldsymbol{\lambda_2 = -1}.$$

Since $s_1 < s_2 < 0$ we have a **sink**. The more negative λ_2 gives slower decay as $t \to \infty$.

13 Why do the eigenvalues have $\lambda_1 + \lambda_2 = -B$? Why is $\lambda_1 \lambda_2 = C$?

Solution This refers to the eigenvalues of the companion matrix:

$A = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$ comes from $\begin{array}{l} y_1' = y_2 \\ y_2' = -Cy - By_2 \end{array}$. Then $y_1'' = y_2'$ is $y_1'' + By_1' + Cy_1 = 0$.

The eigenvalues λ_1 and λ_2 are the roots of $\lambda^2 + B\lambda + C = 0$ just as the roots s_1 and s_2 are the roots of $s^2 + Bs + C = 0$. We know from factoring into $(s - s_1)(s - s_2)$ or $(\lambda - \lambda_1)(\lambda - \lambda_2)$ that the coefficient of λ^2 is 1, the coefficient of λ is $B = -\lambda_1 - \lambda_2$, and the constant form is $C = \lambda_1$ times λ_2.

14 Which second order equations did these matrices come from?

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (saddle)} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ (center)}$$

Solution Write the matrix equation $y' = Ay$ as two coupled first order equations. For A we get

$$y_1' = y_2$$
$$y_2' = y_1$$

Then $y_1'' = y_2' = y_1$ and the second order equation is $\boldsymbol{y'' = y}$.

The second matrix A_2 gives $y_1' = y_2$ and $y_2' = -y_1$.

Then $y_1'' = y_2' = -y_1$ and the second order equation is $\boldsymbol{y'' + y = 0}$. (Notice that we also find $y_2'' = -y_2$.)

15 The equation $y'' = 4y$ produces a saddle point at $(0,0)$. Find $s_1 > 0$ and $s_2 < 0$ in the solution $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. If $c_1 c_2 \neq 0$, this solution will be (large) (small) as $t \to \infty$ and also as $t \to -\infty$.

The only way to go toward the saddle $(y, y') = (0,0)$ as $t \to \infty$ is $c_1 = 0$.

Solution Assuming a solution of the form $y(t) = e^{st}$ gives:

$$\begin{aligned} y'' - 4y &= 0 \\ s^2 e^{st} - 4e^{st} &= 0 \\ s^2 - 4 &= 0 \\ s &= \pm 2 \end{aligned}$$

Therefore $s_1 = 2$ and $s_2 = -2$. The solution becomes $y = \boldsymbol{c_1 e^{2t} + c_2 e^{-2t}}$. As $t \to \infty$, the e^{2t} term will grow unless $c_1 = 0$. In that case $(y, y') = (c_2 e^{-2t}, -2c_2 e^{-2t})$ goes to the saddle point $(0,0)$.

16 If $B = 5$ and $C = 6$ the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. The vectors $\boldsymbol{v} = (1,3)$ and $\boldsymbol{v} = (1,2)$ are *eigenvectors* of the matrix A: Multiply $A\boldsymbol{v}$ to get $3\boldsymbol{v}$ and $2\boldsymbol{v}$.

Solution $v = (1,3)$ is an eigenvector with eigenvalue $\lambda_1 = 3$:

$$Av = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3v.$$

Similarly $v = (1,2)$ is an eigenvector with eigenvalue $\lambda_2 = 2$:

$$\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Notice that these eigenvectors of the companion matrix A have the form $\boldsymbol{v = (1, \lambda)}$.

17 In Problem 16, write the two solutions $\boldsymbol{y} = \boldsymbol{v} e^{\lambda t}$ to the equations $\boldsymbol{y}' = A\boldsymbol{y}$. Write the complete solution as a combination of those two solutions.

Solution The eigenvectors $\boldsymbol{v_1} = (1,3)$ and $\boldsymbol{v_2} = (1,2)$ give two pure exponential solutions $\boldsymbol{y} = \boldsymbol{v} e^{\lambda t}$:

$$\boldsymbol{y_1} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \quad \text{and} \quad \boldsymbol{y_2} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}.$$

The complete solution is $\boldsymbol{y}(t) = c_1 \boldsymbol{y_1} + c_2 \boldsymbol{y_2}$. Two constants to match two components of the initial vector $\boldsymbol{y}(0)$ at $t = 0$. Then $\boldsymbol{y}(0) = c_1 \boldsymbol{v_1} + c_2 \boldsymbol{v_2}$.

18 The eigenvectors of a companion matrix have the form $v = (1, \lambda)$. Multiply by A to show that $Av = \lambda v$ gives one trivial equation and the characteristic equation $\lambda^2 + B\lambda + C = 0$.

$$\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \qquad \text{is} \qquad \begin{aligned} \lambda &= \lambda \\ -C - B\lambda &= \lambda^2 \end{aligned}$$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution The eigenvectors of a companion matrix have the special form $\boldsymbol{v} = (1, \lambda)$, as the problem statement shows—because $-C - B\lambda = \lambda^2$ from the eigenvalue equation $\boldsymbol{\lambda^2 + B\lambda + C = 0}$.

The example A is *not* a companion matrix!

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ has eigenvectors } \boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ with } \lambda_1 = \mathbf{4} \text{ and } \lambda_2 = \mathbf{2}.$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The equation for λ is $\lambda^2 - 6\lambda + 8 = 0$ with 6 coming from the trace $3 + 3$ and 8 coming from the determinant $9 - 1$.

19 An equation is stable and all its solutions $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ go to $y(\infty) = 0$ exactly when

$(s_1 < 0$ **or** $s_2 < 0)$ $\qquad$ $(s_1 < 0$ **and** $s_2 < 0)$ $\qquad$ $(\text{Re } s_1 < 0$ **and** $\text{Re } s_2 < 0)$?

Solution The correct answer is $(\text{Re} s_1 < 0$ **and** $\text{Re} s_2 < 0)$.

20 If $Ay'' + By' + Cy = D$ is stable, what is $y(\infty)$?

Solution The steady state solution to this equation is the constant $\boldsymbol{y(\infty) = D/C}$. Because the equation is stable, the null solution $y_n(t)$ will go to zero as $t \to \infty$. The roots s_1 and s_2 have negative real parts.

Problem Set 3.3, page 182

1 If $y' = 2y + 3z + 4y^2 + 5z^2$ and $z' = 6z + 7yz$, how do you know that $Y = 0$, $Z = 0$ is a critical point ? What is the 2 by 2 matrix A for linearization around $(0, 0)$? This steady state is certainly unstable because _____ .

Solution Here $y' = f(y, z)$ and $z' = g(y, z)$ have $f = g = 0$ at the point $(y, z) = (0, 0)$. Then this point is a critical point (stationary point). The Jacobian matrix of derivatives at that point $(0, 0)$ is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 2 + 8y & 3 + 10z \\ 7z & 6 + 7y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \text{ at } (y, z) = (0, 0).$$

The eigenvalues of this triangular matrix are 2 and 6 (on the diagonal). Any positive eigenvalue means growth and instability.

2 In Problem 1, change $2y$ and $6z$ to $-2y$ and $-6z$. What is now the matrix A for linearization around $(0,0)$? How do you know this steady state is stable ?

Solution
$A = \begin{bmatrix} -2+8y & 3+10z \\ 7z & -6+7y \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & -6 \end{bmatrix}$ now has eigenvalues $\lambda = -2, -6$: **stable**.

3 The system $y' = f(y,z) = 1 - y^2 - z$, $z' = g(y,z) = -5z$ has a critical point at $Y = 1$, $Z = 0$. Find the matrix A of partial derivatives of f and g at that point : stable or unstable ?

Solution Here $f = g = 0$ when $(Y,Z) = (1,0)$.

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} -2y & -1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}. \quad \textbf{Stable}.$$

4 This linearization is wrong but the zero derivatives are correct. *What is missing* ? $Y = 0$, $Z = 0$ is not a critical point of $y' = \cos\,(ay + bz)$, $z' = \cos\,(cy + dz)$.

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} -a\sin 0 & -b\sin 0 \\ -c\sin 0 & -d\sin 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

Solution At the point $(Y,Z) = (0,0)$, the functions $f = \cos(0+0)$ and $g = \cos(0+0)$ are equal to **1**. This is not a critical point.

5 Find the linearized matrix A at every critical point. Is that point stable ?

(a) $\begin{aligned} y' &= 1 - yz \\ z' &= y - z^3 \end{aligned}$ (b) $\begin{aligned} y' &= -y^3 - z \\ z' &= y + z^3 \end{aligned}$

Solution (a) $f(y,z) = 1 - yz$ and $g(y,z) = y - z^3$ are both zero when $y = z^3$ and then $1 - yz = 1 - z^4 = 0$. Then $Z = 1$ goes with $Y = 1$ and $Z = -1$ goes with $Y = -1$: **two critical points**.

$$A = \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} -z & -y \\ 1 & -3z^2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \quad \textbf{OR} \quad \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}.$$

The eigenvalues solve $\det(A - \lambda I) = 0$.

$$\textbf{At } (\mathbf{1},\mathbf{1}) \quad \det \begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 4 = 0, \quad \boldsymbol{\lambda = -2, -2}$$

$$\textbf{At } (\mathbf{-1},\mathbf{-1}) \quad \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 4 = 0, \quad \boldsymbol{\lambda = -1 \pm \sqrt{5}}$$

Then $(Y,Z) = (1,1)$ is **stable** but $(-1,-1)$ is **unstable** (because $-1 + \sqrt{5} > 0$).

(b) $f = -y^3 - z$ and $g = y + z^3$ are both zero at $(Y,Z) = \mathbf{(0,0)}$ and $\mathbf{(1,-1)}$ and $\mathbf{(-1,1)}$: three critical points because $f = 0$ gives $z = -y^3$ and then $g = 0$ gives $y = y^a$, leading to $y = 0, 1$, or -1. The stability test applies to the matrix of derivatives :

$$A = \begin{bmatrix} -3y^2 & -1 \\ 1 & 3z^2 \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 + \lambda(3y^2 - 3z^2) + 1 - 9y^2z^2.$$

At $\mathbf{(0,0)}$ $\lambda^2 + 1 = 0$ and $\lambda = \pm i$ **Unstable** (neutrally stable)

At $\mathbf{(1,-1)}$ **and** $\mathbf{(-1,1)}$ $\lambda^2 - 8 = 0$ **Unstable** with $\lambda = \sqrt{8}$.

6 Can you create two equations $y' = f(y,z)$ and $z' = g(y,z)$ with four critical points: $(1,1)$ and $(1,-1)$ and $(-1,1)$ and $(-1,-1)$?

I don't think all four points could be stable ? This would be like a surface with four minimum points and no maximum.

Solution An example would be $y' = \boldsymbol{y^2 - z^2}$ and $z' = \boldsymbol{1 - z^2}$. Then $z^2 - 1 = 0$ and $y^2 - z^2 = 0$ have the four points $(Y,Z) = (\pm 1, \pm 1)$ as critical points. In this case the linearized matrix (Jacobian matrix) is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 2y & -2z \\ 0 & -2z \end{bmatrix} \text{ and only } (Y,Z) = (-1,1) \text{ is stable.}$$

7 The second order nonlinear equation for a damped pendulum is $y'' + y' + \sin y = 0$. Write z for the damping term y', so the equation is $z' + z + \sin y = 0$.

Show that $Y = 0$, $Z = 0$ is a stable critical point at the bottom of the pendulum.
Show that $Y = \pi$, $Z = 0$ is an unstable critical point at the top of the pendulum.

8 Those pendulum equations $y' = z$ and $z' = -\sin y - z$ have infinitely many critical points! What are two more and are they stable ?

Solutions to 7 and 8 The system $y' = z$ and $z' = -z - \sin y$ has critical points when $z = 0$ and $\sin y = 0$ (**this allows all values $\boldsymbol{y = n\pi}$**).

The Jacobian matrix of derivatives of z and $-z - \sin y$ is a companion matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -\cos y & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \textbf{ or } \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

We have $-\cos y = -1$ at $y = 0, \pm 2\pi, \pm 4\pi, \ldots$ and $-\cos y = +1$ at $y = \pm\pi, \pm 3\pi, \ldots$

The eigenvalues satisfy $\lambda^2 + \lambda + 1 = 0$ or $\lambda^2 + \lambda - 1 = 0$:

$$\lambda = \tfrac{1}{2}(-1 \pm \sqrt{-3}) = \tfrac{1}{2}(-1 \pm i\sqrt{-3}) \text{ is } \textbf{stable at } \boldsymbol{y = 2n\pi}.$$

$$\lambda = \tfrac{1}{2}(-1 \pm \sqrt{5}) \text{ is } \textbf{unstable at } \boldsymbol{y = (2n+1)\pi}.$$

The pendulum is stable hanging straight down (at 6:00) and unstable when balanced directly upward (at 12:00).

9 The Liénard equation $y'' + p(y)y' + q(y) = 0$ gives the first order system $y' = z$ and $z' =$ _____. What are the equations for a critical point ? When is it stable ?

Solution The coupled equations are $y' = z$ and $z' = -p(y)z - q(y)$. These right sides are zero (critical point) when $z = 0$ and $q(y) = 0$.

The first derivative matrix is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -p'z - q' & -p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}.$$

That companion matrix is stable (according to Section 3.1) when $\boldsymbol{B > 0}$ **and** $\boldsymbol{C > 0}$.

10 Are these matrices stable or neutrally stable or unstable (source or saddle) ?

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & -1 \end{bmatrix}$$

Solution The stability tests are **trace** $<$ **0** and **determinant** $>$ **0**. This is because determinant $= (\lambda_1)(\lambda_2)$ and trace $=$ sum down the main diagonal $= \lambda_1 + \lambda_2$. Apply these tests to find

stable, **unstable** (saddle with $\det < 0$), **stable,** **unstable,** **stable**.

The second matrix has $\lambda = \pm 3i$ which gives undamped oscillation and neutral stability.

11 Suppose a predator x eats a prey y that eats a smaller prey z :

$$\begin{array}{ll} dx/dt = -x + xy & \text{Find all critical points } X, Y, Z \\ dy/dt = -xy + y + yz & \text{Find } A \text{ at each critical point} \\ dz/dt = -yz + 2z & \text{(9 partial derivatives)} \end{array}$$

Solution The right hand sides are $x(1-y)$ and $y(-x+1+z)$ and $z(-y+z)$. These are all zero at **three critical points** (X, Y, Z): $\mathbf{(0,0,0)}$ $\mathbf{(0,2,-1)}$, $\mathbf{(1,1,0)}$

(Follow the two possibilities $X = 0$ or $Y = 1$ needed for $X(1-Y) = 0$.) The matrix of first derivatives of those right hand sides is

$$\begin{bmatrix} 1-y & -x & 0 \\ -y & -x+1+z & y \\ 0 & -z & 2-y \end{bmatrix}. \text{ Substitute the three critical vectors } (X, Y, Z) \text{ :}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

12 The damping in $y'' + (y')^3 + y = 0$ depends on the velocity $y' = z$. Then $z' + z^3 + y = 0$ completes the system. Damping makes this nonlinear system stable—is the linearized system stable ?

Solution $y' = z$ and $z' = -y - z^3$ has only $(Y, Z) = (0, 0)$ as critical point:

A = first derivative matrix = $\begin{bmatrix} 0 & 1 \\ -1 & -3z^2 \end{bmatrix}$ has determinant $= 1$, trace $= -3z^2$: **unstable**.

13 Determine the stability of the critical points $(0, 0)$ and $(2, 1)$:

(a) $\begin{array}{l} y' = -y + 4z + yz \\ z' = -y - 2z + 2yz \end{array}$ (b) $\begin{array}{l} y' = -y^2 + 4z \\ z' = y - 2x^4 \end{array}$

Solution (a) The first derivative matrix at $(y, z) = (0, 0)$ or $(2, 1)$ is

$$A = \begin{bmatrix} z-1 & 4+y \\ z-1 & 2y-2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & -2 \end{bmatrix} \textbf{(stable)} \text{ or } \begin{bmatrix} 0 & 6 \\ 1 & 2 \end{bmatrix} \begin{array}{l} \textbf{(unstable)} \\ \text{(trace 2)} \end{array}$$

(b) The first derivative matrix at $(y, z) = (0, 0)$ or $(2, 1)$ is (**replace** $\boldsymbol{x}$ **by** $\boldsymbol{z}$)

$$A = \begin{bmatrix} -2y & 4 \\ 1 & -8z^3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{array}{l} \textbf{(unstable)} \\ \text{(trace 0)} \end{array} \text{ or } \begin{bmatrix} -4 & 4 \\ 1 & -8 \end{bmatrix} \textbf{(stable)}.$$

Problems 14–17 are about Euler's equations for a tumbling box.

14 The correct coefficients involve the moments of inertia I_1, I_2, I_3 around the axes. The unknowns x, y, z give the angular momentum around the three principal axes:

$$\begin{array}{lll} dx/dt = ayz & \text{with} & a = (1/I_3 - 1/I_2) \\ dy/dt = bxz & \text{with} & b = (1/I_1 - 1/I_3) \\ dz/dt = cxy & \text{with} & c = (1/I_2 - 1/I_1). \end{array}$$

Multiply those equations by x, y, z and add. This proves that $x^2 + y^2 + z^2$ is _____.

Solution Multiply by x, y, and z to get

$$\begin{aligned} xx' &= axyz \\ yy' &= bxyz \\ zz' &= cxyz \\ \hline \tfrac{1}{2}(x^2+y^2+z^2)' &= (a+b+c)xyz = \mathbf{0} \quad \text{for the given } a, b, c. \end{aligned}$$

Then $x^2 + y^2 + z^2 =$ **constant** because its derivative is zero.

15 Find the 3 by 3 first derivative matrix from those three right hand sides $f,\ g,\ h$. What is the matrix A in the 6 linearizations at the same 6 critical points?

Solution The first derivative matrix in Problem 14 is

$$\begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} 0 & az & ay \\ bz & 0 & bx \\ cy & cx & 0 \end{bmatrix}.$$

The 3 right sides are zero at the 6 critical points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \pm b \\ 0 & \pm c & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \pm a \\ 0 & 0 & 0 \\ \pm c & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \pm a & 0 \\ \pm b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

All six points are neutrally stable ($\mathrm{Re}\,\lambda = 0$).

16 You almost always catch an unstable tumbling book at a moment when it is flat. That tells us: The point $x(t), y(t), z(t)$ spends most of its time (near) (far from) the critical point $(0, 1, 0)$. This brings the travel time t into the picture.

Solution This neat observation was explained to me by Alar Toomre. The velocity $(f, g, h) = (ayz, bxz, cxy)$ is low near a critical point where x, y, z are small. Then the book spends most time **near** the point where the book is flat and easy to catch.

17 In reality what happens when you

(a) throw a baseball with no spin (a knuckleball)?

(b) hit a tennis ball with overspin?

(c) hit a golf ball left of center?

(d) shoot a basketball with underspin (a free throw)?

Solution (a) The knuckleball is unstable–hard for the batter to judge.

(b) The topspin brings the tennis ball down faster with a higher bounce.

(c) The golf ball slices to the right off the fairway.

(d) The basketball with underspin is more stable with less bounce around the rim. It is more likely to end up in the basket.

Problem Set 3.4, page 189

1 Apply Euler's method $y_{n+1} = y_n + \Delta t f_n$ to find y_1 and y_2 with $\Delta t = \frac{1}{2}$:

(a) $y' = y$ (b) $y' = y^2$ (c) $y' = 2ty$ (all with $y(0) = y_0 = 1$)

Solution (a) $y_1 = y_0 + \Delta t \quad y_0 = 1 + \Delta t = 1.5 \quad y_2 = (1+\Delta t)^2 = y_n = (1+\Delta t)^R = 2.25$

(b) $y_1 = y_0 + \Delta t \quad y_0^2 = 1 + \Delta t = 1.5 \quad y_2 = y_1 + \Delta t y_1^2 = 1 + \Delta t + \Delta t(1 + 2\Delta t + \Delta t^2) = (1+\Delta t)(1 + \Delta t + \Delta t^2) = (1.5)(1.75)$

(c) $y_1 = (1 + 2t + \Delta t)y_0 = 1$ because $t = 0 \quad y_2 = (1 + 2t + \Delta t)y_1 = 1.5$ because $t = \Delta t$.

2 For the equations in Problem 3, find y_1 and y_2 with the step size reduced to $\Delta t = \frac{1}{4}$. Now the value y_2 is an approximation to the exact $y(t)$ at what time t? Then y_2 in this question corresponds to which y_n in Problem 3?

Solution With $\Delta t = \frac{1}{4}, y_2$ will now be an approximation to the true solution $y(\frac{1}{2})$ because $2\Delta t = \frac{1}{2}$.

(a) $y_1 = 1 + \Delta t = 5/4 = 1.35 \qquad y_2 = (1+\Delta t)^2 = 25/16$

(b) $y_1 = 1 + \Delta t = 1.25 \qquad y_2 = \left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{4} + \frac{1}{16}\right) = \left(\frac{5}{4}\right)\left(\frac{21}{16}\right)$

(c) $y_1 = 1 \qquad y_2 = (1 + 2t + \Delta t)y_1 = \left(1 + \frac{2}{19}\right) = \left(\frac{9}{8}\right)$

3 (a) For $dy/dt = y$ starting from $y_0 = 1$, what is Euler's y_n when $\Delta t = 1$?

(b) Is it larger or smaller than the true solution $y = e^t$ at time $t = n$?

(c) What is Euler's y_{2n} when $\Delta t = \frac{1}{2}$? This is closer to the true $y(n) = e^n$.

Solution (a) $y_{n+1} = (1+\Delta t)y_n = 2y_n$ so $y_n = 2^n$

(b) 2^n is smaller than e^n

(c) $y_{n+1} = (1+\Delta t)y_n = \frac{3}{2}y_n$. Then $y_{2n} = \left(1+\frac{1}{2}\right)^{2n}$ is above 2^n because $\left(1+\frac{1}{2}\right)^2 > 2$.

4 For $dy/dt = -y$ starting from $y_0 = 1$, what is Euler's approximation y_n after n steps of size Δt? Find all the y_n's when $\Delta t = 1$. Find all the y_n's when $\Delta t = 2$. Those time steps are *too large* for this equation.

Solution $y_{n+1} = Y_n - \Delta t y_n$ so $y_n = (1-\Delta t)^n y_0$.

If $\Delta t = 1$ then all of $Y_1, Y_2, Y_3, \ldots$ are zero.

If $\Delta t = 2$ then $Y_{n+1} = -y_n$ and $y_n = (-1)^n$.

The approximation will blow up for $\Delta t > 2$.

In reality it seems useless for $\Delta t > 0.1$.

5 The true solution to $y' = y^2$ starting from $y(0) = 1$ is $y(t) = 1/(1-t)$. This explodes at $t = 1$. Take 3 steps of Euler's method with $\Delta t = \frac{1}{3}$ and take 4 steps with $\Delta t = \frac{1}{4}$. Are you seeing any sign of explosion?

Solution With $\Delta t = \frac{1}{3}$, Euler's method for $y' = y^2$ becomes $y_{n+1} = y_n + \Delta t y_n^2$. Three steps with $\Delta t = \frac{1}{3}$ and four steps with $\Delta t = \frac{1}{4}$ give

$y_1 = \frac{4}{3}, \quad y_2 = \frac{52}{27}, \quad y_3 = \_\_$ $\qquad y_1 = \frac{5}{4}, \quad y_2 = \frac{105}{64}, \quad y_3 = \_\_ \quad y_4 = \_\_$

We are not reaching infinity at time $t = n\Delta t = 1$ but as $\Delta t \to 0$ and $n = 1/\Delta t$ the numbers y_n will keep growing past any bound.

6 The true solution to $dy/dt = -2ty$ with $y(0) = 1$ is the bell-shaped curve $y = e^{-t^2}$. It decays quickly to zero. Show that step $n+1$ of Euler's method gives $y_{n+1} = (1 - 2n\Delta t^2)y_n$. Do the y_n's decay toward zero ? Do they stay there ?

Solution A step of Euler's method starting at time $t = n\Delta t$ gives $y_{n+1} = y_n - 2(n\Delta t)y_n$. In the early steps we are multiplying y_n by $1 - 2n\Delta t$ which is normally less than 1. So the y_n are decreasing at first. But when n is larger than $1/\Delta t$, we are multiplying by a number below -1. At that point the y_n start growing and changing sign at every step : serious *instability*.

7 The equations $y' = -y$ and $z' = -10z$ are uncoupled. If we use Euler's method for both equations with the same Δt between $\frac{2}{10}$ and 2, show that $y_n \to 0$ but $|z_n| \to \infty$. The method is failing on the solution $z = e^{-10t}$ that should decay fastest.

Solution The Euler formulas are $y_{n+1} = (1 - \Delta t)y_n$ and $z_{n+1} = (1 - 10\Delta t)z_n$. For time steps Δt between $\frac{2}{10}$ and 2, the y factor has $|1 - \Delta t| < 1$. But the z factor has $|1 - 10\Delta t| > 1$. The true solutions are $y = Ce^{-t}$ and $z = Ce^{-10t}$.

But that quickly decreasing z has a quickly increasing z_n when $|1 - 10\Delta t| > 1$: instability.

8 What values y_1 and y_2 come from *backward Euler* for $dy/dt = -y$ starting from $y_0 = 1$? Show that $y_1^B < 1$ and $y_2^B < 1$ even if Δt is very large. We have *absolute stability* : no limit on the size of Δt.

Solution Backward Euler for $y' = -y$ is $y_{n+1} - y_n = -\Delta t y_{n+1}$ (**not** $\boldsymbol{-\Delta t} y_n$). Then $y_{n+1} = y_n/(1 + \Delta t)$. For any At that factor $1/(1 + \Delta t)$ is **less than** 1 : absolute stability.

9 The logistic equation $y' = y - y^2$ has an S-curve solution in Section 1.7 that approaches $y(\infty) = 1$. This is a steady state because $y' = 0$ when $y = 1$.

Write Euler's approximation $y_{n+1} = \_\_\_\_\_$ to this logistic equation, with stepsize Δt. Show that this has the same steady state : y_{n+1} equals y_n if $y_n = 1$.

Solution $y' = y - y^2$ is approximated by $y_{n+1} = y_n + \Delta t(y_n - y_n^2)$. This equation has a steady state when $y_{n+1} = y_n$—and this requires the Δt factor to be zero : $\boldsymbol{y_n - y_n^2} = 0$. So the two steady states are ($y_n = 1$ forever) and ($y_n = 0$ forever).

10 The important question in Problem 3 is whether the steady state $y_n = 1$ is stable or unstable. Subtract 1 from both sides of Euler's $y_{n+1} = y_n + \Delta t(y_n - y_n^2)$:

$$y_{n+1} - 1 = y_n + \Delta t(y_n - y_n^2) - 1 = (y_n - 1)(1 - \Delta t y_n).$$

Each step multiplies the distance from 1 by $(1 - \Delta t y_n)$. Near the steady $y_\infty = 1$, $1 - \Delta t\, y_n$ has size $|1 - \Delta t|$. For which Δt is this smaller than 1 to give stability ?

Solution $y_n - 1$ is the distance from steady state. The equation in the problem shows that this distance is multiplied at each step by a factor $\boldsymbol{1 - \Delta t y_n}$. This factor has $|1 - \Delta t y_n| < 1$ when $\boldsymbol{0 < \Delta t y_n < 2}$. When y_n is near 1, this means Δt can be almost **2** for stability.

11 Apply backward Euler $y_{n+1}^B = y_n + \Delta t f_{n+1}^B = y_n + \Delta t \left[y_{n+1}^B - \left(y_{n+1}^B \right)^2 \right]$ to the logistic equation $y' = f(y) = y - y^2$. What is y_1^B if $y_0 = \frac{1}{2}$ and $\Delta t = \frac{1}{4}$? You have to solve a quadratic equation to find y_1^B. I am finding two answers for y_1^B. A computer code might choose the answer closer to y_0.

Solution At each new time step, Backward Euler becomes a quadratic equation for y_{n+1} in the logistic equation. If $y_0 = \frac{1}{2}$ and $\Delta t = \frac{1}{4}$ the equation for $y_1 (= y_1^B)$ is

$$\Delta t (y_1)^2 + (1 - \Delta t) y_1 - y_0 = 0 \quad \text{OR} \quad \frac{1}{4} y_1^2 + \frac{3}{4} y_1 - \frac{1}{2} = 0.$$

Multiply by 4. The solutions of $y_1^2 + 3y_1 - 2 = 0$ are

$$y_1 = \frac{-3 \pm \sqrt{17}}{2}. \text{ The better choice } \left(\text{near } \frac{1}{2} \right) \text{ is } y_1^B = \frac{-3 + \sqrt{17}}{2}.$$

12 For the bell-shaped curve equation $y' = -2ty$, show that backward Euler divides y_n by $1 + 2n(\Delta t)^2$ to find y_{n+1}^B. As $n \to \infty$, what is the main difference from forward Euler in Problem 3 ?

Solution Backward Euler for $y' = -2ty$ is $y_{n+1} - y_n = -2t \Delta t y_{n+1}$ or $y_{n+1} = y_n / (1 + 2t + \Delta t)$.

That fraction is smaller than 1 for all t and Δt. Then the numbers y_n are steadily decreasing as $n \to \infty$, like the true solution $y(t) = e^{-t^2}$. (Forward Euler was hopeless in Problem 6, with Y_n increasing and changing sign at every step beyond $n = 1/\Delta t$.)

13 The equation $y' = \sqrt{|y|}$ has *many solutions* starting from $y(0) = 0$. One solution stays at $y(t) = 0$, another solution is $y = t^2/4$. (Then $y' = t/2$ agrees with $\sqrt{y}$.) Other solutions can stay at $y = 0$ up to $t = T$, and then switch to the parabola $y = (t - T)^2/4$. As soon as y leaves the bad point $y = 0$, where $f(y) = y^{1/2}$ has infinite slope, the equation has only one solution.

Backward Euler $y_1 - \Delta t \sqrt{|y_1|} = y_0 = 0$ gives two correct values $y_1^B = 0$ and $y_1^B = (\Delta t)^2$. What are the three possible values of y_2^B ?

Solution Backward Euler for y_2^B will be $\boldsymbol{y_2 - \Delta t \sqrt{|y_2|} = Y_1}$. If $y_1^B = 0$ then y_2^B can be 0 or $(\Delta t)^2$. If $y_1^B = (\Delta t)^2$ then $x = \sqrt{|y_2^B|}$ solves $x^2 - \Delta t x - (\Delta t)^2 = 0$. Again two possibilities :

$$x = \frac{1}{2} \left(1 \pm \sqrt{5} \right) \Delta t.$$

Because $\sqrt{|y|}$ is continuous but its derivative blows up at $y = 0$, multiple solutions are possible.

14 Every finite difference person will think of averaging forward and backward Euler :

$$\textbf{Centered Euler / Trapezoidal} \qquad \boldsymbol{y_{n+1}^C - y_n = \Delta t \left(\frac{1}{2} f_n + \frac{1}{2} f_{n+1}^C \right)}.$$

For $y' = -y$ the key questions are **accuracy** and **stability**. Start with $y(0) = 1$.

$$y_1^C - y_0 = \Delta t \left(-\frac{1}{2} y_0 - \frac{1}{2} y_1^C \right) \text{ gives } \boldsymbol{y_1^C = \frac{1 - \Delta t/2}{1 + \Delta t/2} y_0}.$$

Stability Show that $|1 - \Delta t/2| < |1 + \Delta t/2|$ for all Δt. *No stability limit on* Δt.

Accuracy For $y_0 = 1$ compare the exact $y_1 = e^{-\Delta t} = 1 - \Delta t + \frac{1}{2}\Delta t^2 - \cdots$ with $y_1^C = (1 - \frac{1}{2}\Delta t)/(1 - \frac{1}{2}\Delta t) = (1 - \frac{1}{2}\Delta t)(1 - \frac{1}{2}\Delta t + \frac{1}{4}\Delta t^2 - \cdots)$.

An extra power of Δt is correct: *Second order accuracy*. A good method.

Solution Stability is $|\boldsymbol{y_{n+1}}| \le |\boldsymbol{y_n}|$ for an equation like $y' = -y$ where the true solution $y = e^{-t}$ is decreasing. In this problem

$$y_1^C = \frac{1 - \Delta t/2}{1 + \Delta t/2} y_0 \text{ has growth factor } \left|\frac{1 - \Delta t/2}{1 + \Delta t/2}\right| < 1 \text{ because } \left|1 + \frac{\Delta t}{2}\right| > \left|1 - \frac{\Delta t}{2}\right|$$

Accuracy is decided by comparing y_1^C to the exact y_1. The two series agree in the terms 1 and $-\Delta t$ and $\frac{1}{2}(\Delta t)^2$: **Second order accuracy** because the $(\Delta t)^3$ error appears in $1/\Delta t$ time steps to reach the typical time $t = 1$. Sign correction in text to:

$$y_1^C = \left(1 - \frac{1}{2}\Delta t\right) / \left(1 + \frac{1}{2}\Delta t\right) = \cdots$$

The rest is correct and produces $1 - \Delta t + \frac{1}{2}(\Delta t)^2 \ldots$ as required.

The website has codes for Euler and Backward Euler and Centered Euler. Those methods are slow and steady with first order and second order accuracy. The test problems give comparisons with faster methods like Runge-Kutta.

Problem Set 3.5, page 194

Runge-Kutta can only be appreciated by using it. A simple code is on math.mit.edu/dela. Professional codes are ode 45 (in MATLAB) and ODEPACK and many more.

1 For $y' = y$ with $y(0) = 1$, show that simplified Runge-Kutta and full Runge-Kutta give these approximations y_1 to the exact $y(\Delta t) = e^{\Delta t}$:

$$y_1^S = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 \qquad y_1^{RK} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \frac{1}{24}(\Delta t)^4$$

Solution Simplified Runge-Kutta (equation (1) in this section) when $y' = f(t, y) = \boldsymbol{y}$:

$$\begin{aligned} y_{n+1} &= y_n + \Delta t\left[\frac{1}{2}f(t_n, y_n) + \frac{1}{2}f\left(t_{n+1}, y_{n+1}^{\text{Euler}}\right)\right] \\ &= y_n + \Delta t\left[\frac{1}{2}y_n + \frac{1}{2}\left(y_n + \Delta t y_n\right)\right] \\ &= y_n + \Delta t y_n + \frac{1}{2}(\Delta t^2) y_n \ (\textbf{3 good terms of} \boldsymbol{e^{\Delta t} y_n}) \end{aligned}$$

Full Runge-Kutta is in equation (5)—now applied when $f(t, y) = \boldsymbol{y}$:

$$\boldsymbol{k_1} = \frac{1}{2}y_n \qquad \boldsymbol{k_3} = \frac{1}{2}\left(y_n + \frac{\Delta t}{2}\left(y_n + \frac{\Delta t}{2}y_n\right)\right)$$

$$\boldsymbol{k_2} = \frac{1}{2}\left(y_n + \frac{\Delta t}{2}y_n\right) \quad \boldsymbol{k_4} = \frac{1}{2}\left(y_n + \Delta t\left(y_n + \frac{\Delta t}{2}\left(y_n + \frac{\Delta t}{2}y_n\right)\right)\right)$$

Then the Runge-Kutta choice for y_{n+1} is correct through $(\Delta t)^4$!

$$y_n + \frac{\Delta t}{3}(k_1 + 2k_2 + 2k_3 + k_4) = y_n \left[1 + \frac{\Delta t}{6} + \frac{\Delta t}{3}\left(1 + \frac{\Delta t}{2}\right) + \right.$$

$$\left. \frac{\Delta t}{3}\left(1 + \frac{\Delta t}{2}\left(1 + \frac{\Delta t}{2}\right)\right) + \frac{\Delta t}{6}\left(1 + \Delta t + \frac{(\Delta t)^2}{2}\left(1 + \frac{\Delta t}{2}\right)\right)\right]$$

$$= y_n \left[1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \frac{1}{24}(\Delta t)^4\right].$$

2 With $\Delta t = 0.1$ compute those numbers y_1^S and y_1^{RK} and subtract from the exact $y = e^{\Delta t}$. The errors should be close to $(\Delta t)^3/6$ and $(\Delta t)^5/120$.

Solution When $y_0 = 1$ and $\Delta t = \frac{1}{10}$, the first step in the solution above gives

Simplified Runge-Kutta $1 + \frac{1}{10} + \frac{1}{2}\left(\frac{1}{10}\right)^2 = \mathbf{1.105}$.

Runge-Kutta $1 + \frac{1}{10} + \frac{1}{2}\left(\frac{1}{10}\right)^2 + \frac{1}{6}\left(\frac{1}{10}\right)^3 + \frac{1}{24}\left(\frac{1}{10}\right)^4 = \frac{11}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{1}{240000} = \mathbf{1.1051708}$.

The exact growth factor is $\exp\left(\frac{1}{10}\right) = \mathbf{1.1051709}$. Error 10^{-7} is near $10^{-5}/120$.

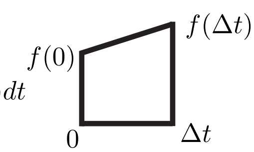
3 Those values y_1^S and y_1^{RK} have errors of order $(\Delta t)^3$ and $(\Delta t)^5$. Errors of this size at every time step will produce total errors of size _____ and _____ at time T, from N steps of size $\Delta t = T/n$.

Those estimates of total error are correct provided errors don't grow (*stability*).

Solution Local errors of size $(\Delta t)^3$ or $(\Delta t)^5$ produce global errors of size $\boldsymbol{(\Delta t)^2}$ **or** $\boldsymbol{(\Delta t)^4}$ after $1/\Delta t$—**provided** the system is stable and local errors don't grow.

4 $dy/dt = f(t)$ with $y(0) = 0$ is solved by integration when f does not involve y. From time $t = 0$ to Δt, simplified Runge-Kutta approximates the integral of $f(t)$:

$$y_1^S = \Delta t\left(\frac{1}{2}f(0) + \frac{1}{2}f(\Delta t)\right) \text{ is close to } y(\Delta t) = \int_0^{\Delta t} f(t)\,dt$$



Suppose the graph of $f(t)$ is a straight line as shown. Then the region is a *trapezoid*. Check that its area is exactly y_1^S. Second order means exact for linear f.

Solution The area of a trapezoid is $(\text{base})(\text{average height}) = (\Delta t)(f(0)+f(\Delta t))/2$. This is exactly the answer chosen by simplified Runge-Kutta.

5 Suppose again that f does not involve y, so $dy/dt = f(t)$ with $y(0) = 0$. Then full Runge-Kutta from $t = 0$ to Δt approximates the integral of $f(t)$ by y_1^{RK} :

$$y_1^{RK} = \Delta t\,(c_1 f(0) + c_2 f(\Delta t/2) + c_3 f(\Delta t))\,. \qquad \textbf{Find } \boldsymbol{c_1, c_2, c_3}.$$

This approximation to $\int_0^{\Delta t} f(t)\,dt$ is called Simpson's Rule. It has 4^{th} order accuracy.

Solution Full Runge-Kutta allows the top edge of the trapezoid to be *curved* : it is the graph of a nonlinear $f(t)$. The area under this curve is well approximated by Simpson's Rule :

$$\text{area} \approx \Delta t\left[\frac{\mathbf{1}}{\mathbf{6}}f(0)+\frac{\mathbf{4}}{\mathbf{6}}f\left(\frac{\Delta t}{2}\right)+\frac{\mathbf{1}}{\mathbf{6}}f(\Delta t)\right].$$

If you apply Runge-Kutta to $y'=f(t)$ from 0 to Δt, with the right hand side independent of y, the result is

$$\boldsymbol{k_1}=\frac{1}{2}f(0)\qquad \boldsymbol{k_2}=\frac{1}{2}f\left(\frac{\Delta t}{2}\right)\qquad \boldsymbol{k_3}=\frac{1}{2}f\left(\frac{\Delta t}{2}\right)\qquad \boldsymbol{k_4}=\frac{1}{2}f\left(\Delta t\right)$$

$$\frac{\Delta t}{3}(k_1+2k_2+2k_3+k_4)=\frac{\Delta t}{\mathbf{6}}f(0)+\frac{4\Delta t}{\mathbf{6}}f\left(\frac{\Delta t}{2}\right)+\frac{\Delta t}{\mathbf{6}}f\left(\Delta t\right)\ :\ \textbf{Simpson's Rule}$$

6 Reduce these second order equations to first order systems $\boldsymbol{y'}=\boldsymbol{f}(t,y)$ for the vector $\boldsymbol{y}=(y,y')$. Write the two components of $\boldsymbol{y}_1^E$ (Euler) and $\boldsymbol{y}_1^S$.

(a) $y''+yy'+y^4=1$ (b) $my''+by'+ky=\cos t$

Solutions to Problems 6 and 7 **Write $\boldsymbol{z}$ for $\boldsymbol{y'}$**. The first order systems are

$$\text{(a)}\ \ y'=z \qquad\qquad \text{(b)}\ \ y'=z$$
$$z'=1-yz-y^4 \qquad\qquad mz'=-ky-bz+\cos t$$

Then Euler's method gives (y_1^E,z_1^E) from (y_0,z_0):

$$\begin{bmatrix} y_1^E \\ z_1^E \end{bmatrix}=\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}+\Delta t\begin{bmatrix} z_0 \\ 1-y_0z_0-(z_0)^4 \end{bmatrix}$$

$$\begin{bmatrix} y_1^E \\ mz_1^E \end{bmatrix}=\begin{bmatrix} y_0 \\ mz_0 \end{bmatrix}+\Delta t\begin{bmatrix} z_0 \\ -ky_0-bz_0+\cos 0 \end{bmatrix}$$

Simplified Runge-Kutta finds (y_1^S,z_1^S) from (y_0,z_0) by adding *half* of those Euler corrections *plus half* of the updated correction:

$$\text{(a)}\ \begin{bmatrix} y_1^S \\ z_1^S \end{bmatrix}=\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}+\frac{\Delta t}{2}\begin{bmatrix} z_0 \\ 1-y_0z_0-(z_0)^4 \end{bmatrix}+\frac{\Delta t}{2}\begin{bmatrix} z_1^E \\ 1-y_1^Ez_1^E-(z_1^E)^4 \end{bmatrix}$$

$$\text{(b)}\ \begin{bmatrix} y_1^S \\ mz_1^S \end{bmatrix}=\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}+\frac{\Delta t}{2}\begin{bmatrix} z_0 \\ -ky_0\ldots \end{bmatrix}+\frac{\Delta t}{2}\begin{bmatrix} z_1^E \\ -ky_1^E-bz_1^E+\mathbf{cos}\,\boldsymbol{\Delta t} \end{bmatrix}$$

8 For $y'=-y$ and $y_0=1$ the exact solution $y=e^{-t}$ is approximated at time Δt by 2 or 3 or 5 terms:

$$y_1^E=1-\Delta t\quad y_1^S=1-\Delta t+\frac{1}{2}(\Delta t)^2\quad y_1^{RK}=1-\Delta t+\frac{1}{2}(\Delta t)^2-\frac{1}{6}(\Delta t)^3+\frac{1}{24}(\Delta t)^4$$

(a) With $\Delta t=1$ compare those three numbers to the exact e^{-1}. What error E?

(b) With $\Delta t=1/2$ compare those three numbers to $e^{-1/2}$. Is the error near $E/16$?

Solution (a) $\Delta t=1$ gives $y_1^E=\mathbf{0}$ $\quad y_1^S=\mathbf{\frac{1}{2}}$ $\quad y_1^{\mathrm{RK}}=\mathbf{\frac{9}{24}}=\mathbf{.375}$ compared to the exact $e^{-1}=\mathbf{.368}$ $\quad E^{RK}=\mathbf{.007}$.

(b) $\Delta t = \frac{1}{2}$ gives $y_1^E = \mathbf{\frac{1}{2}}$ $\quad y_1^S = \mathbf{\frac{5}{8}}$ $\quad y_1^{\text{RK}} = \mathbf{\frac{233}{(24)(16)}} = \mathbf{.60677}$ $\quad e^{-1/2} = \mathbf{.60653}$ $E^{RK} = \mathbf{.00024}$.

Two steps with $\Delta t = \frac{1}{2}$ would leave an error about $2(.00024) = -.00048$ which is close to $.007/16$.

9 For $y' = ay$, simplified Runge-Kutta gives $y_{n+1}^S = (1 + a\Delta t + \frac{1}{2}(a\Delta t)^2)y_n$. This multiplier of y_n reaches $1 - 2 + 2 = \mathbf{1}$ when $a\Delta t = -2$: *the stability limit.*

(**Computer experiment**) For $N = 1, 2, \ldots, 10$ discover the stability limit $L = L_N$ when the series for e^{-L} is cut off after $N + 1$ terms :

$$\left|1 - L + \frac{1}{2}L^2 - \frac{1}{6}L^3 + \cdots \pm \frac{1}{N!}L^N\right| = 1.$$

We know $L = 2$ for $N = 1$ and $N = 2$. Runge-Kutta has $L = 2.78$ for $N = 4$.

Solution The stability limits L_N for $N = 1, \ldots, 10$ come from MATLAB:

$2.0 \quad 2.0 \quad 2.513 \quad 2.785 \quad 3.217 \quad 3.55 \quad 3.954 \quad 4.314 \quad 4.701 \quad 5.070.$