DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

Gilbert Strang

Massachusetts Institute of Technology

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Email address	**diffeqla@gmail.com**
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Problem Set 2.1, page 79

1 Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice):

Simple harmonic motion $\quad y(t) = C\ \cos\,\omega t + D\ \sin\,\omega t.\quad$ What is ω ?

If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero ?

Solution Letting $y(t) = C\cos(\omega t) + D\sin(\omega t)$:

$$\frac{d^2y}{dt^2} + 9y = -\omega^2 C\cos(\omega t) + 9C\cos(\omega t) - \omega^2\sin(\omega t) + 9\sin(\omega t) = 0$$
$$\boldsymbol{\omega = 3}$$

Differentiating $y(t)$ and equating to zero at time $t = 0$ gives us :

$$y'(t) = -C\omega\sin(\omega t) + D\omega\cos(\omega t) = 0$$
$$\text{At } t = 0 : D\omega = 0 \rightarrow D = 0$$

2 In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

Solution $\ y(0) = C\cos(\omega 0) + D\sin(\omega 0) = 0$ gives $C = 0$

Differentiating $y(t)$ and equating to 1 at time $t = 0$ gives us :

$$y'(0) = D\omega = 1 \ \text{ and } \ D = \frac{1}{\omega} = \frac{1}{3}$$

3 Draw Figure 2.3 to show simple harmonic motion $y = A\ \cos\,(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.

Solution Notice that A is the maximum height $y_{\max}$. At $t = 0$ we see $y = A\cos(-\alpha) = A\cos\alpha$.

4 Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\ \cos\,(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x axis ?

Solution $\ f = \omega/2\pi = 60$ Hertz is equivalent to $\omega = 120\pi$ radians per second. With magnitude $A = 3$ and $\alpha = -45\,° = -\pi/4$ radians, $A\cos(\omega t - \alpha)$ becomes $3\cos(120\pi t + \pi/4)$. The point going around the circle hits the x-axis when that angle is a multiple of π. The first hit occurs at $120\pi t + \pi/4 = \pi$ and $120\,t = 3/4$ and $t = 3/480 = 1/160$.

5 If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit ? Your circular frequency is $f =$ _____ and your angular frequency is $\omega =$ _____ (with what units ?). The period is T.

Solution The distance around a circle of radius $R = 3$ miles is $2\pi R = 6\pi$ miles. The time T for a complete circuit at 60 miles per hour is $T = 6\pi/60 = \boldsymbol{\pi/10}$ hours. From $T = 1/f = 2\pi/\omega$ the circular frequency is $f = 10/\pi$ cycles per hour and $\omega = 2\pi f = 2\pi/T = 20$ radians per hour.

6 The total energy E in the oscillating spring-mass system is

$$E = \textbf{kinetic} \text{ energy in mass} + \textbf{potential} \text{ energy in spring} = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2.$$

Compute E when $y = C\,\cos\,\omega t + D\,\sin\,\omega t$. The energy is constant!

Solution $y = C\cos\omega t + D\sin\omega t$ has $dy/dt = -\omega C\sin\omega t + \omega D\cos\omega t$.

The total energy is $E = \frac{1}{2}m\omega^2(C^2\sin^2\omega t - 2CD\sin\omega t\cos\omega t + D^2\cos^2\omega t)$
$+\ \frac{1}{2}k(C^2\cos^2\omega t + 2CD\sin\omega t\cos\omega t + D^2\sin^2\omega t)$.

When $\omega = \sqrt{k/m}$ and $m\omega^2 = k$, use $\sin^2\omega t + \cos^2\omega t = 1$ to find

$$E = \frac{1}{2}k\left(C^2 + D^2\right)(\sin^2\omega t + \cos^2\omega t) = \frac{1}{2}k(C^2 + D^2) = \textbf{ constant}.$$

7 Another way to show that the total energy E is constant:

Multiply $\boldsymbol{my'' + ky = 0}$ by $\boldsymbol{y'}$. Then integrate $my'y''$ and kyy'.

Solution $(my'' + ky)\,y' = 0$ is the same as $\frac{d}{dt}(\frac{1}{2}my'^2 + \frac{1}{2}ky^2) = 0$.

This says that $E = \frac{1}{2}my'^2 + \frac{1}{2}ky^2$ is constant.

8 A **forced oscillation** has another term in the equation and $A\cos\omega t$ in the solution:

$$\frac{d^2y}{dt^2} + 4y = F\,\cos\,\omega t \quad \text{has} \quad y = C\,\cos\,2t + D\,\sin\,2t + A\,\cos\,\omega t.$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n). Find the forced amplitude A in the particular solution $y_p = A\,\cos\,\omega t$.

(b) In case $\omega = 2$ (forcing frequency $=$ natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

Solution (a) The frequency $\omega = 2$ gives the null solutions $y = C\cos 2t + D\sin 2t$: $\boldsymbol{y_n'' + 4y_n = 0}$.

The choice of A gives a particular solution $y_p = A\cos\omega t$. Substitute this y_p:

$$y_p'' + 4y_p = (-\omega^2 + 4)A\cos\omega t = F\cos\omega t \text{ and } A = \boldsymbol{\frac{F}{4-\omega^2}}.$$

(b) $\omega = 2$ leads to $A = \infty$ and that solution y_p breaks down: **resonance**. (The correct y_p will include a factor t)

9 Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F\,\cos\,\omega t \text{ with } \omega \neq \omega_n = \sqrt{k/m} \text{ (no resonance).}$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A *is fixed* by F).

Solution $y = y_n + y_p = C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{A}{k-m\omega^2}\cos\omega t.$

10 Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:

$$my'' = +ky \quad \text{including} \quad y'' = 4y.$$

Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

Solution The exponents in $y_n = Ce^{t\sqrt{k/m}} + De^{-t\sqrt{k/m}}$ are now real. Those numbers $\pm\sqrt{k/m}$ come from substituting $y = e^{st}$ into the differential equation:

$$my'' - ky = (ms^2 - k)e^{st} = 0 \text{ when } s = \sqrt{\boldsymbol{k/m}} \text{ and } s = \boldsymbol{-}\sqrt{\boldsymbol{k/m}}.$$

11 Here is a *fourth order* equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

Solution Substitute $y = e^{st}$ in the differential equation to find $s^4 = 16$. This has four solutions: $s = 2, -2, 2i, -2i$. The constants in $y = c_1e^{2t} + c_2e^{-2t} + c_3e^{2it} + c_4e^{-2it}$ are determined by the initial values $y(0), y'(0), y''(0), y'''(0)$.

12 To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

Solution Substitute $y_p = Ye^{ct}$ to find $(c^2 + 9)Ye^{ct} = e^{ct}$ and $Y = 1/(c^2 + 9)$. This is called the "exponential response function" in Section 2.4. The resonant case $Y = \infty$ occurrs when $c^2 + 9 = 0$ or $c = \pm 3i$. Then a new formula for $y(t)$ involves $\boldsymbol{te^{ct}}$ as well as e^{ct}.

13 In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency $\omega =$ what natural frequency?

Solution Substitute $y_p = Ae^{i\omega t}$ to find $i^2\omega^2Ae^{i\omega t} + 9Ae^{i\omega t} = e^{i\omega t}$. With $i^2 = -1$ this gives $A = 1/(9 - \omega^2)$. This blows up when $9 - \omega^2 = 0$ at the natural frequency $\omega_n = 3$.

14 If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0, 0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

Solution If $y(0) > 0$ and $y'(0) < 0$ then α falls between $3\pi/2$ and 2π. This occurs because the vector from $(0, 0)$ to $(y(0), y'(0)/\omega)$ is in the fourth quadrant.

15 Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping *UP* and bending *UP*.

Solution For $\frac{\pi}{2} < t < \pi$ $(90^\circ$ to $180^\circ)$, $y(t) = \sin t > 0$ but $y'(t) < 0$ and $y''(t) < 0$.

Note that for $\frac{3\pi}{2} < t < 2\pi$, $y(t) < 0$ but $y'(t) > 0$ and $y''(t) > 0$. The point is below the x-axis but the bold sine curve is sloping upwards and bending upwards.

16 (a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is $\boldsymbol{y_n}$**.)

(b) Solve $y'' + 100y = \cos\omega t$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be $\boldsymbol{y_p}$**.)

Solution (a) Substitute $y = e^{ct}$

$$\begin{aligned} y'' + 100y &= 0 \\ c^2e^{ct} + 100e^{ct} &= 0 \\ c^2 &= -100 \\ c &= \pm 10i \\ y &= ce^{10it} + de^{-10it} \end{aligned}$$

This can be rewritten in terms of sines and cosines of $10t$. Introducing the initial conditions we have:

$$\begin{aligned} y(t) &= A\cos(10t) + B\sin(10t) \\ y(0) &= A = 1 \\ y'(0) &= 10B = 10 \to B = 1 \\ y(t) &= \sin(10t) + \cos(10t) \end{aligned}$$

(b) As in equation (11) we assume the particular solution is

$$y(t) = \frac{1}{100 - \omega^2}\cos(\omega t)$$

Adding in the null solution and substituting in the initial conditions gives:

$$\begin{aligned} y(t) &= B\sin(10t) + A\cos(10t) + \frac{1}{100-\omega^2}\cos(\omega t) \\ y(0) &= B\sin(0) + A\cos(0) + \frac{1}{100-\omega^2}\cos(0) = 0 \\ A &= \frac{1}{\omega^2 - 100} \\ y'(0) &= 10B\cos(0) - 10A\sin(0) - \frac{\omega}{100-\omega^2}\sin(0) \\ &= 10B = 0 \to B = 0 \end{aligned}$$

Therefore the solution is:

$$y(t) = \frac{1}{100-\omega^2}(\cos(\omega t) - \cos(10t))$$

17 Find a particular solution $y_p = R\cos(\omega t - \alpha)$ to $y'' + 100y = \cos\omega t - \sin\omega t$.

Solution

$$\begin{aligned} \text{Right side } : \cos\omega t - \sin\omega t &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ \text{Diff. Eqn } : -\omega^2 R\cos(\omega t - \alpha) + 100R\cos(\omega t - \alpha) &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ (100 - \omega^2)R\cos(\omega t - \alpha) &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ \text{Then } \alpha = -\frac{\pi}{4} \text{ and } R &= \frac{\sqrt{2}}{100-\omega^2} \end{aligned}$$

18 Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A \cos \omega t$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

Solution The equation describing Simple Harmonic Motion is:

$$x(t) = A\cos(\omega t - \phi)$$

If the period is $T = 1$ second, the frequency is $f = 1$ Hertz or $\omega = 2\pi$ radians per second.

19 If the phase lag is α, what is the time lag in graphing $\cos(\omega t - \alpha)$?

Solution

$$\cos(\omega t - \alpha) = \cos\left(\omega\left(t - \frac{\alpha}{\omega}\right)\right)$$

Therefore the time lag is α/ω.

20 What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

Solution Similar to equation (15) we have

$$y_p(t) = \frac{\sin(\omega_n(t - T))}{m\omega_n}$$

The conditions at time T are:

$$y_p(T) = 0 \text{ and } y_p'(T) = \frac{1}{m}$$

Note that y_p starts from time $t = T$. We have $y_p = 0$.

21 (Good challenge) Show that $y = \int\limits_0^t g(t-s)f(s)\,ds$ has $my'' + ky = f(t)$.

1 Why is $y' = \int\limits_0^t g'(t-s)f(s)\,ds + g(0)f(t)$? Notice the two t's in y.

Solution 1 The variable t appears twice in the formula for y, so the derivative dy/dt has **two terms** (called the Leibniz rule). One term is the value of $g(t-s)f(s)$ at the upper limit $s = t$; this is from the Fundamental Theorem of Calculus. Since t also appears in the quantity $g(t-s)f(s)$, its derivative $g'(t-s)f(s)$ also appears in y'.

2 Using $g(0) = 0$, explain why $y'' = \int\limits_0^t g''(t-s)f(s)\,ds + g'(0)f(t)$.

Solution 2 Since $g(0) = 0$, part 1 produced $y' = \int\limits_0^t g'(t-s)f(s)ds$. Using the Leibniz rule again (now on y'), we get the two terms in y''.

3 Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

Solution 3 $my''+ky = m\left(\int\limits_0^t g''(t-s)f(s)\,ds + g'(0)f(t)\right)+k\left(\int\limits_0^t g(t-s)f(s)\,ds\right) = m(1/m)f(t)$. The integrals cancelled because $mg'' + kg = 0$.

22 With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y:

$$\textbf{Step response}\quad y(t) = \int_0^t \frac{\sin\omega_n(t-s)}{m\omega_n}\,1\,ds = y_p + y_n \ \text{ equals }\ \boldsymbol{\frac{1}{k} - \frac{1}{k}\cos\omega_n t}.$$

Solution This $y(t)$ certainly solves $my'' + ky = 1$. *Comment*: That formula for $y(t)$ fits with the usual $\int g(t-s)f(s)\,ds$ when $f = 1$ and the impulse response is $g(t) = (\sin\omega_n t)/m\omega_n$ in equation (15). And always this **step response should be the integral of the impulse response**. The natural frequency is $\omega_n = k/m$:

$$y(t) = \int_0^t \frac{\sin(\omega_n(t-s))}{m\omega_n}\,ds = -\left.\frac{\cos(\omega_n(t-s))}{m\omega_n^2}\right]_0^t = \frac{1}{k} - \frac{\cos(\omega_n t)}{k}.$$

Notice that without damping resistance, the step response oscillates forever—not approaching the steady state $y_\infty = 1/k$.

23 (Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

Solution

$$\frac{d^2y}{dt^2} = 0 \ \text{ gives }\ y_n = A + Bt.$$

We get the fundamental solution $\boldsymbol{g(t) = t}$ for $t \geq 0$ by starting the null solution with $g(0) = 0$ and $g'(0) = 1$. Then $g(t) = t$ and $g(t-s) = t-s$. This gives the particular solution for $d^2y/dt^2 = f(t)$ using formula (16):

$$y(t) = \int_0^t (t-s)f(s)\,ds.$$

24 For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

Solution Substitute $y = Ye^{i\omega t}$:

$$\begin{aligned} -Y(\omega)\omega^2 e^{i\omega t} &= e^{i\omega t} \\ Y(\omega) &= -1/\omega^2 \\ y_p(t)_p &= e^{i\omega t}/\omega^2 \end{aligned}$$

The null solution to $y'' = 0$ is $y(t)_n = At + B$.

When $A = 0$ and $B = 1$, we get $y_n = 1$. This causes resonance at $\omega = 0$, the solution formula $y_p = e^{i\omega t}/\omega^2$ breaks down.

25 Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite?

Solution
$$\begin{aligned} \text{Substitute } y(t) = Ye^{i\omega t} &\text{ in } my'' - ky = e^{i\omega t} \\ \text{Then } -Ym\omega^2 e^{i\omega t} - kYe^{i\omega t} &= e^{i\omega t} \\ -Ym\omega^2 - Yk &= 1 \\ Y(\omega) &= \frac{1}{k + m\omega^2} \end{aligned}$$

Y is infinite for $\omega = i\sqrt{\frac{k}{m}}$. No resonance at real frequencies ω, because the equation has $-ky$ instead of ky.

Problem Set 2.2, page 87

1 Mark the numbers $s_1 = 2+i$ and $s_2 = 1-2i$ as points in the complex plane. (The plane has a real axis and an imaginary axis.) Then mark the sum $s_1 + s_2$ and the difference $s_1 - s_2$.

Solution The sum is $s_1 + s_2 = 3 - i$. The difference is $s_1 - s_2 = 1 + 3i$.

2 Multiply $s_1 = 2 + i$ times $s_2 = 1 - 2i$. Check absolute values: $|s_1||s_2| = |s_1 s_2|$.

Solution The product $(2 + i)(1 - 2i)$ is $2 + i - 4i - 2i^2 = 4 - 3i$. The absolute values of $2 + i$ and $1 - 2i$ are $\sqrt{2^2 + 1^2} = \sqrt{5}$. The product $4 - 3i$ has absolute value $\sqrt{4^2 + 3^2} = 5$, agreeing with $(\sqrt{5})(\sqrt{5})$.

3 Find the real and imaginary parts of $1/(2 + i)$. Multiply by $(2 - i)/(2 - i)$:

$$\frac{1}{2+i} \quad \frac{2-i}{2-i} = \frac{2-i}{|2+i|^2} = ?$$

Solution $\dfrac{1}{2+i} \quad \dfrac{2-i}{2-i} = \dfrac{2-i}{5}$ In general $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$ because $z\overline{z} = |z|^2$.

4 *Triple angles* Multiply equation (2.10) by another $e^{i\theta} = \cos\theta + i\sin\theta$ to find formulas for $\cos 3\theta$ and $\sin 3\theta$.

Solution Equation (10) is $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$. Multiply by another $\cos\theta + i\sin\theta$:

$$\begin{aligned} (\cos\theta + i\sin\theta)^3 &= \cos\theta\cos 2\theta + i\sin\theta\cos 2\theta + i\cos\theta\sin 2\theta - \sin\theta\sin 2\theta \\ &= \cos(\theta + 2\theta) + i\sin(\theta + 2\theta) \text{ by sum formulas} \\ &= \cos 3\theta + i\sin 3\theta \end{aligned}$$

Real part $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **Imaginary part** $\sin 3\theta = 3\cos^2\theta\sin\theta - sin^3\theta$.

5 *Addition formulas* Multiply $e^{i\theta} = \cos\,\theta + i\,\sin\,\theta$ times $e^{i\phi} = \cos\,\phi + i\sin\,\phi$ to get $e^{i(\theta+\phi)}$. Its real part is $\cos\,(\theta+\phi) = \cos\,\theta\cos\,\phi - \sin\,\theta\,\sin\,\phi$. What is its imaginary part $\sin\,(\theta+\phi)$?

Solution The imaginary part of $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$ is the coeffieient of i : $\sin\theta\cos\phi + \cos\theta\sin\phi$ must equal $\sin(\theta+\phi)$.

6 Find the real part and the imaginary part of each cube root of 1. Show directly that the three roots add to zero, as equation (2.11) predicts.

Solution The cube roots of 1 are at angles $0, 2\pi/3, 4\pi/3$ (or $0\,°, 120\,°, 240\,°$). They are equally spaced on the unit circle (absolute value 1). The three roots are 1 and

$e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \boldsymbol{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \boldsymbol{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

The sum $1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}$ equals **zero**. Always : n roots of $2^n = 1$ add to zero.

7 The three cube roots of 1 are z and z^2 and 1, when $z = e^{2\pi i/3}$. What are the three cube roots of 8 and the three cube roots of i ? (The angle for i is $90°$ or $\pi/2$, so the angle for one of its cube roots will be _____. The roots are spaced by $120°$.)

Solution The three cube roots of 8 are 2 and $2e^{2\pi i/3} = -1 + \sqrt{3}i$ and $2e^{4\pi i/3} = -1 - \sqrt{3}i$. (They also add to zero.)

The three cube roots of $i = e^{\pi i/2}$ are $e^{\pi i/6}$ and $e^{5\pi i/6}$ and $e^{9\pi i/6}$ still add to zero.

8 (a) The number i is equal to $e^{\pi i/2}$. Then its i^{th} power $\boldsymbol{i^i}$ comes out equal to a real number, using the fact that $(e^s)^t = e^{st}$. What is that real number $\boldsymbol{i^i}$?

(b) $e^{i\pi/2}$ is also equal to $e^{5\pi i/2}$. Increasing the angle by 2π does not change $e^{i\theta}$ — it comes around a full circle and back to i. Then i^i has another real value $(e^{5\pi i/2})^i = e^{-5\pi/2}$. What are all the possible values of i^i ?

Solution (a) The i^{th} power of $i = e^{\pi i/2}$ is $i^i = (e^{\pi i/2})^i = e^{-\pi/2}$ by the ordinary rule for exponents. Surprising that i^i is a real number.

(b) i also equals $e^{5\pi i/2}$ since $\frac{5\pi}{2}$ is a full rotation from $\frac{\pi}{2}$. So i^i also equals $(e^{5\pi i/2})^i = e^{-5\pi/2}$—and infinitely many other possibilities $e^{-(2\pi+1)\pi/2}$ for every whole number n. We are on a "Riemann surface" with an infinity of layers.

9 The numbers $s = 3 + i$ and $\overline{s} = 3 - i$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$ and also $\overline{s}^2 + B\overline{s} + C = 0$. Those numbers s and $\overline{s}$ are the two roots of the quadratic equation $x^2 + Bx + C = 0$.

Solution $-B = s + \overline{s} = (3+i) + (3-i) = \mathbf{6}$. $C = (s)(\overline{s}) = (3+i)(3-i) = \mathbf{10}$.

Then s and $\overline{s}$ are the two roots of $x^2 - Bx + C = x^2 - 6x + 10 = 0$. The usual quadratic formula gives $\frac{6\pm\sqrt{36-40}}{2} = \frac{6\pm 2i}{2} = 3 \pm i$.

10 The numbers $s = a + i\omega$ and $\overline{s} = a - i\omega$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$. The two solutions of $x^2 + Bx + C = 0$ are s and $\overline{s}$.

Solution $-B = (a + i\omega) + (a - i\omega) = \boldsymbol{2a}$ $\quad C = (a+i\omega)(a-i\omega) = \boldsymbol{a^2 + i\omega^2}$.

Then the roots of $x^2 - 2ax + a^2 + \omega^2 = 0$ are $x = \frac{2a\pm\sqrt{-4\omega^2}}{2} = a \pm i\omega$.

11 (a) Find the numbers $(1+i)^4$ and $(1+i)^8$.

(b) Find the polar form $re^{i\theta}$ of $(1+i\sqrt{3})/(\sqrt{3}+i)$.

Solution (a) $(1+i)^4 = (\sqrt{2}e^{i\pi/4})^4 = (\sqrt{2})^4 e^{i\pi} = \mathbf{-4}$

$(1+i)^8 =$ square of $(1+i)^4 =$ (square of $-4) = \mathbf{16}$.

(b) $(1+i\sqrt{3})(\sqrt{3}+i) = \sqrt{3}+3i+i-\sqrt{3} = \boldsymbol{4i}$. Dividing by $(2)(2)=4$ this is

$(\cos\theta + i\sin\theta)(\sin\theta + i\cos\theta) = i(\cos^2\theta + \sin^2\theta) = \boldsymbol{i}$.

The unexpected part is $\boldsymbol{\sin\theta + i\cos\theta} = \cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta) = \boldsymbol{e^{i(\pi/2-\theta)}}$.

Then the product of $e^{i\theta}$ and $e^{i(\pi/2-\theta)}$ is $e^{i\pi/2}$ which equals $\boldsymbol{i}$ as above.

12 The number $z = e^{2\pi i/n}$ solves $z^n = 1$. The number $Z = e^{2\pi i/2n}$ solves $Z^{2n} = 1$. How is z related to Z ? (This plays a big part in the Fast Fourier Transform.)

Solution If $Z = e^{2\pi i/2n}$ then $Z^2 = e^{2\pi i/n} = z$. The square of the $2n$ th root is the n th root. The angle for Z is half the angle for z.

The Fast Fourier Transform connects the transform at level $2n$ to the transform at level n (and on down to $n/2$ and $n/4$ and eventually to 1, if these numbers are powers of 2).

13 (a) If you know $e^{i\theta}$ and $e^{-i\theta}$, how can you find $\sin\theta$?

(b) Find all angles θ with $e^{i\theta} = -1$, and all angles ϕ with $e^{i\phi} = i$.

Solution (a) $\sin\theta = \frac{1}{2i}[(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)] = \boldsymbol{\frac{1}{2i}(e^{i\theta} - e^{-i\theta})}$.

(b) The angles with $e^{i\theta} = -1$ are $\theta = \pi +$ (any multiple of $2\pi) = \boldsymbol{(2n+1)\pi}$.

The angles with $e^{i\phi} = 1$ are $\phi =$ any multiple of $2\pi = \boldsymbol{2n\pi}$.

14 Locate all these points on one complex plane:

(a) $2+i$ (b) $(2+i)^2$ (c) $\dfrac{1}{2+i}$ (d) $|2+i|$

Solution $2+i$ is in quadrant 1. $(2+i)^2$ is in quadrant 2. $\frac{1}{2+i}$ is in quadrant 4. $|2+i| = \sqrt{5}$ is on the positive real axis.

15 Find the absolute values $r = |z|$ of these four numbers. If θ is the angle for $6+8i$, what are the angles for these four numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\dfrac{1}{6-8i}$ (d) $8i+6$

Solution The absolute values are 10 and 100 and $\frac{1}{10}$ and 10.

The angles are $2\pi - \theta$ (or just $-\theta$), $2\pi - 2\theta$ (or just -2θ), θ, and θ.

16 What are the real and imaginary parts of $e^{a\,+\,i\pi}$ and $e^{a\,+\,i\omega}$?

Solution $e^{a+i\pi} = e^a e^{i\pi} = -e^{-a}(real)$ $\qquad e^{a+i\omega} = e^a\cos\omega + ie^a\sin\omega$

17 (a) If $|s| = 2$ and $|z| = 3$, what are the absolute values of sz and s/z ?

(b) Find upper and lower bounds in $L \le |s+z| \le U$. When does $|s+z| = U$?

Solution (a) $|sz| = |s|\,|z| = 6 \qquad |s/z| = |s|/|z| = 2/3$.

(b) The best bounds are $L = 1$ and $U = 5 : 1 \le |s+z| \le 5$.

That bound 5 is reached when s and z have the **same angle** θ.

18 (a) Where is the product $(\sin\,\theta + i\,\cos\,\theta)(\cos\,\theta + i\,\sin\,\theta)$ in the complex plane?

(b) Find the absolute value $|S|$ and the polar angle ϕ for $S = \mathbf{sin}\ \boldsymbol{\theta} + \boldsymbol{i}\ \mathbf{cos}\ \boldsymbol{\theta}$.

This is my favorite problem, because S combines $\cos\theta$ and $\sin\theta$ in a new way. To find ϕ, you could plot S or add angles in the multiplication of part (a).

Solution $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta) = \sin\theta\cos\theta + i(\sin^2\theta + \cos^2\theta) - \cos\theta\sin\theta = \boldsymbol{i}$. The product is imaginary. The angles must add to $90\,^\circ$.

Since $\cos\theta + i\sin\theta$ is at angle θ and the product i is at angle $\pi/2$, the first factor $\sin\theta + i\cos\theta$ must be $e^{i\phi}$ at angle $\boldsymbol{\phi = \frac{\pi}{2} - \theta}$. The absolute value is $\mathbf{1}$. See also Problem 2.2.11.

19 Draw the spirals $e^{(1\,-\,i)t}$ and $e^{(2\,-\,2i)t}$. Do those follow the same curves? Do they go clockwise or anticlockwise? When the first one reaches the negative x-axis, what is the time T? What point has the second one reached at that time?

Solution The spiral $e^{(1-i)t} = e^t e^{-it}$ starts at 1 when $t = 0$. As t increases, it goes outward (absolute value e^t) and clockwise (the angle is $-t$). It reaches the negative X axis when $\boldsymbol{t = \pi}$. The second spiral $e^{(2-2i)t}$ is **the same curve** but traveled twice as fast. Its angle $-2t$ reaches $-\pi$ (the X-axis) at time $t = \pi/2$.

20 The solution to $d^2y/dt^2 = -y$ is $y = \cos\,t$ if the initial conditions are $y(0) =$ _____ and $y'(0) =$ _____. The solution is $y = \sin\,t$ when $y(0) =$ _____ and $y'(0) =$ _____. Write each of those solutions in the form $c_1\,e^{it} + c_2\,e^{-it}$, to see that real solutions can come from complex c_1 and c_2.

Solution $y = \cos t$ has $y(0) = 1$ and $y'(0) = 0$. $y = \sin t$ has $y(0) = 0$ and $y'(0) = 1$. Those solutions are $\cos t = (e^{it} + e^{-it})/2$ and $\sin t = (e^{it} - e^{-it})/2i$.

The complete solution to $y'' = -y$ is $y = C_1\cos t + C_2\sin t$. The same complete solution is $C_1(e^{it} + e^{-it})/2 + C_2(e^{it} - e^{-it})/2i = c_1e^{it} + c_2e^{-it}$ with $c_1 = (C_1 + C_2)/2$ and $c_2 = (C_1 - C_2)/2i$.

21 Suppose $y(t) = e^{-t}\,e^{it}$ solves $y'' + By' + Cy = 0$. What are B and C? If this equation is solved by $y = e^{3it}$, what are B and C?

Solution If $y = e^{st}$ solves $y'' + By' + Cy = 0$ then substituting e^{st} shows that $s^2 + Bs + C = 0$. This problem has $s = -1 + i$. Then the other root is the conjugate $\overline{s} = -1 - i$ (always assuming B and C are real numbers). The sum -2 is $-B$. The product $(s)(\overline{s}) = 2$ is C. So the underlying equation is $y'' + 2y' + 2y = 0$.

22 From the multiplication $e^{iA}\ e^{-iB} = e^{i(A\,-\,B)}$, find the "subtraction formulas" for $\cos\,(A - B)$ and $\sin\,(A - B)$.

Solution Start with the fact that $e^{iA}e^{-iB} = e^{i(A-B)}$. Use Euler's formula:

$$(\cos A + i\sin A)(\cos B - i\sin B) = \cos(A - B) + i\sin(A - B).$$

Compare real parts: $\cos A\cos B + \sin A\sin B = \cos(A - B)$.

Compare imaginary parts: $\sin A\cos B - \cos A\sin B = \sin(A - B)$.

23 (a) If r and R are the absolute values of s and S, show that rR is the absolute value of sS. (Hint: Polar form!)

(b) If $\overline{s}$ and $\overline{S}$ are the complex conjugates of s and S, show that $\overline{s}\overline{S}$ is the complex conjugate of sS. (Polar form!)

Solution (a) Given: $s = re^{i\theta}$ and $S = Re^{i\phi}$ for some angles θ and ϕ. Then $\boldsymbol{sS = rRe^{i(\theta+\phi)}}$. The absolute value of sS is $rR =$ (absolute value of s) (absolute value of S).

(b) Now $\overline{s} = re^{-i\theta}$ and $\overline{S} = Re^{-i\phi}$. Multiply to get $\overline{s}\overline{S} = rRe^{-i(\theta+\phi)}$. This is the complex conjugate of $sS = rRe^{i(\theta+\phi)}$ in part (a).

24 Suppose a complex number s solves a real equation $s^3 + As^2 + Bs + C = 0$ (with A, B, C real). Why does the complex conjugate $\overline{s}$ also solve this equation? "*Complex solutions to real equations come in conjugate pairs s and $\overline{s}$.*"

Solution The complex conjugate of $s^3+As^2+Bs+C=0$ is $\overline{s}^3+A\overline{s}^2+B\overline{s}+C=0$.

We took the conjugate of every term using the fact that A, B, C are real. (The conjugates of s^2 and s^3 are $\overline{s}^2$ and $\overline{s}^3$ by Problem 23).

For quadratic equations $x^2 + Bx + C = 0$, the formula $(-B \pm \sqrt{B^2 - 4C})/2$ is producing **complex conjugates from** $\pm$ when $B^2 - 4C$ is negative.

25 (a) If two complex numbers add to $s + S = 6$ and multiply to $sS = 10$, what are s and S? (They are complex conjugates.)

(b) If two numbers add to $s + S = 6$ and multiply to $sS = -16$, what are s and S? (Now they are real.)

Solution (a) s and S must have the same real part 3. They each have magnitude $\sqrt{10}$. So s and S are $3 + i$ and $3 - i$.

(b) If $s + S = 6$ and $sS = -16$ then s and S are the roots of $x^2 - 6x - 16 = 0$. Factor into $(x-8)(x+2) = 0$ to see that s and S are 8 and -2. (Not complex conjugates! In this example $B^2 - 4AC = 36 + 64 = 100$ and the quadratic has real roots 8 and -2.)

26 If two numbers s and S add to $s + S = -B$ and multiply to $sS = C$, show that s and S solve the quadratic equation $s^2 + Bs + C = 0$.

Solution Just check that $(x - s)(x - S) = x^2 + Bx + C$. The left side is $x^2 - (s + S)x + sS$. Then $s + S$ agrees with $-B$ and sS matches C.

27 Find three solutions to $s^3 = -8i$ and plot the three points in the complex plane. What is the sum of the three solutions?

Solution The three solutions have the same absolute value 2. Their angles are separated by $120^\circ = 2\pi/3$ radians $= 4\pi/6$ radians. The first angle is $\theta = -30^\circ = -\pi/6$ radians (so that $3\theta = -90^\circ = -\pi/2$ radians matches $-i$).

The answers are $\boldsymbol{2e^{-\pi i/6}, 2e^{3\pi i/6}, 2e^{7\pi i/6}}$. They add to $\mathbf{0}$.

28 (a) For which complex numbers $s = a + i\omega$ does e^{st} approach 0 as $t \to \infty$? Those numbers s fill which "half–plane" in the complex plane?

(b) For which complex numbers $s = a + i\omega$ does s^n approach 0 as $n \to \infty$? Those numbers s fill which part of the complex plane? Not a half-plane!

Solution (a) If $s = a + i\omega$, the absolute value of e^{st} is e^{at}. This approaches 0 if a is **negative**. The numbers $s = a + i\omega$ with negative a fill the **left half-plane**.

(b) This part asks about the powers s^n instead of e^{st}. Powers of s approach zero if $|s| < 1$. This is the same as $\boldsymbol{a^2 + \omega^2 < 1}$. These complex numbers fill the **inside of the unit circle**.

Problem Set 2.3, page 101

1 Substitute $y = e^{st}$ and solve the characteristic equation for s:

(a) $2y'' + 8y' + 6y = 0$ (b) $y'''' - 2y'' + y = 0$.

Solution (a) $2s^2 + 8s + 6$ factors into $2(s+3)(s+1)$ so the roots are $s = -3$ and $s = -1$. The null solutions are $y = e^{-3t}$ and $y = e^{-t}$ (and any combination).

(b) $s^4 - 2s^2 + 1$ factors into $(s^2 - 1)^2$ which is $(s-1)^2(s+1)^2$. The roots are $s = 1, 1, -1, -1$. The null solutions are $y = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t}$. (The factor t enters for double roots.)

2 Substitute $y = e^{st}$ and solve the characteristic equation for $s = a + i\omega$:

(a) $y'' + 2y' + 5y = 0$ (b) $y'''' + 2y'' + y = 0$

Solution (a) $s^2 + 2s + 5 = 0$ gives $s = (-2 \pm \sqrt{4 - 20})/2 = -1 \pm 2i = a + i\omega$. Then $y = e^{-t}\cos 2t$ and $y = e^{-t}\sin 2t$ solve the (null) equation.

(b) $s^4 + 2s^2 + 1 = 0$ factors into $(s^2+1)(s^2+1) = 0$. The roots are $i, i, -i, -i$. The solutions are $y = c_1 e^{it} + c_2 te^{it} + c_3 e^{-it} + c_4 te^{-it}$. They can also be written as $y = C_1 \cos t + C_2 t \cos t + C_3 \sin t + C_4 t \sin t$.

3 Which second order equation is solved by $y = c_1 e^{-2t} + c_2 e^{-4t}$? Or $y = te^{5t}$?

Solution If $s = -2$ and $s = 4$ are the exponents, the characteristic equation must be $s^2 + 6s + 8 = 0$ coming from $y'' + 6y' + 8y = 0$.

If $y = te^{5t}$ is a solution, then 5 is a **double root**. The characteristic equation must be $(s-5)^2 = s^2 - 10s + 25 = 0$ coming from $y'' - 10y' + 25y = 0$.

4 Which second order equation has solutions $y = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t$?

Solution Those sine/cosine solutions combine to give $e^{-2t}e^{3it}$ and $e^{-2t}e^{-3it}$. Then $\boldsymbol{s = -2 \pm 3i}$. The sum is -4 and 4, the product is $2^2 + 3^2 = 13$.

The equation must be $\boldsymbol{y'' - 4y' + 13y = 0}$.

5 Which numbers B give (under) (critical) (over) damping in $4y'' + By' + 16y = 0$?

Solution The roots of $4s^2 + Bs + 16$ are $s = (-B \pm \sqrt{B^2 - 16^2})/2$. We have underdamping for $B^2 > 16^2$ (real roots); critical damping for $B^2 = 16^2$ (double root); overdamping for $B^2 < 16^2$ (complex roots).

6 If you want oscillation from $my'' + by' + ky = 0$, then b must stay below _____ .

Solution Oscillations mean underdamping. We need $\boldsymbol{b^2 < 4km}$.

Problems 7–16 are about the equation $\boldsymbol{As^2 + Bs + C = 0}$ and the roots $\boldsymbol{s_1, s_2}$.

7 The roots s_1 and s_2 satisfy $s_1 + s_2 = -2p = -B/2A$ and $s_1 s_2 = \omega_n^2 = C/A$. Show this two ways:

(a) Start from $As^2 + Bs + C = A(s - s_1)(s - s_2)$. Multiply to see $s_1 s_2$ and $s_1 + s_2$.

(b) Start from $s_1 = -p + i\omega_d$, $s_2 = -p - i\omega_d$

Solution (a) Match $As^2 + Bs + C$ to $A(s-s_1)(s-s_2) = As^2 - A(s_1+s_2)s + As_1 s_2$. Then $-B = A(s_1 + s_2)$ and $C = As_1 s_2$. **Error in problem**: $s_1 + s_2$ equals $-B/A$ and not $-B/2A$.

(b) $s_1 + s_2 = (-p + i\omega_d) + (-p - i\omega_d) = -2p = \boldsymbol{-B/A}$. Then $p = B/2A$.

8 Find s and y at the bottom point of the graph of $y = As^2 + Bs + C$. At that minimum point $s = s_{\text{min}}$ and $y = y_{\text{min}}$, the slope is $dy/ds = 0$.

Solution The minimum of $As^2 + Bs + C$ is located by derivative $= 2As + B = 0$. Then $s = -B/2A$ (which is p). The value of $As^2 + Bs + C$ at that minimum point is $A(B^2/4A^2) - (B^2/2A) + C = -(B^2/4A) + C = (4AC - B^2)/4A$.

Notice: If $B^2 < 4AC$ the minimum is > 0. Then $As^2 + Bs + C \neq 0$ for real s.

9 The parabolas in Figure 2.10 show how the graph of $y = As^2 + Bs + C$ is raised by increasing B. Using Problem 8, show that the bottom point of the graph moves left (change in s_{min}) and down (change in y_{min}) when B is increased by ΔB.

Solution For the graph of $y = As^2+Bs+C$, the bottom point is $y = (4AC-B^2)/4A$ at $s = -B/2A$. When B is increased, s moves left and y moves down. (The convention is $A > 0$.)

10 (recommended) Draw a picture to show the paths of s_1 and s_2 when $s^2 + Bs + 1 = 0$ and the damping increases from $B = 0$ to $B = \infty$. At $B = 0$, the roots are on the _____ axis. As B increases, the roots travel on a circle (why?). At $B = 2$, the roots meet on the real axis. For $B > 2$ the roots separate to approach 0 and $-\infty$. *Why is their product s_1s_2 always equal to 1?*

Solution The roots of $s^2 + Bs + 1$ will move as B increases from 0 to ∞. At $B = 0$, the roots of $s^2 + 1 = 0$ are **imaginary**: $s = \pm i$. As B increases, the roots are complex conjugates always multiplying to $s_1s_2 = 1$. They are on the **unit circle**. When B reaches 2, the roots of $s^2 + 2s + 1 = (s + 1)^2$ meet at $s = -1$. (Each root traveled a quarter-circle, from $\pm i$ to -1.) For larger B and overdamping $B^2 > 4AC = 4(1)(1)$, the roots s_1s_2 are **real**. One root moves from -1 toward $s = 0$, the other moves from -1 toward $-\infty$. **At all times** $s_1s_2 = C/A = 1/1$.

11 (this too if possible) Draw the paths of s_1 and s_2 when $s^2+2s+k = 0$ and the stiffness increases from $k = 0$ to $k = \infty$. When $k = 0$, the roots are _____ . At $k = 1$, the roots meet at $s =$ _____. For $k \to \infty$ the two roots travel up/down on a _____ in the complex plane. *Why is their sum $s_1 + s_2$ always equal to -2?*

Solution This problem changes k in $s^2 + 2s + k = 0$. So the **sum** $s_1 + s_2$ stays at -2, the **product** $s_1s_2 = k/1$ increases from 0 to ∞.

When $k = 0$, the roots -2 and 0 are **real**. When $k = 1$, the roots are -1 and -1 (**repeated**). When $k \to \infty$, then $B^2 - 4AC = 4 - 4k$ is negative and the roots $s = -1 \pm i\omega$ are **complex conjugates**. They lie on the vertical line $x = \text{Re}\, s = -1$ in the complex plane.

12 If a polynomial $P(s)$ has a double root at $s = s_1$, then $(s - s_1)$ is a double factor and $P(s) = (s - s_1)^2Q(s)$. Certainly $P = 0$ at $s = s_1$. Show that also $dP/ds = 0$ at $s = s_1$. Use the product rule to find dP/ds.

Solution $P = (s - s_1)^2Q(s)$ has a double root $s = s_1$, together with the roots of $Q(s)$. The derivative is

$$\frac{dP}{ds} = (s - s_1)^2 \frac{dQ}{ds} + 2(s - s_1)Q(s). \text{ This is zero at } s = s_1.$$

13 Show that $y'' = 2ay' - (a^2 + \omega^2)y$ leads to $s = a \pm i\omega$. Solve $y'' - 2y' + 10y = 0$.

Solution Substitute $y = e^{st}$ in the differential equation. Cancel e^{st} from every term to leave $s^2 = 2as - (a^2 + \omega^2)$.

The roots are $a \pm i\omega$, their sum is $2a$, their product is $a^2 + \omega^2$.

For $y'' - 2y' + 10y = 0$ (negative damping!) the sum is $s_1 + s_2 = 2$ and the product is 10. The roots are $\boldsymbol{s = 1 \pm 3i}$. The solution $y(t)$ is $c_1 e^{(1+3i)t} + c_2 e^{(1-3i)t}$.

14 The undamped *natural frequency* is $\omega_n = \sqrt{k/m}$. The two roots of $ms^2 + k = 0$ are $s = \pm i\omega_n$ (pure imaginary). With $p = b/2m$, the roots of $ms^2 + bs + k = 0$ are $\boldsymbol{s_1, s_2 = -p \pm \sqrt{p^2 - \omega_n^2}}$. The coefficient $p = b/2m$ has the units of $1/\text{time}$.

Solve $s^2 + 0.1s + 1 = 0$ and $s^2 + 10s + 1 = 0$ with numbers correct to two decimals.

Solution $s^2 + 0.1s + 1 = 0$ gives $s = (-0.1 \pm \sqrt{0.01 - 4})/2 = (-0.1 \pm i\sqrt{3.99})/2$.

How to approximate that square root?

The square root of $4 - x$ is close to $2 - \frac{1}{4}x$. Computing $(2 - \frac{1}{4}x)^2 = 4 - x + x^2/16$ we see the small error $x^2/16$. Our problem has $4 - x = 3.99$ and $x = 1/100$. So the square root is close to $2 - \frac{1}{400}$. The roots are $s \approx \left(-0.1 \pm i\left(2 - \frac{1}{400}\right)\right)/2$. In other words $s = \boldsymbol{-0.05 + i(1 - 0.00125)}$.

For $s^2 + 10s + 1 = 0$, the roots are $s = (-10 \pm \sqrt{(100 - 4})/2 = -5 \pm \sqrt{25 - 1}$. The square root of $25 - x$ is close to $5 - \frac{1}{10}x$, because squaring the approximation gives $25 - x + (x^2/100)$. Our example has $x = 1$ and $s \approx -5 \pm (5 - \frac{1}{10})$, which gives the two approximate roots $s = -\frac{1}{10}$ and $-10 + \frac{1}{10}$.

These add to -10 (correct) and multiply to .99 (almost correct).

15 With large overdamping $\boldsymbol{p >> \omega_n}$, the square root $\sqrt{p^2 - \omega_n^2}$ is close to $p - \omega_n^2/2p$. Show that the roots of $ms^2 + bs + k$ are $s_1 \approx \boldsymbol{-\omega_n^2/2p}$ = (small) and $s_2 \approx -2p = \boldsymbol{-b/m}$ (large).

Solution Use that approximate square root $p - \omega_n^2/2p$ in the quadratic formula:

$$s = -p \pm \sqrt{p^2 - \omega_n^2} \approx -p \pm \left(p - \frac{\omega_n^2}{2p}\right). \text{ Then } s = -\frac{\omega_n^2}{2p} \text{ and } -2p + \frac{\omega_n^2}{2p}.$$

When p is large and ω_n is small, a small root is near $-\omega_n^2/2p$ and a large root is near $-2p$. (Their product is the correct ω_n^2, their sum is close to the correct $-2p$.)

16 With small underdamping $\boldsymbol{p << \omega_n}$, the square root of $p^2 - \omega_n^2$ is approximately $i\omega_n - ip^2/2\omega_n$. Square that to come close to $p^2 - \omega_n^2$. Then the frequency for small underdamping is reduced to $\omega_d \approx \omega_n - p^2/2\omega_n$.

Solution Now p is much **smaller** than ω_n. So the roots $s = -p \pm \sqrt{p^2 - \omega_n^2}$ are complex. The damped frequency $\omega_d = \sqrt{\omega_n^2 - p^2}$ is close to ω_n and the correction term is $-p^2/2\omega_n$ from the approximation $\omega_n - p^2/2\omega_n$ to the square root. (Square that approximation to see $\omega_n^2 - p^2 + (p^4/4\omega_n^2)$.

17 Here is an 8th order equation with eight choices for solutions $y = e^{st}$:

$$\frac{d^8y}{dt^8} = y \quad \text{becomes} \quad s^8 e^{st} = e^{st} \quad \text{and} \quad \boldsymbol{s^8 = 1} : \quad \text{Eight roots in Figure 2.6.}$$

Find two solutions e^{st} that don't oscillate (s is real). Find two solutions that only oscillate (s is imaginary). Find two that spiral in to zero and two that spiral out.

Solution The equation $s^8 = 1$ has 8 roots. Two of them are $s = 1$ and $s = -1$ (**real**: no oscillation). Two are $s = i$ and $s = -i$ (**imaginary**: pure oscillation). Two are $s = e^{2\pi i/8}$ and $s = e^{-2\pi i/8}$ (positive real parts $\cos\frac{\pi}{4}$: (oscillating growth, spiral out). Two are $s = e^{3\pi i/4}$ and $s = e^{-3\pi i/4}$ (negative real parts: oscillating decay, spiral in).

18 $A_n \dfrac{d^n y}{dt^n} + \cdots + A_1 \dfrac{dy}{dt} + A_0 y = 0$ leads to $\boldsymbol{A_n s^n + \cdots + A_1 s + A_0 = 0}$.

The n roots $s_1, \ldots, s_n$ produce n solutions $y(t) = e^{st}$ (if those roots are distinct). Write down n equations for the constants c_1 to c_n in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$ by matching the n initial conditions for $y(0)$, $y'(0)$, …, $D^{n-1}y(0)$.

Solution The n roots give n solutions $y = e^{st}$ (when the roots s are all different). There are n constants in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$. These constants are found by matching the n initial conditions $y(0), y'(0), \ldots$ **Take derivatives of y and set $t = 0$:**

$$\begin{aligned} c_1 + c_2 + \cdots + c_n &= y(0) \\ c_1 s_1 + c_2 s_2 + \cdots + c_n s_n &= y'(0) \\ c_1 s_1^2 + c_2 s_2^2 + \cdots + c_n s_n^2 &= y''(0) \\ \cdots &= \cdots \end{aligned}$$

The n by n matrix A in those equations is the transpose of a **Vandermonde matrix** :

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ \cdot & \cdot & \cdots & \cdot \end{bmatrix}$$

19 **Find two solutions to $\boldsymbol{d^{2015}y/dt^{2015} = dy/dt}$.** Describe all solutions to $s^{2015} = s$.

Solution With $y = e^{st}$ we find $\boldsymbol{s^{2015} = s}$. One solution has $s = 1$ and $y = e^t$. The other 2014 solutions have $\boldsymbol{s^{2014} = 1}$ ($s = 1$ is double! Second solution $y = te^t$.) The 2014 values of s are equally spaced around the unit circle, separated by the angle $2\pi/2014$.

20 The solution to $y'' = 1$ starting from $y(0) = y'(0) = 0$ is $y(t) = t^2/2$. The fundamental solution to $g'' = \delta(t)$ is $g(t) = t$ by Example 5. Does the integral $\int g(t-s)f(s)ds = \int (t-s)ds$ from 0 to t give the correct solution $y = t^2/2$?

Solution The main formula for a particular solution is correct :

$$y_p(t) = \int_0^t g(t-s)f(s)\,ds = \int_0^t (t-s)\,ds = -\left.\frac{(t-s)^2}{2}\right]_{s=0}^{t} = \frac{t^2}{2}.$$

21 The solution to $y'' + y = 1$ starting from $y(0) = y'(0) = 0$ is $y = 1 - \cos t$. The solution to $g'' + g = \delta(t)$ is $\boldsymbol{g(t) = \sin t}$ by equation (13) with $\omega = 1$ and $A = 1$. Show that $1 - \cos t$ agrees with the integral $\int g(t-s)f(s)ds = \int \sin(t-s)ds$.

Solution The formula for a particular solution is again correct :

$$y_p(t) = \int_0^t g(t-s)f(s)\,ds = \int_0^t \sin(t-s)\,ds = \cos(t-s)\Big]_{s=0}^{t} = 1 - \cos t.$$

Then $y_p'' + y_p = 1$.

22 The step function $H(t) = 1$ for $t \geq 0$ is the integral of the delta function. **So the step response $\boldsymbol{r(t)}$ is the integral of the impulse response.** This fact must also come from our basic solution formula :

$$Ar'' + Br' + Cr = 1 \text{ with } r(0) = r'(0) = 0 \text{ has } \boldsymbol{r(t) = \int_0^t g(t-s)\,1\,ds}$$

Change $t - s$ to τ and change ds to $-d\tau$ to confirm that $r(t) = \int_0^t g(\tau)d\tau$.

Section 2.5 will find two good formulas for the step response $r(t)$.

Solution For any equation $Ar'' + Br' + Cr = 1$ with $f(t) = 1$, y_p comes from the integral formula:

$$y_p = \int_0^t g(t-s)f(s)\,ds = \int_0^t g(t-s)\,ds. \text{ Change to } t-s=\tau \text{ and } -ds = d\tau \text{ and}$$

$$-\int_t^0 g(\tau)d\tau = +\int_0^t \boldsymbol{g(\tau)d\tau} = \text{ step response}$$

Problem Set 2.4, page 114

Problems 1-4 use the exponential response $y_p = e^{ct}/P(c)$ to solve $P(D)y = e^{ct}$.

1 Solve these constant coefficient equations with exponential driving force:

(a) $y_p'' + 3y_p' + 5y_p = e^t$ (b) $2y_p'' + 4y_p = e^{it}$ (c) $y'''' = e^t$

Solution (a) Substitute $y = Ye^t$ to find Y:

$$Ye^t + 3Ye^t + 5Ye^t = e^t \text{ gives } 9Y = 1 \text{ and } Y = 1/9 : y = e^t/9$$

(b) Substitute $y = Ye^{it}$: $2i^2Ye^{it} + 4Ye^{it} = e^{it}$: $2Y = 1$: $y = e^{it}/2$

(c) Substitute $y = Ye^t$ to find $Y = 1$ and $y = e^t$.

2 These equations $P(D)y = e^{ct}$ use the symbol D for d/dt. Solve for $y_p(t)$:

(a) $(D^2 + 1)y_p(t) = 10e^{-3t}$ (b) $(D^2 + 2D + 1)y_p(t) = e^{i\omega t}$

(c) $(D^4 + D^2 + 1)y_p(t) = e^{i\omega t}$

Solution (a) Substitute $y = Ye^{-3t}$ to find $9Y + Y = 10$: $Y = 1$ and $y = e^{-3t}$.

(b) Substitute $y = Ye^{i\omega t}$ to find $((i\omega)^2 + 2i\omega + 1)Y = 1$ and $Y = 1/(1 - \omega^2 + 2i\omega)$.

(c) Substitute $y = Ye^{i\omega t}$ to find $((i\omega)^4 + (i\omega)^2 + 1)Y = 1$ and $Y = 1/(1 - \omega^2 + \omega^4)$.

3 How could $y_p = e^{ct}/P(c)$ solve $y'' + y = e^te^{it}$ and then $y'' + y = e^t \cos t$?

Solution First, $y'' + y = e^{(1+i)t}$ has $c = 1 + i$ and $y = Ye^{ct} = e^{(1+i)t}/((1+i)^2 + 1) = e^te^{it}/(1 + 2i)$. The **real part** of that y solves the equation driven by $e^t \cos t$:

$$y = \text{Re}\left[e^t(\cos t + i\sin t)\left(\frac{1-2i}{1^2+2^2}\right)\right] = \frac{1}{5}e^t(\cos t + 2\sin t).$$

4 (a) What are the roots s_1 to s_3 and the null solutions to $y_n''' - y_n = 0$?

(b) Find particular solutions to $y_p''' - y_p = e^{it}$ and to $y_p''' - y_p = e^t - e^{i\omega t}$.

Solution (a) $y = e^{st}$ leads to $s^3 - 1 = 0$. The three roots $s = 1, s = e^{2\pi i/3} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}, s = e^{-2\pi i/3} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$ give three null solutions $y_n = e^t, e^{-t/2}\cos\frac{\sqrt{3}}{2}t, e^{-t/2}\sin\frac{\sqrt{3}}{2}t$.

(b) The particular solution with $f = e^{it}$ is $y_p = e^{it}/(i^3 - 1)$.

The particular solution with $f = e^t - e^{i\omega t}$ looks like $y = e^t/(1^3 - 1) - e^{i\omega t}((i\omega)^3 - 1)$. But the first part has $1^3 - 1 = 0$ and resonance : then $e^t/(1^3 - 1)$ changes by equation (19) to $te^t/3$: (The differential equation has $y''' - y = (D^3 - 1)y = P(D)y$ and is $P'(D) = 3D^2$ and $P'(c) = 3$ because e^t has $c = 1$.)

Problems 5-6 involve repeated roots s in y_n and resonance $P(c) = 0$ in y_p.

5 Which value of C gives resonance in $y'' + Cy = e^{i\omega t}$? Why do we never get resonance in $y'' + 5y' + Cy = e^{i\omega t}$?

Solution $y'' + Cy = e^{i\omega t}$ has resonance when $e^{i\omega t}$ solves the null equation, so $(i\omega)^2 + C = 0$ and $\boldsymbol{C = \omega^2}$. For this C the particular solution must change from $y_p = e^{i\omega t}/0$ to $y_p = \boldsymbol{te^{i\omega t}/2i\omega}$ (because the derivative of $P(D) = D^2 + C$ is $P'(D) = 2D$ and then $P'(i\omega) = 2i\omega$).

We never get resonance with $P(D) = D^2 + 5D + C$ because $P(i\omega) = (i\omega)^2 + 5i\omega + C$ is never zero and $y = e^{i\omega t}$ is never a null solution.

6 Suppose the third order equation $P(D)y_n = 0$ has solutions $y = c_1e^t + c_2e^{2t} + c_3e^{3t}$. What are the null solutions to the sixth order equation $P(D)P(D)y_n = 0$?

Solution The three roots of $P(s)$ must be $s = 1, 2, 3$. The sixth order equation $P(D)P(D)y = 0$ has those as **double roots** of $P(s)^2$. So the null solutions are

$$y = c_1e^t + c_2te^t + c_3e^{2t} + c_4te^{2t} + c_5e^{3t} + c_6te^{3t}$$

7 Complete this table with equations for s_1 and s_2 and y_n and y_p :

Undamped free	$my'' + ky = 0$	$\boldsymbol{y_n = c_1e^{i\omega_n t} + c_2e^{-i\omega_n t}}$
Undamped forced	$my'' + ky = e^{i\omega t}$	$\boldsymbol{y_p = e^{i\omega t}/m(\omega_n^2 - \omega^2)}$
Damped free	$my'' + by' + ky = 0$	$\boldsymbol{y_n = c_1e^{s_1 t} + c_2e^{s_2 t}}$
Damped forced	$my'' + by' + ky = e^{ct}$	$\boldsymbol{y_p = e^{ct}/(mc^2 + bc + k)}$

Here s_1 and s_2 are $-b/2m \pm \sqrt{b^2 - 4mk}/2m$.

8 Complete the same table when the coefficients are 1 and $2Z\omega_n$ and ω_n^2 with $Z < 1$.

Undamped free	$y'' + \omega_n^2 y = 0$	$\boldsymbol{y_n = c_1e^{i\omega_n t} + c_2e^{-i\omega_n t}}$
Undamped forced	$y'' + \omega_n^2 y = e^{i\omega t}$	$\boldsymbol{y_p = e^{i\omega t}/m(\omega_n^2 - \omega^2)}$
Underdamped free	$y'' + 2Z\omega_n y' + \omega_n^2 y = 0$	$\boldsymbol{y_n = c_1e^{s_1 t} + c_2e^{s_2 t}}$
Underdamped forced	$y'' + 2Z\omega_n y' + \omega_n^2 y = e^{ct}$	$\boldsymbol{y_p = e^{ct}/(c^2 + 2Z\omega_n c + \omega_n^2)}$

Those use equations (20) in 2.3 and (32-33) in 2.4.

9 What equations $y'' + By' + Cy = f$ have these solutions ? Hint: Find B and C from the exponents s in y_n : $s_1 + s_2 = -B$ and $s_1s_2 = C$. Find f by substituting y_p.

(a) $y = c_1\cos 2t + c_2\sin 2t + \cos 3t$ $\quad\boldsymbol{y'' + 4y = -5\cos 3t}$

(b) $y = c_1e^{-t}\cos 4t + c_2e^{-t}\sin 4t + \cos 5t$ $\quad\boldsymbol{y'' + 2y' + 17y = -8\cos 5t - 10\sin 5t}$

(c) $y = c_1e^{-t} + c_2te^{-t} + e^{i\omega t}$ $\quad\boldsymbol{y'' + 2y' + y = [(i\omega)^2 + 2i\omega + 1]e^{i\omega t}}$.

10 If $y_p = te^{-6t}\cos 7t$ solves a second order equation $Ay'' + By' + Cy = f$, what does that tell you about A, B, C, and f ?

Solution This particular y_p is showing **resonance** from the factor t. (If this was y_n, we would be seeing a double root of $As^2 + Bs + C = 0$.) The root is $s = -6 + 7i$ from the other factors of y_p.

So I believe that

$$\begin{aligned} As^2 + Bs + C &= A(s + 6 - 7i)(s + 6 + 7i) = A(s^2 + 12s + 36 + 49) \\ f &= Fe^{-6t}(A\cos 7t + B\sin 7t) \end{aligned}$$

11 (a) Find the steady oscillation $y_p(t)$ that solves $y'' + 4y' + 3y = 5\cos\omega t$.

(b) Find the amplitude A of $y_p(t)$ and its phase lag α.

(c) Which frequency ω gives maximum amplitude (maximum gain) ?

Solution (a) y_p has $\sin\omega t$ as well as $\cos\omega t$. Use equations (22-23) for $y_p = M\cos\omega t + N\sin\omega t$:

$$D = (3 - \omega^2)^2 + 16\omega^2 \qquad M = \frac{3 - \omega^2}{D} \qquad N = \frac{4\omega}{D}$$

(b) From equation (26) and the page 112 table :

Amplitude $= G = \frac{1}{\sqrt{D}}$ and the angle α has tangent $= \frac{N}{M} = \frac{4\omega}{3-\omega^2}$.

(c) The maximum gain G and the minimum of $D = (3 - \omega^2)^2 + 16\omega^2$ will occur when

$$\frac{dD}{d\omega} = -4\omega(3 - \omega^2) + 32\omega = 0 \text{ and } 3 - \omega^2 = 8 \text{ and } \omega = \pm\sqrt{5}.$$

This “practical resonance frequency” is computed at the end of section 2.5.

12 Solve $y'' + y = \sin\omega t$ starting from $y(0) = 0$ and $y'(0) = 0$. Find the limit of $y(t)$ as ω approaches 1, and the problem approaches resonance.

Solution The solution is $y = y_n + y_p = c_1\cos t + c_2\sin t + Y\sin\omega t$. Substituting into the equation gives $-\omega^2 Y\sin\omega t + Y\sin\omega t = \sin\omega t$ and $Y = \frac{1}{1-\omega^2}$.

$y(0) = 0$ gives $c_1 = 0$. And $y'(0) = c_2 + \omega Y = 0$ gives $c_2 = -\omega Y$:

$$y(t) = \frac{-\omega}{1 - \omega^2}\sin t + \frac{1}{1 - \omega^2}\sin\omega t = \frac{\sin\omega t - \omega\sin t}{1 - \omega^2}.$$

As ω goes to 1, this goes to $0/0$. Then the l’Hopital Rule finds the ratio of ω-derivatives at $\omega = 1$:

$$\frac{t\cos\omega t - \sin t}{-2\omega} \quad\longrightarrow\quad \boldsymbol{\frac{t\cos t - \sin t}{-2}} = \textbf{Resonant solution}$$

13 Does critical damping and a double root $s = 1$ in $y'' + 2y' + y = e^{ct}$ produce an extra factor t in the null solution y_n or in the particular y_p (proportional to e^{ct}) ? What is y_n with constants c_1, c_2 ? What is $y_p = Ye^{ct}$?

Solution Critical damping is shown in the double root $s = -1, -1$ in $s^2 + 2s + 1 = 0$ and in the **null solutions** $y_n = c_1e^{-t} + c_2te^{-t}$. (Resonance would come when c is also -1 in the right hand side.) The solution $y_p = Ye^{ct}$ has $y'' + 2y' + y = e^{ct}$ and $(c^2Y + 2cY + Y) = 1$ and $Y = 1/(c^2 + 2c + 1)$.

14 If $c = i\omega$ in Problem 13, the solution y_p to $y'' + 2y' + y = e^{i\omega t}$ is _____. That fraction Y is the transfer function at $i\omega$. What are the magnitude and phase in $Y = Ge^{-i\alpha}$?

Solution Set $c = i\omega$ in the solution to Problem 13:

$$y_p + Ye^{ct} = e^{i\omega t}/(i^2\omega^2 + 2i\omega + 1) = Ge^{-i\alpha}e^{i\omega t}.$$

Then $G = 1/(1 - \omega^2 + 2i\omega)$ has magnitude $|G| = 1/\sqrt{(1-\omega^2)^2 + 4\omega^2} = 1/\sqrt{D}$. The phase angle has $\tan\alpha = \frac{2\omega}{1-\omega^2}$.

By rescaling both t and y, we can reach $A = C = 1$. Then $\omega_n = 1$ and $B = 2Z$. The model problem is $y'' + 2Zy' + y = f(t)$.

15 What are the roots of $s^2 + 2Zs + 1 = 0$? Find two roots for $Z = 0, \frac{1}{2}, 1, 2$ and identify each type of damping. The natural frequency is now $\omega_n = 1$.

Solution The roots are $s = -Z \pm \sqrt{Z^2 - 1}$. (All factors 2 will cancel.)

$Z = 0 : s = \pm i$	No damping
$Z = \frac{1}{2} : s = (-1 \pm \sqrt{3}i)/2$	Underdamping
$Z = 1 : s = -1, -1$	Critical damping
$Z = 2 : s = -2 \pm \sqrt{3}$	Overdamping

16 Find two solutions to $y'' + 2Zy' + y = 0$ for every Z except $Z = 1$ and -1. Which solution $g(t)$ starts from $g(0) = 0$ and $g'(0) = 1$? What is different about $Z = 1$?

Solution If $Z^2 \neq 1$ the solutions are $y = c_1e^{s_1t} + c_2e^{s_2t}$. The **impulse response** $g(t)$ on page 97 comes from $s = -Z \pm r$:

$$g(t) = \frac{e^{s_1t} - e^{s_2t}}{s_1 - s_2} = e^{-Zt}(e^{rt} - e^{-rt})/2r \text{ with } r = \sqrt{Z^2 - 1} \text{ in formula (2.3.12).}$$

If $Z = 1$ (critical) then $s_1 = s_2$ and $r = 0$ and $g(t)$ changes to te^{-t} (formula 2.3.15).

17 The equation $my'' + ky = \cos\omega_n t$ is exactly at resonance. The driving frequency on the right side equals the natural frequency $\omega_n = \sqrt{k/m}$ on the left side. Substitute $y = Rt\sin(\sqrt{k/m}\,t)$ to find R. This resonant solution grows in time because of the factor t.

Solution $y' = R\sin\sqrt{\frac{k}{m}}t + R\sqrt{\frac{k}{m}}t\cos\sqrt{\frac{k}{m}}t$ and $y'' = 2R\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t - R\frac{k}{m}t\sin\sqrt{\frac{k}{m}}t$.

Then $my'' + ky = 2R\sqrt{km}\cos\sqrt{\frac{k}{m}}t - Rkt\sin\sqrt{\frac{k}{m}}t + kRt\sin\sqrt{\frac{k}{m}}t = 2R\sqrt{km}\cos\sqrt{\frac{k}{m}}t$.

This agrees with $\cos\omega_n t$ on the right side of the differential equation if $\boldsymbol{R = 1/2\sqrt{km}}$.

18 Comparing the equations $Ay'' + By' + Cy = f(t)$ and $4Az'' + Bz' + (C/4)z = f(t)$, what is the difference in their solutions?

Correction The forcing term in the z-equation should be $f(\frac{t}{4})$.

Solution $\boldsymbol{z(t)}$ **will be** $\boldsymbol{4y(\frac{t}{4})}$. Then $z' = y'(\frac{t}{4})$ and $z'' = \frac{1}{4}y''(\frac{t}{4})$.

$4Az'' + Bz' + \frac{C}{4}z$ equals term by term to $Ay''(\frac{t}{4}) + By'(\frac{t}{4}) + Cy(\frac{t}{4}) = f(\frac{t}{4})$.

19 Find the fundamental solution to the equation $g'' - 3g' + 2g = \delta(t)$.

Solution The roots of $s^2 - 3s + 2 = 0$ are $s = 2$ and $s = 1$: **Real roots**. Use formula 2.3.12 on page 97 to find $g(t)$:

$$g(t) = \frac{e^{s_1t} - e^{s_2t}}{A(s_2 - s_1)} = e^{2t} - e^t.$$

Notice that $g(0) = 0$ and $g'(0) = 1$ (and $A = 1$ in the differential equation).

20 (Challenge problem) Find the solution to $y''+By'+y = \cos\,t$ that starts from $y(0) = 0$ and $y'(0) = 0$. Then let the damping constant B approach zero, to reach the resonant equation $y'' + y = \cos\,t$ in Problem 17, with $m = k = 1$.

Show that your solution $y(t)$ is approaching the resonant solution $\frac{1}{2}t\,\sin\,t$.

Solution The particular solution is $y_p = \frac{\sin t}{B}$. Then $y_p'' + y_p = 0$ and $By_p' = \cos t$. The roots of $s^2 + Bs + 1 = 0$ are $s = (-B \pm \sqrt{B^2 - 4})/2 = (-B \pm i\sqrt{4 - B^2})/2$.

Then $y = c_1e^{s_1t} + c_2e^{s_2t} + \frac{1}{B}\sin t$. At $t = 0$ we must have $c_1 + c_2 = 0$ and $s_1c_1 + s_2c_2 + \frac{1}{B} = 0$. Put $c_2 = -c_1$ to find $(s_1 - s_2)c_1 = i\sqrt{4 - B^2}c_1 = -1/B$.

$$\text{Solution near } B = 0 \quad y = \frac{i}{B\sqrt{4 - B^2}}(e^{s_1t} - e^{s_2t}) + \frac{1}{B}\sin t.$$

At $B = 0$ the roots are $s_1 = i$ and $s_2 = -i$, and $\sqrt{4 - B^2} = 2$.

The solution $y(t)$ approaches $y = \frac{i}{2B}2i\sin t + \frac{1}{B}\sin t = \frac{0}{0}$ (sign of resonance).

l'Hopital asks for the ratio of the B-derivatives. Certainly B in the denominator has B-derivative equal to 1. And $\sqrt{4 - B^2}$ approaches 2. So we want the ***B*-derivative of the numerator**, where s_1, s_2 depend on B. Then as $B \to 0, y$ approaches $\frac{d}{dB}\frac{i}{2}\left(e^{s_1t} - e^{s_2t}\right) = \frac{it}{2}\left[e^{s_1t}\frac{ds_1}{dB} - e^{s_2t}\frac{ds_2}{dB}\right] \to \frac{it}{2}\left(-\frac{1}{2}\right)e^{it} - \frac{it}{2}\left(-\frac{1}{2}\right)e^{-it} = \boldsymbol{\frac{1}{2}t\sin t}$. Wow!

21 Suppose you know three solutions y_1, y_2, y_3 to $y'' + B(t)y' + C(t)y = f(t)$. (Recommended) How could you find $B(t)$ and $C(t)$ and $f(t)$?

Solution The differences $u = y_1 - y_2$ and $v = y_1 - y_3$ are null solutions:

$$\begin{aligned} u'' + B(t)u' + C(t)u = 0 \\ v'' + B(t)v' + C(t)v = 0 \end{aligned}$$

Solve those two linear equations for the numbers $B(t)$ and $C(t)$ at each time t. Then y_1 is a particular solution so $y_1'' + B(t)y_1' + C(t)y_1$ gives $f(t)$.

Problem Set 2.5, page 127

1 (Resistors in parallel) Two parallel resistors R_1 and R_2 connect a node at voltage V to a node at voltage zero. The currents are V/R_1 and V/R_2. What is the total current I between the nodes? Writing R_{12} for the ratio V/I, what is R_{12} in terms of R_1 and R_2?

Solution Currents V/R_1 and V/R_2 in parallel give total current $I = V/R_1 + V/R_2$. Then the effective resistance in $I = V/R$ has

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1R_2} \quad \text{and} \quad R = \frac{R_1R_2}{R_1 + R_2}.$$

2 (Inductor and capacitor in parallel) Those elements connect a node at voltage $Ve^{i\omega t}$ to a node at voltage zero (grounded node). The currents are $(V/i\omega L)e^{i\omega t}$ and $V(i\omega C)e^{i\omega t}$. The total current $Ie^{i\omega t}$ between the nodes is their sum. Writing $Z_{12}e^{i\omega t}$ for the ratio $Ve^{i\omega t}/Ie^{i\omega t}$, what is Z_{12} in terms of $i\omega L$ and $i\omega C$?

Solution This is like Problem 1 with impedances $i\omega L$ and $1/i\omega C$ in parallel, instead of resistances R_1 and R_2. The effective impedance imitates that previous formula for $R = R_1R_2/(R_1 + R_2)$:

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{i\omega L(1/i\omega C)}{i\omega L + (i\omega C)^{-1}} = \frac{i\omega L}{1 - \omega^2 LC}.$$

3 The impedance of an RLC loop is $Z = i\omega L + R + 1/i\omega C$. This impedance Z is real when $\omega =$ _____. This impedance is pure imaginary when _____. This impedance is zero when _____.

Solution Z is real when $i\omega L$ cancels with $1/i\omega C = -i/\omega C$. Then $\omega L = 1/\omega C$ and $\boldsymbol{\omega^2 = 1/LC}$. Z is imaginary when $\boldsymbol{R = 0}$. The impedance is zero when both $R = 0$ and $\boldsymbol{\omega^2 = 1/LC}$.

4 What is the impedance Z of an RLC loop when $R = L = C = 1$? Draw a graph that shows the magnitude $|Z|$ as a function of ω.

Solution An RLC loop adds the impedances $R + i\omega L + i/(i\omega C)$. In case $R = L = C = 1$, the total impedance in series is $Z = \boldsymbol{1 + i\omega + 1/i\omega}$. The magnitude $|Z| = (1 + (\omega - 1/\omega)^2)^{1/2}$ will equal 1 at $\omega = 1$. For large ω, $|Z|$ is asymptotic to the line $|Z| = \omega$. For small ω, $|Z|$ is asymptotic to the curve $|Z| = 1/\omega$.

5 Why does an LC loop with no resistor produce a $90°$ phase shift between current and voltage ? Current goes around the loop from a battery of voltage V in the loop.

Solution The phase shift is the angle of the complex impedance Z. With no resistor, $R = 0$ and $Z = i\omega L + (1/i\omega C) = i(\omega L - (1/\omega C))$. This pure imaginary number has angle $\theta = \pm\pi/2 = \pm 90\,°$ in the complex plane.

6 The mechanical equivalent of zero resistance is zero damping : $my'' + ky = \cos\omega t$. Find c_1 and Y starting from $y(0) = 0$ and $y'(0) = 0$ with $\omega_n^2 = k/m$.

$$y(t) = c_1 \cos\omega_n t + Y\, \cos\omega t.$$

That answer can be written in two equivalent ways :

$$y = Y(\cos\omega t - \cos\omega_n t) = 2Y \sin\frac{(\omega_n - \omega)t}{2} \sin\frac{(\omega_n + \omega)t}{2}.$$

Solution The complete solution is $y = c_1 \cos\omega_n t + c_2 \sin\omega_n t + (\cos\omega t)/(k - m\omega^2)$. The initial conditions $y = y' = 0$ determine c_1 and c_2 :

$$y(0) = 0 \qquad c_1 = -1/(k - m\omega^2) \qquad y'(0) = 0 \qquad c_2 = 0.$$

Then $y(t) = \boldsymbol{(\cos\omega t - \cos\omega_n t)/(k - m\omega^2)}$. The identity $\cos\omega t - \cos\omega_n t = 2\sin\frac{(\omega - \omega_n)t}{2}\sin\frac{(\omega + \omega_n)t}{2}$ expresses y as the product of two oscillations.

7 Suppose the driving frequency ω is close to ω_n in Problem 2. A fast oscillation $\sin[(\omega_n + \omega)t/2]$ is multiplying a very slow oscillation $2Y\sin[(\omega_n - \omega)t/2]$. By hand or by computer, draw the graph of $y = (\sin t)(\sin 9t)$ from 0 to 2π.

You should see a fast sine curve inside a slow sine curve. This is a **beat**.

Solution When ω is close to ω_n, the first (bold) formula in Problem 6 is near $0/0$. The second formula is much better :

$$2\sin\frac{(\omega - \omega_n)t}{2} \approx (\omega - \omega_n)t \quad \sin\frac{(\omega + \omega_n)t}{2} \approx \sin\omega_n t \quad \boldsymbol{y \approx (\omega - \omega_n)t \sin\omega_n t}$$

This shows the typical t factor for resonance. The graph of $y = (\sin t)(\sin 9t)$ has $\omega = 10$ and $\omega_n = 8$, so that $(10 - 8)/2 = 1$ and $(10 + 8)/2 = 9$. The graph shows a fast "$\sin 9t$" curve inside a slow "$\sin t$" curve : good to draw by computer.

8 What m, b, k, F equation for a mass-dashpot-spring-force corresponds to Kirchhoff's Voltage Law around a loop ? What force balance equation on a mass corresponds to Kirchhoff's Current Law ?

Solution The Voltage Law says that **voltage drops add to zero** around a loop :

$$\text{Equation (5) is } L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt = Ve^{i\omega t}.$$

This corresponds to $my'' + by' + ky = f$. The Current Law says that "flow in equals flow out" at every node. The mechanical analog is that "**forces balance**" at every node.

In a static structure (no movement) we can have force balance equations in the x, y, and z direction. In a dynamic structure (with movement) the forces include the inertia term my'' and the friction term by'.

9 If you only know the natural frequency ω_n and the damping coefficient b for one mass and one spring, why is that *not enough* to find the damped frequency ω_d ? If you know all of m, b, k what is ω_d ?

Solution If we only know $\omega_n^2 = k/m$ and b, that does not determine the damping ratio $Z = b^2/4mk$ or the damped frequency $\omega_d = \sqrt{p^2 - \omega_n^2}$ with $p = B/2A = b/2m = \omega_n Z$ in equation (2.4.30). We need *three numbers* as in m, b, k or *two ratios* as in $\omega_n^2 = k/m$ and $2p = b/m$.

10 Varying the number a in a first order equation $y' - ay = 1$ changes the *speed* of the response. Varying B and C in a second order equation $y'' + By' + Cy = 1$ changes the *form* of the response. Explain the difference.

Solution The growth factor in a first order equation is e^{at}. The units of a are $1/\text{time}$ and this controls the speed. For a second-order equation $y'' + By' + Cy' = f$, the coefficients B and C control not only the frequency $\omega_n = \sqrt{C}$ but also the form of $y(t)$: damped oscillation if $B^2 < 4C$ and overdamping if $B^2 > 4C$.

11 Find the step response $r(t) = y_p + y_n$ for this overdamped system :

$$r'' + 2.5r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0.$$

Solution The roots of $s^2 + 2.5s + 1 = (s + 2)(s + \frac{1}{2})$ are $s_1 = -2$ and $s_2 = -\frac{1}{2}$. Then equation (18) for the step response gives

$$r(t) = 1 + \left(-\frac{1}{2}e^{-2t} + 2e^{-t/2}\right)/(-3/2) = 1 + \frac{1}{3}e^{-2t} - \frac{4}{3}e^{-t/2}.$$

Check that $r(0) = 0$ and $r'(0) = 0$ (and $r(\infty) = 1$).

12 Find the step response $r(t) = y_p + y_n$ for this critically damped system. The double root $s = -1$ produces what form for the null solution ?

$$r'' + 2r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0.$$

Solution The characteristic equation $s^2 + 2s + 1 = 0$ has a double root $s = -1$. The null solution is $\boldsymbol{y_n = c_1 e^{-t} + c_2 t e^{-t}}$. The particular solution with $f = 1$ is $y_p = 1$. The initial conditions give c_1 and c_2 :

$$\begin{aligned} r(t) &= c_1 e^{-t} + c_2 t e^{-t} + 1 & \\ r(0) &= c_1 + 1 = 0 & \boldsymbol{c_1 = -1} \\ r'(0) &= -c_1 + c_2 + 1 = 0 & \boldsymbol{c_2 = -2} \\ r(t) &= \boldsymbol{1 - (1 + 2t)e^{-t}} & \end{aligned}$$

13 Find the step response $r(t)$ for this underdamped system using equation (22):

$$r'' + r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0.$$

Solution Equation (22) gives the step response for an underdamped system.

$$r(t) = 1 - \frac{\omega_n}{\omega_d} e^{-pt} \sin(\omega_d t + \phi).$$

Then $r'' + r' + r = 1$ has $m = b = k = 1$ and $b^2 < 4mk$ (underdamping).

$$p = \frac{b}{2m} = \frac{\mathbf{1}}{\mathbf{2}} \quad \omega_n^2 = \frac{k}{m} = \mathbf{1} \quad \omega_d^2 = \omega_n^2 - p^2 = \frac{\mathbf{3}}{\mathbf{4}} \quad \cos\phi = \frac{p}{\omega_n} = \frac{1}{2} \quad \phi = \frac{\pi}{3}.$$

Substituting in the formula gives $r(t) = 1 - \dfrac{2}{\sqrt{3}} e^{-t/2} \sin\left(\dfrac{\sqrt{3}}{2} t + \dfrac{\pi}{3}\right)$.

14 Find the step response $r(t)$ for this undamped system and compare with (22):

$$r'' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0.$$

Solution Now $r'' + r = 1$ has $m = k = 1$ and $b = 0$ (no damping):

In this case $\quad p = 0 \quad \omega_n^2 = 1 \quad \omega_d = \omega_n \quad \cos\phi = \frac{p}{\omega_n} = 0 \quad \phi = \frac{\pi}{2}$.

Substituting into (22) gives $r(t) = 1 - \sin\left(t + \frac{\pi}{2}\right) = \mathbf{1 - \cos t}$.

15 For $b^2 < 4mk$ (underdamping), what parameter decides the speed at which the step response $r(t)$ rises to $r(\infty) = 1$? Show that the **peak time** is $T = \pi/\omega_d$ when $r(t)$ reaches its maximum before settling back to $r = 1$. At peak time $r'(T) = 0$.

Solution With underdamping, formula (22) has the decay factor e^{-pt}. Then $p = B/2A = b/2m$ is the decay rate. The "peak time" is the time when r reaches its maximum (its peak). That time T has $dr/dt = 0$.

$$\frac{dr}{dt} = -\frac{\omega_n}{\omega_d}\left(-pe^{-pt}\sin(\omega_d t + \phi) + \omega_d e^{-pt}\cos(\omega_d t + \phi)\right) = 0 \text{ at } t = T \text{ (peak time).}$$

$$-p\sin(\omega_d T + \phi) + \omega_d \cos(\omega_d T + \phi) = 0$$

$$\tan(\omega_d T + \phi) = \omega_d/p \text{ which is } \tan\phi$$

Then $\omega_d T = \pi$ and $\boldsymbol{T = \pi/\omega_d}$. (Note: I seem to get $2\pi/\omega_d$.)

16 If the voltage source $V(t)$ in an RLC loop is a unit step function, what resistance R will produce an overshoot to $r_{\text{max}} = 1.2$ if $C = 10^{-6}$ Farads and $L = 1$ Henry? (Problem 15) found the peak time T when $r(T) = r_{\text{max}}$).

Sketch two graphs of $r(t)$ for $p_1 < p_2$. Sketch two graphs as ω_d increases.

Solution The peak time is $T = \pi/\omega_d$. Then $\omega_d T = \pi$ and we want $r = 1.2$:

$$\begin{aligned} r_{\max}(T) &= 1 - \tfrac{\omega_n}{\omega_d} e^{-pT} \sin(\pi + \phi) \\ 1.2 &= 1 + \tfrac{\omega_n}{\omega_d} e^{-pT} \sin(\phi) = 1 + e^{-pT} \\ 0.2 &= e^{-p\pi/\omega_d} \\ p\pi/\omega_d &= -\ln(0.2) = \ln 5 \end{aligned}$$

We substitute $p = B/2A = R/2\omega L$ and $\omega_d = \sqrt{\omega_n^2 - \omega^2} = \sqrt{(1/LC) - \omega^2}$. With known values of L and C **and $\boldsymbol{\omega}$** we can find R.

17 What values of m, b, k will give the step response $r(t) = 1 - \sqrt{2}e^{-t}\sin(t + \frac{\pi}{4})$?

Solution This response $r(t)$ matches equation (22) when $\boldsymbol{\omega_n = \sqrt{2}\omega_d}$ and $\boldsymbol{p = 1}$ and $\boldsymbol{\phi = \pi/4}$. Then

$$\omega_d^2 = \omega_n^2 - p^2 = 2\omega_d^2 - 1 \text{ gives } \boldsymbol{\omega_d = 1} \text{ and } \boldsymbol{\omega_n = \sqrt{2}}.$$

Therefore $\omega_n^2 = k/m = 2$ and $p = b/2m = 1$. The numbers m, b, k are proportional to $\mathbf{1, 2, 2}$.

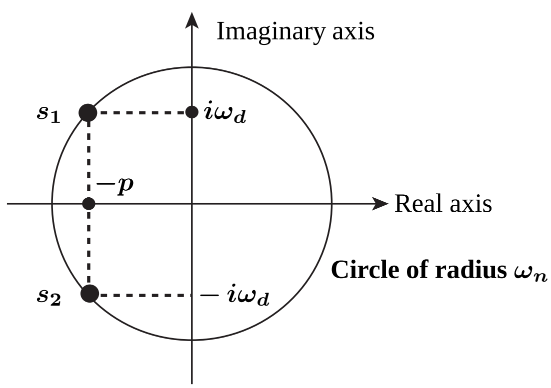
18 What happens to the $p - \omega_d - \omega_n$ right triangle as the damping ratio ω_n/p increases to 1 (critical damping)? At that point the damped frequency ω_d becomes _____. The step response becomes $r(t) =$ _____.

Solution Critical damping has equal roots $s_1 = s_2$ and $b^2 = 4mk$ and damping ratio $Z = 1$ and $\omega_d = \omega_n\sqrt{1 - Z^2} = 0$. (The oscillation disappears and the damped frequency goes to $\omega_d = 0$ so that $\phi = 0$.) Then the step response is

$$r(t) = 1 - \frac{\omega_n t}{\omega_d t^e} - pt\sin(\omega_d t) \longrightarrow \boldsymbol{1 - \omega_n t e^{-pt}}.$$

19 **The roots $\boldsymbol{s_1, s_2 = -p \pm i\omega_d}$ are poles of the transfer function $\boldsymbol{1/(As^2 + Bs + C)}$**

Show directly that the product of the roots $s_1 = -p + i\omega_d$ and $s_2 = -p - i\omega_d$ is $s_1 s_2 = \omega_n^2$. The sum of the roots is $-2p$. The quadratic equation with those roots is $s^2 + 2ps + \omega_n^2 = 0$.



Solution Multiplying the complex conjugate number $s = -p \pm i\omega_d$ gives $|s|^2 = (-p + i\omega_d)(-p - i\omega_d) = p^2 + \omega_d^2 = \omega_n^2$.

For any quadratic $As^2 + Bs + C = A(s - s_1)(s - s_2), C$ matches $As_1 s_2$. Then $s_1 s_2 = C/A = \omega_n^2$. Complex roots **stay on the circle of radius $\boldsymbol{\omega_n}$,** as in the picture.

Adding $-p + i\omega$ to $-p - i\omega$ gives $s_1 + s_2 = -2p$. This always equals $-B/A$.

20 Suppose p is increased while ω_n is held constant. How do the roots s_1 and s_2 move?

Solution Increasing p will make both roots go along the circle in the direction of $-\omega_n$. Problem 19 showed that they stay on the circle of radius ω_n until they meet at $-\omega_n$. At that point $s_1 + s_2 = -2\omega_n = -2p$. Therefore that value of p is ω_n.

Increasing p beyond ω_n will give **two negative real roots** that add to $-2\omega_n$.

21 Suppose the mass m is increased while the coefficients b and k are unchanged. What happens to the roots s_1 and s_2 ?

Solution The key number $B^2 - 4AC = b^2 - 4mk$ will eventually go negative when m is increased. The roots will be complex (a conjugate pair). Further increasing the mass m will decrease both $p = b/2m$ and $\omega_n^2 = k/m$. The roots approach zero.

22 **Ramp response** How could you find $y(t)$ when $F = t$ is a ramp function?

$$y'' + 2py' + \omega_n^2 y = \omega_n^2 t \text{ starting from } y(0) = 0 \text{ and } y'(0) = 0.$$

A particular solution (straight line) is $y_p =$ ______. The null solution still has the form $y_n =$ ______. Find the coefficients c_1 and c_2 in the null solution from the two conditions at $t = 0$.

This ramp response $y(t)$ can also be seen as the integral of ______.

Solution A particular solution is $y_p = C + t$. Substitute into the equation:

$y'' + 2py' + \omega_n^2 y = 0 + 2p + \omega_n^2(C + t) = \omega_n^2 t$. Thus $C = -2p/\omega_n^2$.

The null solution is still $y_n = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. We find c_1 and c_2 at $\boldsymbol{t = 0}$:

$$\begin{aligned} y &= c_1 e^{s_1 t} + c_2 e^{s_2 t} + C + t = c_1 + c_2 + C &&= 0 \\ y' &= c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t} + 1 = c_1 s_1 + c_2 s_2 + 1 &&= 0 \end{aligned}$$

Solving those equations gives $c_1 = \frac{Cs_2 - 1}{s_1 - s_2}$ and $c_2 = \frac{1 - Cs_1}{s_1 - s_2}$ with $C = -2p/\omega_n^2$.

The ramp response is also the integral of the **step response**.

Problem Set 2.6, page 137

Find a particular solution by inspection (or the method of undetermined coefficients)

1 (a) $y'' + y = 4$ (b) $y'' + y' = 4$ (c) $y'' = 4$

Solution (a) $y_p = 4$ (b) $y_p = 4t$ (c) $y_p = 2t^2$

2 (a) $y'' + y' + y = e^t$ (b) $y'' + y' + y = e^{ct}$

Solution (a) $y_p = \frac{1}{3}e^t$ (b) $y_p = e^{ct}/(c^2 + c + 1)$

3 (a) $y'' - y = \cos t$ (b) $y'' + y = \cos 2t$ (c) $y'' + y = t + e^t$

Solution (a) $y_p = -\frac{1}{2}\cos t$ (b) $y_p = -\frac{1}{3}\cos 2t$ (c) $y_p = t + \frac{1}{2}e^t$

4 For these $f(t)$, predict the form of $y(t)$ with undetermined coefficients:

(a) $f(t) = t^3$ (b) $f(t) = \cos 2t$ (c) $f(t) = t \cos t$

Solution (a) $y_p = at^3 + bt^2 + ct + d$ (b) $y_p = a\cos 2t + b\sin 2t$
(c) $y_p = (At + B)\cos t + (Ct + D)\sin t$

5 Predict the form for $y(t)$ when the right hand side is

(a) $f(t) = e^{ct}$ (b) $f(t) = te^{ct}$ (c) $f(t) = e^t \cos t$

Solution (a) $y_p = Ye^{ct}$ (b) $y_p = (Yt + Z)e^{ct}$ (c) $y_p = ae^t\cos t + be^t \sin t$

6 For $f(t)=e^{ct}$ when is the prediction for $y(t)$ different from Ye^{ct} ?

Solution There will be a te^{ct} term in y_p when e^{ct} is a null solution. This is resonance:

$$Ac^2+Bc+C=0 \text{ and } c \text{ is } s_1 \text{ or } s_2.$$

Problems 7-11 : Use the method of undetermined coefficients to find a solution $\boldsymbol{y_p(t)}$.

7 (a) $y''+9y=e^{2t}$ (b) $y''+9y=te^{2t}$

Solution (a) $y_p=Ye^{2t}$ with $4Ye^{2t}+9Ye^{2t}=e^{2t}$ and $Y=\frac{1}{13}$

(b) $y_p=(Yt+Z)e^{2t}$ with $y'=(2Yt+Y+2Z)e^{2t}$ and $y''=(4Yt+4Y+4Z)e^{2t}$.

The equation $y''+9y=te^{2t}$ gives $(4Yt+4Y+4Z+9Yt+9Z)e^{2t}=te^{2t}$.

Then $13Yt=t$ and $4Y+13Z=0$ give $Y=\frac{1}{13}$ and $Z=-\frac{4}{13}Y$ and $y_p=\boldsymbol{\frac{1}{13}(t-\frac{4}{13})e^{2t}}$.

8 (a) $y''+y'=t+1$ (b) $y''+y'=t^2+1$

Solution (a) $y_p=at^2+bt$ and $y''+y'=2a+2at+b=t+1$.

Then $a=\frac{1}{2}$ and $b=0$ and $\boldsymbol{y_p=\frac{1}{2}t^2}$.

*Notice that $y_p=$ constant is a null solution so we needed to assume $y_p=at^2+bt$.

(b) $y_p=at^3+bt^2+ct$ (NOT $+d$) and $y''+y'=(6at+2b)+(3at^2+2bt+c)=t^2+1$.

Then $3a=1$ and $6a+2b=0$ and $2b+c=1$: $\boldsymbol{y_p=\frac{1}{3}t^3-1t^2+3t}$.

9 (a) $y''+3y=\cos\,t$ (b) $y''+3y=t\,\cos\,t$

Solution (a) $y_p=A\cos t+B\sin t$.

$y_p''+3y_p=-A\cos t-B\sin t+3A\cos t+3B\sin t=\cos t$.

Then $2A=1$ and $2B=0$ and $\boldsymbol{y_p=\frac{1}{2}\cos t}$.

(b) $y_p=(At+B)\cos t+(Ct+D)\sin t$.

$y_p'=(A+Ct+D)\cos t+(-At-B+C)\sin t$.

$y_p''+3y_p=C\cos t-A\sin t+(-A-Ct-D)\sin t+(-At-B+C)\cos t+3(At+B)\cos t+3(Ct+D)\sin t=t\cos t$.

Match $3At-At=t$ and $C-B+C+3B=0$ and $-Ct+3Ct=0$ and $-A-A-D+3D=0$.

Then $A=\frac{1}{2},\ C=0,\ B=0,\ D=A=\frac{1}{2}$ gives $\boldsymbol{y_p=\frac{1}{2}t\cos t+\frac{1}{2}\sin t}$.

10 (a) $y''+y'+y=t^2$ (b) $y''+y'+y=t^3$

Solution (a) $y_p=at^2+bt+c$ give $y_p''+y_p'+y=(2a)+(2at+b)+(at^2+bt+c)=t^2$.

Then $a=1$ and $2a+b=0$ and $2a+b+c=0$ give $a=1,\ b=-2,\ c=0$: $\boldsymbol{y_p=t^2-2t}$.

(b) Now $y_p=at^2+bt+c+dt^3$. Added into part (a), the new dt^3 produces

$$y''+y'+y=(2a)+(2at+b)+(at^2+bt+c)+d(6t+3t^2+t^3)=t^3+c=0$$

Then $d=1,\quad 3d+a=0,\quad 6d+b+2a=0,\quad 2a+b+c=0$ give $d=1,a=-3,b=0,c=6$: $\boldsymbol{y_p=t^3-3t^2+6}$.

11 (a) $y'' + y' + y = \cos\, t$ (b) $y''+y'+y = t\, \sin\, t$

Solution (a) $y_p = A\cos t + B\sin t$.

$y_p'' + y_p' + y_p = (-A + B + A)\cos t + (-B - A + B)\sin t = \cos t$.

Then $B = 1$ and $A = 0$ and $\boldsymbol{y_p = \sin t}$.

(b) The forms for y_p and y_p' and y_p'' are the same as in 2.6.9 (b). Then y_p''+y_p'+y_p equals $C\cos t - A\sin t + (-A - Ct - D)\sin t + (-At - B + C)\cos t + (A + Ct + D)\cos t$ $+(-At - B + C)\sin t + (Ct + D)\sin t = t\sin t$.

Match coefficients of $t\cos t, t\sin t, \cos t, \sin t$:

$-A + C + A = 0 \qquad -C - A + C = 1 \qquad C - B + C + A + D + B = 0$
$-A - A - D - B + C + D = 0$.

Then $A = -1,\ C = 0,\ B = 2,\ D = 1$ give $\boldsymbol{y_p = -t\cos t + 2\cos t}$.

Problems 12–14 involve resonance. Multiply the usual form of $\boldsymbol{y_p}$ by $\boldsymbol{t}$.

12 (a) $y'' + y = e^{it}$ (b) $y'' + y = \cos\, t$

Solution (a) Look for $y_p = Yte^{it}$. Then $y_p' = Y(it + 1)e^{it}$.

$y_p'' + y_p = Y(i^2t + 2ie^{it}) + Yte^{it} = 2iYe^{it}$.

This matches e^{it} on the right side when $Y = 1/2i$ and $\boldsymbol{y_p = te^{it}/2i = -ite^{it}/2}$.

(b) Look for $y_p = At\cos t+Bt\sin t$. Then $y_p' = A\cos t - At\sin t + B\sin t + Bt\cos t$.

$y_p'' + y = -2A\sin t - At\cos t + 2B\cos t - Bt\sin t + At\cos t + Bt\sin t = \cos t$.

Then $A = 0$ and $B = \frac{1}{2}$ and $\boldsymbol{y_p = \frac{1}{2}t\sin t}$.

13 (a) $y'' - 4y' + 3y = e^t$ (b) $y'' - 4y' + 3y = e^{3t}$

Solution (a) Look for $y_p = cte^t$ with $y_p' = c(t + 1)e^t$ and $y_p'' = c(t + 2)e^t$.

$$y_p'' - 4y_p' + 3y_p = (2c - 4c)e^t = e^t \text{ with } c = -\frac{1}{2} \text{ and } \boldsymbol{y_p = -\frac{1}{2}te^t}.$$

(b) Look for $y_p = cte^{3t}$ with $y_p' = c(3t + 1)e^{3t}$ and $y_p'' = c(9t + 6)e^{3t}$.

$$y_p'' - 4y_p' + 3y_p = (6c - 4c)e^{3t} = e^{3t} \text{ with } c = \frac{1}{2} \text{ and } \boldsymbol{y_p = \frac{1}{2}te^{3t}}.$$

14 (a) $y' - y = e^t$ (b) $y' - y = te^t$ (c) $y' - y = e^t\, \cos\, t$

Solution (a) Look for $y_p = cte^t$ with $y_p' = c(t + 1)e^t$.

$$\text{Then } y_p' - y_p = ce^t = e^t \text{ when } c = 1 \text{ and } \boldsymbol{y_p = te^t}.$$

(b) Look for $y_p = ct^2e^t$ with $y_p' = c(t^2 + 2t)e^t$.

$$\text{Then } y_p' - y_p = c(t^2 + 2t - t^2)e^t = te^t \text{ when } c = \frac{1}{2} \text{ and } \boldsymbol{y_p = \frac{1}{2}t^2e^t}.$$

(c) Look for $y_p = Ae^t\cos t + Be^t\sin t$. Then

$y_p' = Ae^t\cos t - Ae^t\sin t + Be^t\sin t + Be^t\cos t$.

$y_p' - y_p = -Ae^t\sin t + Be^t\cos t = e^t\cos t$ when $A = 0, B = 1,$ and $\boldsymbol{y_p = e^t\sin t}$.

15 For $y'' + 4y = e^t \sin t$ (exponential times sinusoidal) we have two choices:

1 (Real) Substitute $y_p = Me^t \cos t + Ne^t \sin t$: determine M and N
2 (Complex) Solve $z'' + 4z = e^{(1+i)t}$. Then y is the imaginary part of z.

Use both methods to find the same $y(t)$—which do you prefer?

Solution Method 1 has $y_p' = Me^t \cos t - Me^t \sin t + Ne^t \sin t + Ne^t \cos t = (M+N)e^t \cos t + (-M+N)e^t \sin t$.

$$\text{Then } y_p'' + 4y_p = (M+N)\, e^t \cos t - (M+N)\, e^t \sin t + (-M+N)\, e^t \sin t + (-M+N)\, e^t \cos t + 4M\, e^t \cos t + 4N\, e^t \sin t.$$

This equals $e^t \sin t$ when $2N + 4M = 0$ and $-2M + 4N = 1$.

Then $N = -2M$ and $-2M - 8M = 1$ and $M = -\frac{1}{10}, N = \frac{2}{10}$, $y_p = \boldsymbol{-\frac{1}{10}e^t \cos t + \frac{2}{10}e^t \sin t}$.

Method 2 Look for $z_p = Ze^{(1+i)t}$. Then $z_p'' + 4z_p = Z\left[(1+i)^2 + 4\right] e^{(1+i)t} = e^{(1+i)t}$ gives $Z = 1/(4+2i)$.

Take the imaginary part of z_p:

$$\mathrm{Im}\frac{e^{(1+i)t}}{4+2i} = \mathrm{Im}\frac{e^t(\cos t + i \sin t)(4-2i)}{16+4} = \boldsymbol{\frac{e^t}{20}(-2\cos t + 4 \sin t)}.$$

This complex method was shorter and easier. It produced the same y_p.

16 (a) Which values of c give resonance for $y'' + 3y' - 4y = te^{ct}$?

Solution $c^2 + 3c - 4 = (c-1)(c+4)$. So $c = 1$ and $c = -4$ will give resonance.

(b) What form would you substitute for $y(t)$ if there is no resonance?

Solution With no resonance look for $y_p = (at+b)e^{ct}$.

(c) What form would you use when c produces resonance?

Solution With resonance look for $y_p = (at^2 + bt)e^{ct}$. If we also look for de^{ct}, this will be a null solution and we cannot determine d.

17 This is the rule for equations $P(D)y = e^{ct}$ with resonance $P(c) = 0$:

If $P(c) = 0$ and $P'(c) \neq 0$, look for a solution $y_p = Cte^{ct}$ $(m = 1)$
If c is a root of multiplicity m, then y_p has the form _____.

Solution If c is a root of P with multiplicity m, then multiply the usual Ye^{ct} by $\boldsymbol{t^m}$.

18 (a) To solve $d^4y/dt^4 - y = t^3e^{5t}$, what form do you expect for $y(t)$?

(b) If the right side becomes $t^3 \cos 5t$, which 8 coefficients are to be determined?

Solution (a) The exponent $c = 5$ is not a root of $P(D) = D^4 - 1$ $(5^4 \neq 1)$. So look for $y_p = (at^3 + bt^2 + ct + d)e^{5t}$.

(b) If the right side is $t^3 \cos 5t$ then

$y_p = (at^3 + bt^2 + ct + d)\cos 5t + (et^3 + ft^2 + gt + h)\sin 5t$.

19 For $y' - ay = f(t)$, the method of undetermined coefficients is looking for all right hand sides $f(t)$ so that the usual formula $y_p = e^{at}\int e^{-as}f(s)ds$ is easy to integrate. Find these integrals for the "nice functions" $f = e^{ct}$, $f = e^{i\omega t}$, and $f = t$:

$$\int e^{-as}e^{cs}ds \qquad \int e^{-as}e^{i\omega s}ds \qquad \int e^{-as}s\,ds$$

Solution The equation has $y' - ay$ so the growth factor (the impulse response) is $g(t) = e^{at}$. This problem connects the method of undetermined coefficients to the ordinary formula $y_p = \int g(t-s)f(s)\,ds$. The integral $\int e^{a(t-s)}f(s)\,ds$ is easy for :

$$\int e^{-as}e^{cs}\,ds = \frac{e^{(c-a)s}}{(c-a)} \qquad \int e^{-as}e^{i\omega s}\,ds = \frac{e^{(i\omega-a)s}}{i\omega - a}$$

$$\int se^{-as}\,ds = -\left(\frac{s}{a} + \frac{1}{a^2}\right)e^{-as}.$$

Problems 20–27 develop the method of variation of parameters.

20 Find two solutions y_1, y_2 to $y'' + 3y' + 2y = 0$. Use those in formula (13) to solve

(a) $y'' + 3y' + 2y = e^t$ (b) $y'' + 3y' + 2y = e^{-t}$

Solution (a) $y'' + 3y' + 2y$ leads to $s^2 + 3s + 2 = (s+1)(s+2)$. The null solutions are $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. The Variation of Parameters formula is

$$y_p = -y_1 \int \frac{y_2 f}{W} + y_2 \int \frac{y_1 f}{W} \text{ with } W = y_1y_2' - y_2y_1' = (-2-1)e^{-t}e^{-2t} = \boldsymbol{-3e^{-3t}}.$$

$f = e^t$ gives $y_p = +\dfrac{e^{-t}}{3}\displaystyle\int \frac{e^{-2t}e^t}{e^{-3t}} - \frac{e^{-2t}}{3}\int \frac{e^{-t}e^t}{e^{-3t}} = \frac{e^{-t}}{3}\frac{e^{2t}}{2} - \frac{e^{-2t}}{3}\frac{e^{3t}}{3} = \left(\frac{1}{6} - \frac{1}{9}\right)e^t = \boldsymbol{\frac{1}{18}e^t}.$

(b) Again $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. Now $f = e^{-t}$ gives resonance and t appears :

$$y_p = +\frac{e^{-t}}{3}\int \frac{e^{-2t}e^{-t}}{e^{-3t}} - \frac{e^{-2t}}{3}\int \frac{e^{-t}e^{-t}}{e^{-3t}} = \frac{e^{-t}}{3}\boldsymbol{t} - \frac{e^{-2t}}{3}e^t = \boldsymbol{\frac{1}{3}(t-1)e^{-t}}.$$

21 Find two solutions to $y'' + 4y' = 0$ and use variation of parameters for

(a) $y'' + 4y' = e^{2t}$ (b) $y'' + 4y' = e^{-4t}$

Solution (a) $y'' + 4y' = 0$ has null solutions $y_1 = 1 = e^{0t}$ and $y_2 = e^{-4t}$. Then $W = y_1y_2' - y_2y_1' = \boldsymbol{-4e^{-4t}}$. The equation has $f = e^{2t}$.

From (13) : $y_p = -1\displaystyle\int \frac{e^{-4t}e^{2t}}{-4e^{-4t}} + e^{-4t}\int \frac{(1)e^{2t}}{-4e^{-4t}} = \frac{e^{2t}}{8} + e^{-4t}\left(\frac{e^{6t}}{-24}\right) = \boldsymbol{\frac{e^{2t}}{12}}.$

(b) $f = e^{-4t}$ is also a null solution : expect resonance and a factor t.

$$y_p = -1\int \frac{e^{-4t}e^{-4t}}{-4e^{-4t}} + e^{-4t}\int \frac{(1)e^{-4t}}{-4e^{-4t}} = \boldsymbol{-\frac{e^{-4t}}{16} - e^{-4t}\left(\frac{t}{4}\right)}.$$

22 Find an equation $y'' + By' + Cy = 0$ that is solved by $y_1 = e^t$ and $y_2 = te^t$. If the right side is $f(t) = 1$, what solution comes from the VP formula (13)?

Solution With $y_1 = e^t$ and $y_2 = te^t$, the exponent $s = 1$ must be a double root:

$$As^2 + Bs + C = A(s-1)^2 \text{ and the equation can be } y'' - 2y' + y = f(t).$$

With $f(t) = 1$ and $W = y_1 y_2' - y_2 y_1' = e^t(e^t + te^t) - te^t(e^t) = \boldsymbol{e^{2t}}$, eq. (13) gives

$$y_p = -e^t \int \frac{te^t(1)}{e^{2t}} + te^t \int \frac{e^t(1)}{e^{2t}} = -e^t(-te^{-t} - e^{-t}) + te^t(-e^{-t}) = \mathbf{1}$$

$y_p = 1$ is a good solution to $y'' - 2y' + y = 1$.

23 $y'' - 5y' + 6y = 0$ is solved by $y_1 = e^{2t}$ and $y_2 = e^{3t}$, because $s = 2$ and $s = 3$ come from $s^2 - 5s + 6 = 0$. Now solve $y'' - 5y' + 6y = 12$ in two ways:

1. Undetermined coefficients (or inspection) **2.** Variation of parameters using (13)

The answers are different. Are the initial conditions different?

Solution Solving $y'' - 5y' + 6y = 12$ gives $\boldsymbol{y_p = 2}$ by inspection or undetermined coefficients.

Using $s^2 - 5s + 6 = (s-2)(s-3)$ we have $y_1 = e^{2t}$ and $y_2 = e^{3t}$ and $W = e^{5t}$. Then set $f = 12$:

$$y_p = -e^{2t} \int \frac{e^{3t}(12)}{e^{5t}} + e^{3t} \int \frac{e^{2t}(12)}{e^{5t}} = -e^{2t}\left(\frac{12e^{-2t}}{-2}\right) + e^{3t}\left(\frac{12e^{-3t}}{-3}\right) = 6 - 4 = \mathbf{2}$$

But if those two integrals are computed from 0 to t, the lower limit gives a different y_p:

$$-e^{2t}\int_0^t e^{-2t}(12) + e^{3t}\int_0^t e^{-3t}(12) = e^{2t}\left[\frac{12e^{-2t}}{-2}\right]_0^t + e^{3t}\left[\frac{12e^{-3t}}{-3}\right]_0^t$$

$$= \mathbf{2 - 6e^{2t} + 4e^{3t} = 2 +} \textbf{ null solution}.$$

24 What are the initial conditions $y(0)$ and $y'(0)$ for the solution (13) coming from variation of parameters, starting from any y_1 and y_2?

Solution Every integral $I(t) = \displaystyle\int_0^t h(s)\,ds$ starts from $I(0) = 0$ and $I'(0) = h(0)$ by the Fundamental Theorem of Calculus. For equation (13), this gives $y_p(0) = 0$ and $y_p'(0) = 0$ (which can be checked for $y_p = 2 - 6e^{2t} + 4e^{3t}$ in Problem 23).

25 The equation $y'' = 0$ is solved by $y_1 = 1$ and $y_2 = t$. Use variation of parameters to solve $y'' = t$ and also $y'' = t^2$.

Solution Those null solutions $y_1 = 1$ and $y_2 = t$ give $W = y_1 y_2' = \mathbf{1}$. Then

$$\textbf{for } \boldsymbol{f = t} \quad y_p = -1\int t^2 + t\int t \quad = -\tfrac{t^3}{3} + \tfrac{t^3}{2} = \boldsymbol{t^3/6}$$

$$\textbf{for } \boldsymbol{f = t^2} \quad y_p = -1\int t\,t^2 + t\int t^2 = -\tfrac{t^4}{4} + \tfrac{t^4}{3} = \boldsymbol{t^4/12}$$

Those are correct solutions to $y'' = t$ and $y'' = t^2$.

26 Solve $y_s{}'' + y_s = 1$ for the step response using variation of parameters, starting from the null solutions $y_1 = \cos\, t$ and $y_2 = \sin\, t$.

Solution The Wronskian of $y_1 = \cos t$ and $y_2 = \sin t$ is $W = (\cos t)(\sin t)' - (\sin t)(\cos t)' = \mathbf{1}$. Set $f = 1$ and $W = 1$ in equation (13):

$$y_p = -\cos t \int_0^t \frac{(\sin t)(1)}{1} + \sin t \int_0^t \frac{(\cos t)(1)}{1} = -\cos t(-\cos t + 1) + \sin t(\sin t)$$
$$= \mathbf{1 - \cos t} \;:\; \text{Step response}$$

27 Solve $y_s{}'' + 3y_s{}' + 2y_s = 1$ for the step response starting from the null solutions $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.

Solution The Wronskian of $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ is

$W = e^{-t}(-2e^{-2t}) - e^{-2t}(-e^{-t}) = \boldsymbol{-e^{-3t}}$. Set $f = 1$ in (13):

$$y_p = -e^{-t} \int_0^t \frac{e^{-2t}(1)}{-e^{-3t}}\, dt + e^{-2t} \int_0^t \frac{e^{-t}(1)}{-e^{-3t}}\, dt = +e^{-t}[e^t - 1] + e^{-2t}\left[\frac{1}{2}e^{2t} + \frac{1}{2}\right]$$
$$= \boldsymbol{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}.$$

The steady state is $y_p(\infty) = \frac{1}{2}$. This agrees with $y'' + 3y' + 2y = 1$ when $y =$ constant.

28 Solve $Ay'' + Cy = \cos \omega t$ when $A\omega^2 = C$ (the case of resonance). Example 4 suggests to substitute $y = Mt \cos \omega t + Nt \sin \omega t$. Find M and N.

Solution $y = Mt \cos \omega t + Nt \sin \omega t$ has

$y' = M(\cos \omega t - \omega t \sin \omega t) + N(\sin \omega t + \omega t \cos \omega t)$.

Now compute $Ay'' + Cy$ when $C = A\omega^2$. The result is

$AM(-2\omega \sin \omega t - \omega^2 t \cos \omega t) + A\omega^2 Mt \cos \omega t + AN(2\omega \cos \omega t - \omega^2 t \sin \omega t) + A\omega^2 N \sin \omega t = \cos \omega t$.

Simplify to $AM(-2\omega \sin \omega t) + AN(2\omega \cos \omega t) = \cos \omega t$. Then $\boldsymbol{M = 0}$ and $\boldsymbol{N = 1/2A\omega}$.

29 Put $g(t)$ into the great formulas (17)-(18) to see the equations above them.

Solution The equation above (17) came from the V of P equation (13):

Particular solution
Constant coefficients
$$y_p(t) = \frac{-e^{s_1 t}}{s_2 - s_1} \int_0^t e^{-s_1 T} f(T)dT + \frac{e^{s_2 t}}{s_2 - s_1} \int_0^t e^{-s_2 T} f(T)dT$$

This is the integral of $\dfrac{-e^{s_1(t-T)}}{s_2 - s_1} f(T) + \dfrac{e^{s_2(t-T)}}{s_2 - s_1} f(T)$ which is exactly $g(t-T)f(T)$.

For equal roots $s_1 = s_2$, the equation after (17) is the V of P equation:

Particular solution $\boldsymbol{y_p}$
Null solutions $\boldsymbol{e^{st}, te^{st}}$
$$y_p(t) = -e^{st} \int_0^t Te^{-sT} f(T)dT + te^{st} \int_0^t e^{-sT} f(T)dT$$

This is the integral of $-Te^{s(t-T)}f(T) + te^{s(t-T)}f(T)\,dt = (t-T)e^{s(t-T)}f(T)$.
This is exactly $g(t-T)f(T)$ when $g(t) = te^{st}$ in the equal roots case.
Neat conclusion: **Variation of Parameters gives exactly** $\boldsymbol{\int g(t-T)f(T)dT}$.

Problem Set 2.7, page 148

1 Take the Laplace transform of each term in these equations and solve for $Y(s)$, with $y(0) = 0$ and $y'(0) = 1$. Find the roots s_1 and s_2 — the poles of $Y(s)$:

Undamped	$y'' + 0y' + 16y = 0$
Underdamped	$y'' + 2y' + 16y = 0$
Critically damped	$y'' + 8y' + 16y = 0$
Overdamped	$y'' + 10y' + 16y = 0$

For the overdamped case use PF2 to write $Y(s) = A/(s-s_1) + B/(s-s_2)$.

Solution (a) Taking the Laplace Transform of $y'' + 0y' + 16y = 0$ gives:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 0\cdot sY(s) - 0\cdot y(0) + 16Y(s) &= 0 \\ s^2Y(s) - 1 + 16Y(s) &= 0 \\ Y(s)(s^2+16) &= 1 \\ Y(s) &= \boldsymbol{\frac{1}{s^2+16}} \end{aligned}$$

The poles of Y = roots of s^2+16 are $s = 4i$ and $-4i$.

(b) Taking the Laplace Transform of $y'' + 2y' + 16y = 0$ gives:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 2\cdot sY(s) - 2\cdot y(0) + 16Y(s) &= 0 \\ s^2Y(s) - 1 + 2sY(s) + 16Y(s) &= 0 \\ Y(s)(s^2+2s+16) &= 1 \\ Y(s) &= \boldsymbol{\frac{1}{s^2+2s+16}} \end{aligned}$$

The roots of $s^2+2s+16$ are $-1-i\sqrt{15}$ and $-1+i\sqrt{15}$. Underdamping.

(c) Taking the Laplace Transform of $y'' + 8y' + 16y = 0$ gives:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 8\cdot sY(s) - 2\cdot y(0) + 16Y(s) &= 0 \\ s^2Y(s) - 1 + 8sY(s) + 16Y(s) &= 0 \\ Y(s)(s^2+8s+16) &= 1 \\ Y(s) = \frac{1}{s^2+8s+16} &= \boldsymbol{\frac{1}{(s+4)^2}} \end{aligned}$$

There is a double pole at $s = -4$. Critical damping.

(d) Taking the Laplace Transform of $y'' + 10y' + 16y = 0$ gives:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 10 \cdot sY(s) - 10 \cdot y(0) + 16Y(s) &= 0 \\ s^2Y(s) - 1 + 10sY(s) + 16Y(s) &= 0 \\ Y(s)(s^2 + 10s + 16) &= 1 \end{aligned}$$

$$Y(s) = \frac{1}{s^2 + 10s + 16} = \frac{1}{(s+2)(s+8)} = \frac{1}{6(s+2)} - \frac{1}{6(s+8)}$$

The poles of $Y(s)$ are -2 and -8: Overdamping.

2 Invert the four transforms $Y(s)$ in Problem 1 to find $y(t)$.

Solution (a) $Y(s) = \dfrac{1}{s^2+16} = \dfrac{1}{4} \cdot \dfrac{4}{s^2+16}$ inverts to $y(t) = \boldsymbol{\dfrac{1}{4}\sin(4t)}$.

(b) $Y(s) = \dfrac{1}{s^2+2s+16} = \dfrac{1}{(s+1)^2+15}$ inverts by equation (28) to $y(t) = \boldsymbol{e^{-t}\cos(\sqrt{15}t)/\sqrt{15}}$.

(c) $Y(s) = \dfrac{1}{(s+4)^2}$ inverts to $y(t) = \boldsymbol{te^{-4t}}$.

(d) $Y(s) = \dfrac{1}{6(s+2)} - \dfrac{1}{6(s+8)}$ inverts to $y(t) = \boldsymbol{\dfrac{1}{6}e^{-2t} - \dfrac{1}{6}e^{-8t}}$.

3 (a) Find the Laplace Transform $Y(s)$ from the equation $y' = e^{at}$ with $y(0) = A$.

(b) Use PF2 to break $Y(s)$ into two fractions $C_1/(s-a) + C_2/s$.

(c) Invert $Y(s)$ to find $y(t)$ and check that $y' = e^{at}$ and $y(0) = A$.

Solution (a) Taking the Laplace Transform of $y' = e^{at}$ gives:

$$\begin{aligned} sY(s) - y(0) &= \frac{1}{s-a} \\ sY(s) - A &= \frac{1}{s-a} \\ Y(s) &= \frac{A}{s} + \frac{1}{s(s-a)} \end{aligned}$$

(b) By using partial fractions $Y(s) = \dfrac{A}{s} + \dfrac{\frac{1}{a}}{(s-a)} + \dfrac{\frac{-1}{a}}{s}$

(c) The inverse Laplace Transform of each term gives:

$$\boldsymbol{y(t) = A + \frac{1}{a}e^{at} - \frac{1}{a}}$$

Differentiating gives: $y'(t) = a\dfrac{1}{a}e^{at} = e^{at}$ with $y(0) = A + \dfrac{1}{a} - \dfrac{1}{a} = A$.

4 (a) Find the transform $Y(s)$ when $y'' = e^{at}$ with $y(0) = A$ and $y'(0) = B$.

(b) Split $Y(s)$ into $C_1/(s-a) + C_2/(s-a)^2 + C_3/s$.

(c) Invert $Y(s)$ to find $y(t)$. Check $y'' = e^{at}$ and $y(0) = A$ and $y'(0) = B$.

Solution (a) The Laplace Transform of $y'' = e^{at}$ gives:

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) &= \frac{1}{s-a} \\
s^2Y(s) &= sA + B + \frac{1}{s-a} \\
Y(s) &= \frac{A}{s} + \frac{B}{s^2} + \frac{1}{s^2(s-a)}
\end{aligned}
$$

(b) $\dfrac{1}{s^2(s-a)} = \dfrac{Cs+D}{s^2} + \dfrac{E}{s-a} = \dfrac{(s-a)(Cs+D)+Es^2}{s^2(s-a)}.$

That numerator matches 1 when $D = -\dfrac{1}{a}, C = -\dfrac{1}{a^2}, E = \dfrac{1}{a^2}.$

(c) $y(t) = A + Bt + C + Dt + Ee^{at} = \boldsymbol{A + Bt - \dfrac{1}{a^2} - \dfrac{t}{a} + \dfrac{1}{a^2}e^{at}}.$

5 Transform these differential equations to find $Y(s)$:

(a) $y'' - y' = 1$ with $y(0) = 4$ and $y'(0) = 0$

(b) $y'' + y = \cos \omega t$ with $y(0) = y'(0) = 0$ and $\omega \neq 1$

(c) $y'' + y = \cos t$ with $y(0) = y'(0) = 0$. What changed for $\omega = 1$?

Solution (a) The Laplace Transform of $y'' - y' = 1$ is

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) &= \frac{1}{s} \\
s^2Y(s) - 4s - sY(s) + 4 &= \frac{1}{s} \\
Y(s)(s^2 - s) &= \frac{1}{s} + 4s - 4 \\
Y(s) &= \frac{\frac{1}{s} + 4s - 4}{s^2 - s} \\
Y(s) &= \frac{4s^2 - 4s + 1}{s^3 - s^2} \\
Y(s) &= \frac{(2s-1)^2}{s^2(s-1)} \\
Y(s) &= \boldsymbol{-\frac{1}{s^2} + \frac{3}{s} + \frac{1}{s-1}}
\end{aligned}
$$

(b) The Laplace Transform of $y'' + y = \cos \omega t$ with $y(0) = 0$ and $y'(0) = 0$:

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{s}{s^2 + \omega^2} \\
s^2Y(s) + Y(s) &= \frac{s}{s^2 + \omega^2} \\
Y(s) &= \frac{s}{(s^2 + \omega^2)(s^2 + 1)}
\end{aligned}
$$

(c) The Laplace Transform of $y'' + y = \cos t$ with $y(0) = 0$ and $y'(0) = 0$:

$$s^2Y(s) + Y(s) = \frac{s}{s^2+1}$$
$$Y(s) = \boldsymbol{\frac{s}{(s^2+1)^2}} \text{ : Double poles from resonance}$$

6 Find the Laplace transforms F_1, F_2, F_3 of these functions f_1, f_2, f_3 :

(a) $f_1(t) = e^{at} - e^{bt}$ (b) $f_2(t) = e^{at} + e^{-at}$ (c) $f_3(t) = t\cos t$

Solution (a) The Laplace Transform of $e^{at} - e^{bt}$ is $\dfrac{1}{s-a} - \dfrac{1}{s-b} = \dfrac{a-b}{(s-a)(s-b)}$.

(b) The Laplace Transform of $e^{at} + e^{-at}$ is $\dfrac{1}{s-a} + \dfrac{1}{s+a} = \dfrac{2s}{s^2-a^2}$.

(c) The Laplace Transform of te^{at} is $\frac{1}{(s-a)^2}$ by equation (19). With $a = i$, write $t\cos t = \frac{1}{2}te^{it} + \frac{1}{2}te^{-it}$. Then the transform of $t\cos t$ is

$$\frac{1}{2}\frac{1}{(s-i)^2} + \frac{1}{2}\frac{1}{(s+i)^2} = \frac{1}{2}\frac{(s+i)^2+(s-i)^2}{(s-i)^2(s+i)^2} = \boldsymbol{\frac{s^2-1}{(s^2+1)^2}}.$$

7 For any real or complex a, the transform of $f = te^{at}$ is _____. By writing $\cos\omega t$ as $(e^{i\omega t} + e^{-i\omega t})/2$, transform $g(t) = t\cos\omega t$ and $h(t) = te^t\cos\omega t$. (*Notice that the transform of h is new.*)

Solution The transform of te^{at} is $\frac{1}{(s-a)^2}$ by equation (19). Here $a = i\omega$. Then $t\cos\omega t = \frac{1}{2}te^{i\omega t} + \frac{1}{2}te^{-i\omega t}$ transforms to

$$\frac{1}{2}\frac{1}{(s-i\omega)^2} + \frac{1}{2}\frac{1}{(s+i\omega)^2} = \frac{1}{2}\frac{(s+i\omega)^2+(s-i\omega)^2}{(s-i\omega)^2(s+i\omega)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

Similarly $te^t\cos\omega t = \frac{1}{2}te^{(1+i\omega)t} + \frac{1}{2}te^{(1-i\omega)t}$ transforms to

$$\frac{1}{2}\frac{1}{(s-1-i\omega)^2} + \frac{1}{2}\frac{1}{(s-1+i\omega)^2} = \frac{1}{2}\frac{(s-1+i\omega)^2+(s-1-i\omega)^2}{(s-1-i\omega)^2(s-1+i\omega)^2} = \frac{(s-1)^2-\omega^2}{((s-1)^2+\omega^2)^2}.$$

8 Invert the transforms F_1, F_2, F_3 using PF2 and PF3 to discover f_1, f_2, f_3 :

(a) $F_1(s) = \dfrac{1}{(s-a)(s-b)}$ (b) $F_2(s) = \dfrac{s}{(s-a)(s-b)}$ (c) $F_3(s) = \dfrac{1}{s^3-s}$

Solution (a) $F_1(s) = \dfrac{1}{(s-a)(s-b)} = \dfrac{1}{(a-b)(s-a)} + \dfrac{1}{(b-a)(s-b)}$.

The inverse transform is $f_1 = \boldsymbol{\dfrac{1}{(a-b)}e^{at} + \dfrac{1}{(b-a)}e^{bt}}$.

(b) $F_2(s) = \dfrac{s}{(s-a)(s-b)} = \dfrac{a}{(a-b)(s-a)} + \dfrac{b}{(b-a)(s-b)}$.

The inverse transform is $f_2 = \boldsymbol{\dfrac{a}{(a-b)}e^{at} + \dfrac{b}{(b-a)}e^{bt}}$.

(c) $F_3(s) = \dfrac{1}{s^3 - s} = \dfrac{1}{(s-1)(s+1)s} = -\dfrac{1}{s} + \dfrac{\frac{1}{2}}{s+1} + \dfrac{\frac{1}{2}}{s-1}$ using **PF3**.

The inverse transform is $f_3 = \boldsymbol{-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}}$.

9 Step 1 transforms these equations and initial conditions. Step 2 solves for $Y(s)$. Step 3 inverts to find $y(t)$:

(a) $y' - ay = t$ with $y(0) = 0$

(b) $y'' + a^2y = 1$ with $y(0) = 1$ and $y'(0) = 2$

(c) $y'' + 3y' + 2y = 1$ with $y(0) = 4$ and $y'(0) = 5$.

What particular solution y_p to (c) comes from using "undetermined coefficients"? $\boldsymbol{y_p = \frac{1}{2}}$.

Solution (a) $y' - ay = t$ transforms to $sY(s) - y(0) - aY(s) = \dfrac{1}{s^2}$ with $y(0) = 0$.

$$Y(s) = \frac{1}{s^2(s-a)} = \frac{-\frac{1}{a^2}}{s} + \frac{-\frac{1}{a}}{s^2} + \frac{\frac{1}{a^2}}{s-a}$$

The inverse transform is $y(t) = \boldsymbol{-\dfrac{1}{a^2} - \dfrac{1}{a}t + \dfrac{1}{a^2}e^{at}}$.

(b) $y'' + a^2y = 1$ transforms to $s^2Y(s) - sy(0) - y'(0) + a^2Y(s) = \dfrac{1}{s}$ with $y(0) = 1$ and $y'(0) = 2$. This is $(s^2 + a^2)Y(s) = y'(0) + sy(0) + \dfrac{1}{s}$:

$$Y(s) = \frac{2}{s^2+a^2} + \frac{s}{s^2+a^2} + \frac{1}{s(s^2+a^2)} = \frac{2}{a}\frac{a}{s^2+a^2} + \frac{s}{s^2+a^2} + \frac{1}{a^2 s} - \frac{1}{a^2}\frac{s}{s^2+a^2}.$$

The inverse transform is $y(t) = \boldsymbol{\dfrac{2}{a}\sin(at) + \cos(at) + \dfrac{1}{a^2} - \dfrac{1}{a^2}\cos(at)}$.

(c) $y''+3y'+2y = 1$ becomes $s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = \dfrac{1}{s}$.

Then $y(0) = 4$ and $y'(0) = 5$ give

$$Y(s) = \frac{1}{s(s^2+3s+2)} + \frac{4s+5}{(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} + \frac{4(s+1)+1}{(s+1)(s+2)}.$$

The inverse transform can come from **PF3** on page 143. It comes much more quickly and directly (without Laplace transforms!) from knowing that $y = y_p + y_n = \frac{1}{2} + c_1e^{-t} + c_2e^{-2t}$:

$y(0) = \frac{1}{2} + c_1 + c_2 = 4$ and $y'(0) = -c_1 - 2c_2 = 5$ add to $\frac{1}{2} - c_2 = \frac{18}{2}$ and $y(t) = \boldsymbol{\frac{1}{2} + 12e^{-t} - \frac{17}{2}e^{-2t}}$.

Questions 10-16 are about partial fractions.

10 Show that PF2 in equation (9) is correct. Multiply both sides by $(s-a)(s-b)$:

$$(*) \qquad 1 = \_\_\_\_\_ + \_\_\_\_\_ .$$

(a) What do those two fractions in $(*)$ equal at the points $s = a$ and $s = b$?

(b) The equation $(*)$ is correct at those two points a and b. It is the equation of a straight _____. So why is it correct for every s?

Solution (using b instead of c in PF2):

$\mathbf{1 = \dfrac{s-b}{a-b} + \dfrac{s-a}{b-a}}$ after multiplying equation (9) by $(s-a)(s-b)$.

(a) At $s=a$ we get $1 = \frac{a-b}{a-b}$. At $s=b$ we get $1 = \frac{b-a}{b-a}$.

(b) When the equation of a *straight line* is correct for two values $s=a$ and $s=b$, it is correct for all values of s.

11 Here is the PF2 formula with numerators. Formula $(*)$ had $K=1$ and $H=0$:

$$\textbf{PF2}' \qquad \frac{Hs+K}{(s-a)(s-b)} = \frac{Ha+K}{(s-a)(a-b)} + \frac{Hb+K}{(b-a)(s-b)}$$

To show that PF2$'$ is correct, multiply both sides by $(s-a)(s-b)$. You are left with the equation of a straight _____. Check your equation at $s=a$ and at $s=b$. Now it must be correct for all s, and PF2$'$ is proved.

Solution Multiplying by $(s-a)(s-b)$ produces

$$\textbf{(*)}\ Hs+K = \frac{(Ha+K)(s-b)}{a-b} + \frac{(Hb+K)(s-a)}{b-a}.$$

At $s=a$ this is $Ha+K = Ha+K+0$: correct. Similarly correct at $s=b$. Since **(*)** is linear in s, it is the equation of a straight line. When correct at 2 points $s=a$ and $s=b$, it is correct for every s.

12 Break these functions into two partial fractions using PF2 and PF2$'$:

(a) $\dfrac{1}{s^2-4}$ (b) $\dfrac{s}{s^2-4}$ (c) $\dfrac{Hs+K}{s^2-5s+6}$

Solution (a) $$\frac{1}{s^2-4} = \frac{1}{(s-2)(s+2)} = \frac{1}{(s-2)(2+2)} + \frac{1}{(s+2)(-4)} = \mathbf{\frac{1}{4(s-2)} - \frac{1}{4(s+2)}}$$

(b) $$\frac{s}{s^2-4} = \frac{s}{(s-2)(s+2)} = \frac{2}{(s-2)(2+2)} + \frac{-2}{(-4)(s+2)} = \mathbf{\frac{1}{2(s-2)} + \frac{1}{2(s+2)}}$$

(c) $$\frac{Hs+K}{s^2-5s+6} = \frac{Hs+K}{(s-2)(s-3)} = \frac{2H+K}{(s-2)(2-3)} + \frac{3H+K}{(3-2)(s-3)} = \mathbf{-\frac{2H+K}{s-2} + \frac{3H+K}{s-3}}$$

13 Find the integrals of $(a)(b)(c)$ in Problem 12 by integrating each partial fraction. The integrals of $C/(s-a)$ and $D/(s-b)$ are logarithms.

Solution (a) $$\int \frac{1}{s^2-4}\,ds = \int \frac{1}{4(s-2)} - \frac{1}{4(s+2)}\,ds$$
$$= \frac{1}{4}\ln(s-2) - \frac{1}{4}\ln(s+2) = \boldsymbol{\frac{1}{4}\ln\frac{s-2}{s+2}}$$

(b) $$\int \frac{s}{s^2-4}\,ds = \int \frac{1}{2(s-2)} + \frac{1}{2(s+2)}\,ds$$
$$= \frac{1}{2}\ln(s-2) + \frac{1}{2}\ln(s+2) = \boldsymbol{\frac{1}{2}\ln(s^2-4)}$$

(c) $$\int \frac{Hs+K}{s^2-5s+6}\,ds = \int -\frac{2H+K}{s-2} + \frac{3H+K}{s-3}\,ds$$
$$= \boldsymbol{-(2H+K)\ln(s-2) + (3H+K)\ln(s-3)}$$

14 Extend PF3 to PF3$'$ in the same way that PF2 extended to PF2$'$:

$$\textbf{PF3}' \qquad \frac{Gs^2+Hs+K}{(s-a)(s-b)(s-c)} = \frac{Ga^2+Ha+K}{(s-a)(a-b)(a-c)} + \frac{?}{?} + \frac{?}{?}.$$

Solution We want $\dfrac{Gs^2+Hs+K}{(s-a)(s-b)(s-c)} = \dfrac{A}{s-a} + \dfrac{B}{s-b} + \dfrac{C}{s-c}$.

We can multiply both sides by $(s-a)(s-b)(s-c)$ and solve for A, B, C. Or we can use A as given in the problem statement—and permute letters a, b, c to get B and C from A. That way is easier, and our three fractions are

$$\boldsymbol{\frac{a^2G+aH+K}{(a-b)(a-c)}\frac{1}{s-a} + \frac{b^2G+bH+K}{(b-a)(b-c)}\frac{1}{s-b} + \frac{c^2G+cH+K}{(c-a)(c-b)}\frac{1}{s-c}}$$

15 The linear polynomial $(s-b)/(a-b)$ equals 1 at $s=a$ and 0 at $s=b$. Write down a quadratic polynomial that equals 1 at $s=a$ and 0 at $s=b$ and $s=c$.

Solution $\dfrac{(s-b)(s-c)}{(a-b)(a-c)}$ equals 0 for $s=b$ and $s=c$. It equals 1 for $s=a$.

16 What is the number C so that $C(s-b)(s-c)(s-d)$ equals 1 at $s=a$?

Note A complete theory of partial fractions must allow double roots (when $b=a$). The formula can be discovered from l'Hôpital's Rule (in PF3 for example) when b approaches a. Multiple roots lose the beauty of PF3 and PF3$'$—we are happy to stay with simple roots a, b, c.

Solution Choose $C = \boldsymbol{\dfrac{1}{(a-b)(a-c)(a-d)}}$.

Questions 17-21 involve the transform $\boldsymbol{F(s)=1}$ of the delta function $\boldsymbol{f(t)=\delta(t)}$.

17 Find $F(s)$ from its definition $\int\limits_0^\infty f(t)e^{-st}dt$ when $f(t) = \delta(t-T)$, $T \geq 0$.

Solution The transform of $\delta(t-T)$ is $F(s) = \displaystyle\int_0^\infty \delta(t-T)\, e^{-st}dt = \boldsymbol{e^{-sT}}$.

18 Transform $y'' - 2y' + y = \delta(t)$. The **impulse response** $y(t)$ transforms into $Y(s) =$ **transfer function**. The double root $s_1 = s_2 = 1$ gives a double pole and a new $y(t)$.

Solution With $y(0) = y'(0) = 0$, the transform is $(s^2 - 2s + 1)Y(s) = 1$. Then $Y(s) = \frac{1}{(s-1)^2}$ and the inverse transform is the impulse response $y(t) = g(t) = \boldsymbol{te^t}$.

19 Find the inverse transforms $y(t)$ of these transfer functions $Y(s)$:

(a) $\dfrac{s}{s-a}$ (b) $\dfrac{s}{s^2-a^2}$ (c) $\dfrac{s^2}{s^2-a^2}$

Solution (a) $Y(s) = \dfrac{s}{s-a} = \dfrac{s-a+a}{s-a} = 1 + \dfrac{a}{s-a}$

$$y(t) = \boldsymbol{\delta(t) + ae^{at}}$$

(b) Using **PF2** we have $Y(s) = \dfrac{s}{s^2-a^2} = \dfrac{s}{(s-a)(s+a)} = \dfrac{1}{2(s-a)} + \dfrac{1}{2(s+a)}$

The inverse transform is $y(t) = \boldsymbol{\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}} = \cosh at$

(c) $Y(s) = \dfrac{s^2}{s^2-a^2} = \dfrac{s^2-a^2+a^2}{s^2-a^2} = 1 + \dfrac{a^2}{s^2-a^2} = 1 + \dfrac{a}{2(s-a)} - \dfrac{a}{2(s+a)}$

$$y(t) = \boldsymbol{\delta(t) + \frac{a}{2}e^{at} - \frac{a}{2}e^{-at} = \delta(t) + a\sinh(at)}$$

20 Solve $y'' + y = \delta(t)$ by Laplace transform, with $y(0) = y'(0) = 0$. If you found $y(t) = \sin t$ as I did, this involves a serious mystery: *That sine solves* $y'' + y = 0$, and it doesn't have $y'(0) = 0$. *Where does* $\delta(t)$ *come from*? In other words, what is the derivative of $y' = \cos t$ if all functions are zero for $t < 0$?

If $y = \sin t$, explain why $y'' = -\sin t + \delta(t)$. Remember that $y = 0$ **for** $t < 0$.

Problem (20) connects to a remarkable fact. The same impulse response $y = g(t)$ solves both of these equations: **An impulse at $t = 0$ makes the velocity $y'(0)$ jump by 1**. Both equations start from $y(0) = 0$.

$$y'' + By' + Cy = \boldsymbol{\delta(t)} \text{ with } y'(0) = \mathbf{0} \quad y'' + By' + Cy = \mathbf{0} \text{ with } y'(0) = \mathbf{1}.$$

Solution $y'' + y = \delta(t)$ transforms into $s^2Y(s) + Y(s) = 1$.

Then $Y(s) = \frac{1}{s^2+1}$ has the inverse transform $y(t) = \boldsymbol{\sin t}$.

At time $t = 0$ the derivative of $y' = \cos(t)$ is not $y'' = \sin(0) = 0$, but rather $\boldsymbol{y'' = \sin(0) + \delta(t)}$, since the function $y' = \cos(t)$ jumps from 0 to 1 at $t = 0$.

21 (Similar mystery) These two problems give the same $Y(s) = s/(s^2+1)$ and the same impulse response $y(t) = g(t) = \cos t$. How can this be?

(a) $y' = -\sin t$ with $y(0) = \mathbf{1}$ (b) $y' = -\sin t + \boldsymbol{\delta(t)}$ with "$y(0) = 0$"

Solution (a) The Laplace transform of $y'(t) = -\sin(t)$ with $y(0) = 1$ is

$$\begin{aligned} sY(s) - 1 &= -\frac{1}{s^2+1} \\ sY(s) &= 1 - \frac{1}{s^2+1} = \frac{s^2+1-1}{s^2+1} = \frac{s^2}{s^2+1} \\ Y(s) &= \frac{s}{s^2+1} \end{aligned}$$

(b) The Laplace transform of $y'(t) = -\sin(t) + \delta(t)$ with $y(0) = 0$ is

$$\begin{aligned} sY(s) - y(0) &= -\frac{1}{s^2+1^2} + 1 \\ sY(s) - 0 &= \frac{s^2+1-1}{s^2+1} = \frac{s^2}{s^2+1} \\ Y(s) &= \frac{s}{s^2+1} \end{aligned}$$

These two problems (a) and (b) give the same $Y(s)$ and therefore the same $y(t)$. The reason is that $\delta(t)$ in the derivative y' gives the same result as an initial condition $y(0) = 1$. Both cause a jump from $y = 0$ before $t = 0$ to $y = 1$ right after $t = 0$. And both transform to 1.

Problems 22-24 involve the Laplace transform of the integral of $\boldsymbol{y(t)}$.

22 If $f(t)$ transforms to $F(s)$, what is the transform of the integral $h(t) = \int_0^t f(T)dT$? Answer by transforming the equation $dh/dt = f(t)$ with $h(0) = 0$.

Solution If $h(t) = \int_0^t f(T)\,dT$ then $dh/dt = f(t)$ with $h(0) = 0$. Taking the Laplace Transform gives:

$$sH(s) = F(s) \text{ and } H(s) = \frac{\boldsymbol{F(s)}}{\boldsymbol{s}}.$$

23 Transform and solve the integro-differential equation $y' + \int_0^t y\,dt = 1$, $y(0) = 0$.

A mystery like Problem 20: $y = \cos t$ seems to solve $y' + \int_0^t y dt = 0, y(0) = 1$.

Solution The Laplace transform of $y' + \int_0^t y\,dt = 1$ with $y(0) = 0$ is

$$sY(s) - y(0) + \frac{Y(s)}{s} = \frac{1}{s}$$
$$Y(s) = \frac{1}{\left(s + \frac{1}{s}\right)s} = \frac{1}{s^2+1}$$

The inverse transform of $Y(s)$ is $\boldsymbol{y(t) = \sin(t)}$

About the mystery: The derivative of $\cos t$ is $-\sin t + \delta(t)$ because $\cos t$ jumps at $t = 0$ from zero for $t < 0$ (by convention) to 1. But I am not seeing a new mystery.

24 Transform and solve the amazing equation $dy/dt + \int_0^t y\,dt = \delta(t)$.

Solution The transform of $\dfrac{dy}{dt} + \displaystyle\int_0^t y\,dt = \delta(t)$ is $sY(s) + \dfrac{Y(s)}{s} = 1$.

Then $Y(s) = \dfrac{1}{\left(s + \frac{1}{s}\right)s} = \dfrac{s}{s^2+1}$ and $\boldsymbol{y(t) = \cos t}$.

Note that this follows from Problem 20, where we found that $\cos(t)$ has integral $\sin(t)$ and derivative $-\sin(t) + \delta(t)$.

25 The derivative of the delta function is not easy to imagine—it is called a "doublet" because it jumps up to $+\infty$ and back down to $-\infty$. Find the Laplace transform of the doublet $d\delta/dt$ from the rule for the transform of a derivative.

A doublet $\delta'(t)$ is known by its integral: $\int \delta'(t)F(t)dt = -\int \delta(t)F'(t)dt = \boldsymbol{-F'(0)}$.

Solution The Laplace transform of $\delta(t)$ is 1. The Laplace transform of the derivative is $sY(s) - y(0)$. The Laplace transform of the doublet $\delta'(t) = d\delta/dt$ is therefore $\boldsymbol{s}$.

26 (Challenge) What function $y(t)$ has the transform $Y(s) = 1/(s^2+\omega^2)(s^2+a^2)$? First use partial fractions to find H and K:

$$Y(s) = \frac{H}{s^2+\omega^2} + \frac{K}{s^2+a^2}$$

Solution $Y(s) = \dfrac{1}{(s^2+\omega^2)(s^2+a^2)} = \dfrac{1}{(s^2+\omega^2)(a^2-\omega^2)} - \dfrac{1}{(s^2+a^2)}\dfrac{1}{(a^2-\omega^2)}$.

Then $y(t) = \dfrac{\boldsymbol{\sin \omega t}}{\boldsymbol{\omega(a^2-\omega^2)}} - \dfrac{\boldsymbol{\sin at}}{\boldsymbol{a(a^2-\omega^2)}}$.

27 Why is the Laplace transform of a unit step function $H(t)$ the same as the Laplace transform of a constant function $f(t) = 1$?

Solution The step function and the constant function are the same for $t \geq 0$.