DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 3

1 Draw the graph of $y = e^t$ by hand, for $-1 \le t \le 1$. What is its slope dy/dt at $t = 0$? Add the straight line graph of $y = et$. Where do those two graphs cross?

Solution The derivative of e^t has slope 1 at $t = 0$. The graphs meet at $t = 1$ where their value is e. They don't actually "cross" because the line is tangent to the curve: both have slope $y' = e$ at $t = 1$.

2 Draw the graph of $y_1 = e^{2t}$ on top of $y_2 = 2e^t$. Which function is larger at $t = 0$? Which function is larger at $t = 1$?

Solution From the graphs we see that at $t = 0$, the function $2e^t$ is larger whereas at $t = 1$, e^{2t} is larger. (e times e is larger than 2 times e).

3 What is the slope of $y = e^{-t}$ at $t = 0$? Find the slope dy/dt at $t = 1$.

Solution The slope of e^{-t} is $-e^{-t}$. At $t = 0$ this is -1. The slope at $t = 1$ is $-e^{-1}$.

4 What "logarithm" do we use for the number t (the exponent) when $e^t = 4$?

Solution We use the natural logarithm to find t from the equation $e^t = 4$. We get that $t = \ln 4 \approx 1.386$.

5 State the chain rule for the derivative dy/dt if $y(t) = f(u(t))$ (chain of f and u).

Solution The chain rule gives:

$$\frac{dy}{dt} = \frac{df(u(t))}{du(t)}\frac{du(t)}{dt}$$

6 The *second* derivative of e^t is again e^t. So $y = e^t$ solves $d^2y/dt^2 = y$. A second order differential equation should have another solution, different from $y = Ce^t$. What is that second solution?

Solution The second solution is $y = e^{-t}$. The second derivative is $-(-e^{-t}) = e^{-t}$.

7 Show that the nonlinear example $dy/dt = y^2$ is solved by $y = C/(1 - Ct)$ for every constant C. The choice $C = 1$ gave $y = 1/(1 - t)$, starting from $y(0) = 1$.

Solution Given that $y = C/(1 - Ct)$, we have:

$$y^2 = C^2/(1 - Ct)^2$$
$$\tfrac{dy}{dt} = C \cdot (-1) \cdot (-C)1/(1 - Ct)^2 = C^2/(1 - Ct)^2$$

8 Why will the solution to $dy/dt = y^2$ grow faster than the solution to $dy/dt = y$ (if we start them both from $y = 1$ at $t = 0$)? The first solution blows up at $t = 1$. The second solution e^t grows exponentially fast but it never blows up.

Solution The solution of the equation $dy/dt = y^2$ for $y(0) = 1$ is $y = 1/(1-t)$, while the solution to $dy/dt = y$ for $y(0) = 1$ is $y = e^t$. Notice that the first solution blows up at $t = 1$ while the second solution e^t grows exponentially fast but never blows up.

9 Find a solution to $dy/dt = -y^2$ starting from $y(0) = 1$. Integrate dy/y^2 and $-dt$. (Or work with $z = 1/y$. Then $\boldsymbol{dz/dt} = (dz/dy)\,(dy/dt) = (-1/y^2)(-y^2) = \mathbf{1}$. From $dz/dt = 1$ you will know $z(t)$ and $y = 1/z$.)

Solution The first method has

$$\frac{dy}{y^2} = -dt$$

$$\int_{y(0)}^{y} \frac{du}{u^2} = -\int_0^t dv \quad (u, v \text{ are integration variables})$$

$$\frac{-1}{y} + \frac{1}{y(0)} = -t$$

$$\frac{-1}{y} = -t - 1$$

$$y = \frac{1}{1+t}$$

The approach using $z = 1/y$ leads to $dz/dt = 1$ and $z(0) = 1/1$.
Then $z(t) = 1 + t$ and $y = 1/z = \frac{1}{1+t}$.

10 Which of these differential equations are linear (in y)?

(a) $y' + \sin y = t$ (b) $y' = t^2(y - t)$ (c) $y' + e^t y = t^{10}$.

Solution (a) Since this equation solves a $\sin y$ term, it is not linear in y.

(b) and (c) Since these equations have no nonlinear terms in y, they are linear.

11 The product rule gives what derivative for $e^t e^{-t}$? This function is constant. At $t = 0$ this constant is 1. Then $e^t e^{-t} = 1$ for all t.

Solution $(e^t e^{-t})' = e^t e^{-t} - e^t e^{-t} = 0$ so $e^t e^{-t}$ is a constant (1).

12 $dy/dt = y + 1$ is not solved by $y = e^t + t$. Substitute that y to show it fails. We can't just add the solutions to $y' = y$ and $y' = 1$. What number c makes $y = e^t + c$ into a correct solution?

Solution

$$\frac{dy}{dt} = y + 1 \qquad\qquad \frac{d(e^t+c)}{dt} = e^t + c + 1$$

$$\textbf{Wrong } \frac{d(e^t+t)}{dt} \neq e^t + t + 1 \qquad\qquad \textbf{Correct } \boldsymbol{c = -1}$$

Problem Set 1.3, page 15

1 Set $t = 2$ in the infinite series for e^2. The sum must be e times e, close to 7.39. How many terms in the series to reach a sum of 7? How many terms to pass 7.3?

Solution The series for e^2 has $t = 2$: $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots$

If we include five terms we get: $e^2 \approx 1 + 2 + 2 + \frac{8}{6} + \frac{16}{24} = 7.0$

If we include seven terms we get: $e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{120} + \frac{2^6}{720} = 7.35556.$

2 Starting from $y(0) = 1$, find the solution to $dy/dt = y$ at time $t = 1$. Starting from that $y(1)$, solve $dy/dt = -y$ to time $t = 2$. Draw a rough graph of $y(t)$ from $t = 0$ to $t = 2$. What does this say about e^{-1} times e ?

Solution $y = e^t$ up to $t = 1$, so that $y(1) = e$. Then for $t > 1$ the equation $dy/dt = -y$ has $y = Ce^{-t}$. At $t = 1$, this becomes $e = Ce^{-1}$ so that $C = e^2$. The solution of $dy/dt = -y$ up to $t = 2$ is $y = e^{2-t}$. At $t = 2$ we have returned to $y(2) = y(0) = 1$. Then $(e^{-1})(e) = 1$.

3 Start with $y(0) = \$5000$. If this grows by $dy/dt = .02y$ until $t = 5$ and then jumps to $a = .04$ per year until $t = 10$, what is the account balance at $t = 10$?

Solution

$$t \le 5: \frac{dy}{dt} = .02y \qquad y = 5000e^{.02t} \qquad y(5) = 5000e^{.1}$$

$$5 \le t \le 10: \frac{dy}{dt} = .04y \text{ gives } y = Ce^{.04t}$$
$$y(5) = Ce^{-2} = 5000e^{.1} \text{ gives } C = 5000e^{-.1}$$
$$y(t) = 5000(e^{.04t-0.1})$$
$$y(10) = 5000e^{.3}$$

4 Change Problem 3 to start with \$5000 growing at $dy/dt = .04y$ for the first five years. Then drop to $a = .02$ per year until year $t = 10$. What is the account balance at $t = 10$?

Solution

$$\begin{aligned} \frac{dy}{dt} &= .04y \\ y &= C_1e^{.04t} \\ y(0) &= C_1 = 5000 \\ y(t) &= 5000e^{.04t} \text{ for } t \le 5 \\ y(5) &= 5000e^{.2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= .02y \text{ for } 5 \le t \le 10 \\ y &= C_2e^{.02t} \\ y(5) &= C_2e^{.1} = 5000e^{.2} \\ C_2 &= 5000e^{.1} \\ y(t) &= 5000(e^{.02t+0.1}) \\ y(10) &= 5000e^{.3} = \text{ same as in 1.3.3.} \end{aligned}$$

Problems 5–8 are about $y = e^{at}$ and its infinite series.

5 Replace t by at in the exponential series to find e^{at} :

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \cdots$$

Take the derivative of every term (keep five terms). Factor out a to show that *the derivative of* e^{at} *equals* ae^{at}. At what time T does e^{at} reach 2 ?

Solution The derivative of this series is obtained by differentiating the terms individually:

$$\begin{aligned} \frac{dy}{dt} &= a + at + \cdots + \frac{1}{(n-1)!}a^n t^{n-1} + \cdots \\ &= a\left(1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{(n-1)!}a^{n-1}t^{n-1} + \cdots\right) = ae^{at} \end{aligned}$$

If $e^{a\mathrm{T}} = 2$ then $a\mathrm{T} = \ln 2$ and $\mathrm{T} = \dfrac{\ln 2}{a}$.

6 Start from $y' = ay$. Take the derivative of that equation. Take the n^{th} derivative. Construct the Taylor series that matches all these derivatives at $t = 0$, starting from $1 + at + \frac{1}{2}(at)^2$. Confirm that this series for $y(t)$ is exactly the exponential series for e^{at}.

Solution The derivative of $y' = ay$ is $y'' = ay' = a^2y$. The next derivative is $y''' = ay''$ which is a^3y. When $y(0) = 1$, the derivatives at $t = 0$ are $a, a^2, a^3, \ldots$ so the Taylor series is $y(t) = 1 + at + \dfrac{1}{2}a^2t^2 + \cdots = e^{at}$.

7 At what times t do these events happen ?
(a) $e^{at} = e$ (b) $e^{at} = e^2$ (c) $e^{a(t+2)} = e^{at}e^{2a}$.

Solution
(a) $e^{at} = e$ at $t = 1/a$.
(b) $e^{at} = e^2$ at $t = 2/a$.
(c) $e^{a(t+2)} = e^{at}e^{2a}$ at all t.

8 If you multiply the series for e^{at} in Problem 5 by itself you should get the series for e^{2at}. Multiply the first 3 terms by the same 3 terms to see the first 3 terms in e^{2at}.

Solution $(1 + at + \frac{1}{2}a^2t^2)(1 + at + \frac{1}{2}a^2t^2) = 1 + 2at + \left(1 + \frac{1}{2} + \frac{1}{2}\right)a^2t^2 + \cdots$

This agrees with $e^{2at} = 1 + 2at + \frac{1}{2}(2at)^2 + \cdots$

9 (recommended) Find $y(t)$ if $dy/dt = ay$ and $\boldsymbol{y(T) = 1}$ (instead of $y(0) = 1$).

Solution $\frac{dt}{dt} = ay$ gives $y(t) = Ce^{at}$. When $Ce^{a\mathrm{T}} = 1$ at $t = \mathrm{T}$, this gives $C = e^{-a\mathrm{T}}$ and $y(t) = e^{a(t-\mathrm{T})}$.

10 (a) If $dy/dt = (\ln 2)y$, explain why $y(1) = 2y(0)$.
(b) If $dy/dt = -(\ln 2)y$, how is $y(1)$ related to $y(0)$?

Solution
(a) $\frac{dy}{dt} = (\ln 2)y \to y(t) = y(0)e^{t(\ln 2)} \to y(1) = y(0)e^{\ln 2} = 2y(0)$.
(b) $\frac{dy}{dt} = -(\ln 2)y \to y(t) = y(0)e^{-t(\ln 2)} \to y(1) = y(0)e^{-\ln 2} = \frac{1}{2}y(0)$.

11 In a one-year investment of $y(0) = \$100$, suppose the interest rate jumps from 6% to 10% after six months. Does the equivalent rate for a whole year equal 8%, or more than 8%, or less than 8% ?

Solution We solve the equation in two steps, first from $t = 0$ to $t = 6$ months, and then from $t = 6$ months to $t = 12$ months.

$$y(t) = y(0)e^{at} \qquad\qquad y(t) = y(0.5)e^{at}$$
$$y(0.5) = \$100e^{0.06\times0.5} = \$100e^{.03} \qquad\qquad y(1) = \$103.05e^{0.1\times0.5} = \$103.05e^{.05}$$
$$= \$103.05 \qquad\qquad = \$108.33$$

If the money was invested for one year at 8% the amount at $t = 1$ would be:

$y(1) = \$100e^{0.08\times1} = \$108.33$.

The equivalent rate for the whole year is indeed exactly 8%.

12 If you invest $y(0) = \$100$ at 4% interest compounded continuously, then $dy/dt = .04y$. Why do you have more than $\$104$ at the end of the year ?

Solution The quantitative reason for why this is happening is obtained from solving the equation:

$$\frac{dy}{dt} = 0.04y \to y(t) = y(0)e^{.04t}$$
$$y(1) = 100e^{0.04} \approx \$104.08.$$

The intuitive reason is that **the interest accumulates interest**.

13 What linear differential equation $dy/dt = a(t)y$ is satisfied by $y(t) = e^{\cos t}$?

Solution The chain rule for $f(u(t))$ has $y(t) = f(u) = e^u$ and $u(t) = \sin t$:

$$\frac{dy}{dt} = \frac{df(u(t))}{dt} = \frac{df}{dt}\frac{du}{dt} = e^u \cos t = y \cos t. \text{ Then } \boldsymbol{a(t) = \cos(t)}.$$

14 If the interest rate is $a = 0.1$ per year in $y' = ay$, how many years does it take for your investment to be multiplied by e ? How many years to be multiplied by e^2 ?

Solution If the interest rate is $a = 0.1$, then $y(t) = y(0)e^{0.1t}$. For $t = 10$, the value is $y(t) = y(0)\,e$. For $t = 20$, the value is $y(t) = y(0)\,e^2$.

15 Write the first four terms in the series for $y = e^{t^2}$. Check that $dy/dt = 2ty$.

Solution

$$y = e^{t^2} = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots$$

$$\frac{dy}{dt} = 2t + 2t^3 + t^5 + \cdots = 2t\left(1 + t^2 + \frac{1}{2}t^4 + \cdots\right) = 2te^{t^2}.$$

16 Find the derivative of $Y(t) = \left(1 + \frac{t}{n}\right)^n$. If n is large, this dY/dt is close to Y !

Solution The derivative of $Y(t) = \left(1 + \frac{t}{n}\right)^n$ with respect to t is $n(\frac{1}{n})\left(1 + \frac{t}{n}\right)^{n-1} = \left(1 + \frac{t}{n}\right)^{n-1}$. For large n the extra factor $1 + \frac{t}{n}$ is nearly 1, and dY/dt is near Y.

17 (Key to future sections). Suppose the exponent in $y = e^{u(t)}$ is $u(t) =$ integral of $a(t)$. What equation $dy/dt =$ _____ y does this solve ? If $u(0) = 0$ what is the starting value $y(0)$?

Solution Differentiating $y = e^{\int a(t)\,dt}$ with respect to t by the chain rule yields $y' = a(t)e^{\int a(t)\,dt}$. Therefore $\boldsymbol{dy/dt = a(t)y}$. If $u(0) = 0$ we have $y(0) = e^{u(0)} = 1$.

18 The Taylor series comes from $e^{d/dx}f(x)$, when you write out $e^{d/dx} = 1 + d/dx + \frac{1}{2}(d/dx)^2 + \cdots$ as a sum of higher and higher derivatives. Applying the series to $f(x)$ at $x = 0$ would give the value $f + f' + \frac{1}{2}f'' + \cdots$ at $\boldsymbol{x = 0}$.
The Taylor series says: This is equal to $f(x)$ at $x =$ _____.

Solution

$$\boldsymbol{f(1)} = f(0) + 1f'(0) + \frac{1}{2}1^2 f''(0) + \cdots \quad \text{This is exactly}$$

$$f(1) = \left(1 + \frac{d}{dx} + \frac{1}{2}\left(\frac{d}{dx}\right)^2 + \cdots\right) f(x) \text{ at } x = 0.$$

19 (Computer or calculator, 2.xx is close enough) Find the time t when $e^t = 10$. The initial $y(0)$ has increased by an order of magnitude—a factor of 10. The exact statement of the answer is $t =$ _____. At what time t does e^t reach 100 ?

Solution The exact time when $e^t = 10$ is $t = \ln 10$. This is $t \approx 2.30$ or 2.3026.

Then the time when $e^{\mathrm{T}} = 100$ is $\mathrm{T} = \ln 100 = \ln 10^2 = 2\ln 10 \approx 4.605$.

Note that the time when $e^t = \frac{1}{10}$ is $t = -\ln 10$ and not $t = \frac{1}{\ln 10}$.

20 The most important curve in probability is the bell-shaped graph of $e^{-t^2/2}$. With a calculator or computer find this function at $t = -2, -1, 0, 1, 2$. Sketch the graph of $e^{-t^2/2}$ from $t = -\infty$ to $t = \infty$. *It never goes below zero.*

Solution At $t = 1$ and $t = -1$, we have $e^{-t^2/2} = e^{-1/2} = 1/\sqrt{e} \approx \mathbf{.606}$

At $t = 2$ and $t = -2$, we have $e^{-t^2/2} = e^{-2} \approx \mathbf{.13}$.

21 Explain why $y_1 = e^{(a+b+c)t}$ is the same as $y_2 = e^{at}e^{bt}e^{ct}$. They both start at $y(0) = 1$. They both solve what differential equation?

Solution The exponent rule is used twice to find $e^{(a+b+c)\,t} = e^{at+bt+ct} = e^{at+bt}e^{ct} = e^{at}e^{bt}e^{ct}$.

This function must solve $\boldsymbol{\frac{dy}{dt} = (a+b+c)y}$. The product rule confirms this.

22 For $y' = y$ with $a = 1$, Euler's first step chooses $Y_1 = (1+\Delta t)Y_0$. Backward Euler chooses $Y_1 = Y_0/(1-\Delta t)$. Explain why $1+\Delta t$ is smaller than the exact $e^{\Delta t}$ and $1/(1-\Delta t)$ is larger than $e^{\Delta t}$. (Compare the series for $1/(1-x)$ with e^x.)

Solution $1+\Delta t$ is certainly smaller than $e^{\Delta t} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \cdots$

$\frac{1}{1-\Delta t} = 1 + \Delta t + (\Delta t)^2 + (\Delta t)^3 + \cdots$ is larger than $e^{\Delta t}$, because the coefficients drop below 1 in $e^{\Delta t}$.

Problem Set 1.4, page 27

1 All solutions to $dy/dt = -y + 2$ approach the steady state where dy/dt is zero and $y = y_\infty =$ ___. That constant $y = y_\infty$ is a particular solution y_p.

Which $y_n = Ce^{-t}$ combines with this steady state y_p to start from $y(0) = 4$? This question chose $y_p + y_n$ to be $y_\infty +$ *transient* (decaying to zero).

Solution $y_\infty = 2 = y_p$ at the steady state when $\frac{dy}{dt} = 0$. Then $y_n = 2e^{-t}$ gives $y = y_n + y_p = 2 + 2e^{-t} = 4$ at $t = 0$.

2 For the same equation $dy/dt = -y + 2$, choose the null solution y_n that starts from $y(0) = 4$. Find the particular solution y_p that starts from $y(0) = 0$.
This splitting chooses y_n and y_p as $e^{at}y(0)$ + integral of $e^{a(t-T)}q$ in equation (4).

Solution For the same equation as 11.4.1, $y_n = 4e^{-t}$ has the correct $y(0) = 4$. Now y_p must be $2 - 2e^{-t}$ to start at $y_p(0) = 0$. Of course $y_n + y_p$ is still $2 + 2e^{-t}$.

3 The equation $dy/dt = -2y + 8$ also has two natural splittings $\boldsymbol{y_S + y_T = y_N + y_P}$:

1. Steady ($\boldsymbol{y_S = y_\infty}$) + Transient ($\boldsymbol{y_T \to 0}$). What are those parts if $y(0) = 6$?

2. ($y_N' = -2y_N$ from $\boldsymbol{y_N(0) = 6}$) + ($y_P' = -2y_P + 8$ starting from $\boldsymbol{y_P(0) = 0}$).

Solution **1.** $y_S = 4$ (when $\frac{dy}{dt} = 0$: steady state) and $y_T = 2e^{-2t}$.

2. $y_N = 6e^{-2t}$ and $y_P = 4 - 4e^{-2t}$ starts at $y_P(0) = 0$.

Again $y_S + y_T = y_N + y_P$: two splittings of y.

4 All null solutions to $u - 2v = 0$ have the form $(u, v) = (c,$ _____$)$.

One particular solution to $u - 2v = 3$ has the form $(u, v) = (7,$ _____$)$.

Every solution to $u - 2v = 3$ has the form $(7,$ _____$) + c(1,$ _____$)$.

But also every solution has the form $(3,$ _____$) + C(1,$ _____$)$ for $C = c + 4$.

Solution All null solutions to $u - 2v = 0$ have the form $(u, v) = \boldsymbol{(c, \frac{1}{2}c)}$.

One particular solution to $u - 2v = 3$ has the form $(u, v) = \boldsymbol{(7, 2)}$.

Every solution to $u - 2v = 3$ has the form $\boldsymbol{(7, 2) + c(1, \frac{1}{2})}$.

But also every solution has the form $\boldsymbol{(3, 0) + C(1, \frac{1}{2})}$. Here $C = c + 4$.

5 The equation $dy/dt = 5$ with $y(0) = 2$ is solved by $y =$ _____. A natural splitting $y_n(t) =$ ___ and $y_p(t) =$ ___ comes from $y_n = e^{at}y(0)$ and $y_p = \int e^{a(t-T)} 5\, dT$. This small example has $\boldsymbol{a = 0}$ (so ay is absent) and $\boldsymbol{c = 0}$ (the source is $q = 5e^{0t}$). When $a = c$ we have "resonance." A factor t will appear in the solution y.

Solution $dy/dt = 5$ with $y(0) = 2$ is solved by $y = 2+5t$. A natural splitting $y_n(t) = 2$ and $y_p(t) = 5t$ comes from $y_n(0) = y(0)$ and $y_p = \int e^{a(t-s)} 5ds = 5t$ (since $a = 0$).

Starting with Problem 6, choose the very particular $\boldsymbol{y_p}$ that starts from $\boldsymbol{y_p(0) = 0}$.

6 For these equations starting at $y(0) = 1$, find $y_n(t)$ and $y_p(t)$ and $y(t) = y_n + y_p$. (a) $y' - 9y = 90$ (b) $y' + 9y = 90$

Solution (a) Since the forcing function is a we use equation 6:

$$y_n(t) = e^{9t}$$
$$y_p(t) = \tfrac{90}{9}(e^{9t} - 1) = 10(e^{9t} - 1)$$
$$y(t) = y_n(t) + y_p(t) = e^{9t} + 10(e^{9t} - 1) = 11e^{9t} - 10.$$

(b) We again use equation 6, noting that $a = -9$. The steady state will be $y_\infty = 10$.

$$y_n(t) = e^{-9t}$$
$$y_p(t) = \tfrac{90}{-9}(e^{-9t} - 1)$$
$$y(t) = y_n(t) + y_p(t) = e^{-9t} - 10(e^{-9t} - 1) = 10 - 9e^{-9t}.$$

7 Find a linear differential equation that produces $y_n(t) = e^{2t}$ and $y_p(t) = 5(e^{8t} - 1)$.

Solution $y_n = e^{2t}$ needs $a = 2$. Then $y_p = 5(e^{8t} - 1)$ starts from $y_p(0) = 0$, telling us that $y(0) = y_n(0) = 1$. This y_p is a response to the forcing term $(e^{8t} + 1)$. So the equation for $y = e^{2t} + 5e^{8t} - 5$ must be $\frac{dy}{dt} = 2y + (e^{8t} + 1)$. Substitute y:

$$2e^{2t} + 40e^{8t} = 2e^{2t} + 10e^{8t} - 10 + (e^{8t} + 1).$$

Comparing the two sides, $C = 30$ and $D = 10$. Harder than expected.

8 Find a resonant equation ($a = c$) that produces $y_n(t) = e^{2t}$ and $y_p(t) = 3te^{2t}$.

Solution Clearly $a = c = 2$. The equation must be $dy/dt = 2y + Be^{2t}$. Substituting $y = e^{2t} + 3te^{2t}$ gives $2e^{2t} + 3e^{2t} + 6te^{2t} = 2(e^{2t} + 3te^{2t}) + Be^{2t}$ and then $B = \mathbf{3}$.

9 $y' = 3y + e^{3t}$ has $y_n = e^{3t}y(0)$. Find the resonant y_p with $y_p(0) = 0$.

Solution The resonant y_p has the form Cte^{3t} starting from $y_p(0) = 0$. Substitute in the equation:

$$\frac{dy}{dt} = 3y + e^{3t} \text{ is } Ce^{3t} + 3Cte^{3t} = 3Cte^{3t} + e^{3t} \text{ and then } C = \mathbf{1}.$$

Problems 10–13 are about $\boldsymbol{y' - ay =}$ constant source $\boldsymbol{q}$.

10 Solve these linear equations in the form $y = y_n + y_p$ with $y_n = y(0)e^{at}$.

(a) $y' - 4y = -8$ (b) $y' + 4y = 8$ Which one has a steady state?

Solution (a) $y' - 4y = -8$ has $a = 4$ and $y_p = 2$. But 2 is not a steady state at $t = \infty$ because the solution $y_n = y(0)e^{4t}$ is exploding.

(b) $y' + 4y = 8$ has $a = -4$ and again $y_p = 2$. This 2 is a steady state because $a < 0$ and $y_n \to 0$.

11 Find a formula for $y(t)$ with $y(0) = 1$ and draw its graph. What is y_∞ ?

(a) $y' + 2y = 6$ (b) $y' + 2y = -6$

Solution (a) $y' + 2y = 6$ has $a = -2$ and $y_\infty = 3$ and $y = y(0)e^{-2t} + 3$.

(b) $y' + 2y = -6$ has $a = -2$ and $y_\infty = -3$ and $y = y(0)e^{-2t} - 3$.

12 Write the equations in Problem 11 as $Y' = -2Y$ with $Y = y - y_\infty$. What is $Y(0)$?

Solution With $Y = y - y_\infty$ and $Y(0) = y(0) - y_\infty$, the equations in 1.4.11 are $Y' = -2Y$. (The solutions are $Y(t) = Y(0)e^{-2t}$ which is $y(t) - y_\infty = (y(0) - y_\infty)e^{-2t}$ or $y(t) = y(0)e^{-2t} + y_\infty(1 - e^{-2t})$.

13 If a drip feeds $q = 0.3$ grams per minute into your arm, and your body eliminates the drug at the rate $6y$ grams per minute, what is the steady state concentration y_∞ ? Then *in* = *out* and y_∞ is constant. Write a differential equation for $Y = y - y_\infty$.

Solution The steady state has $y_{\text{in}} = y_{\text{out}}$ or $0.3 = 6y_\infty$ or $y_\infty = 0.05$. The equation for $Y = y - y_\infty$ is $Y' = aY = -6Y$. The solution is $Y(t) = Y(0)e^{-6t}$ or $y(t) = y_\infty + (y(0) - y_\infty)e^{-6t}$.

Problems 14–18 are about $y' - ay =$ step function $H(t - T)$:

14 Why is y_∞ the same for $y' + y = H(t - 2)$ and $y' + y = H(t - 10)$?

Solution Notice $a = -1$. The steady states are the same because the step functions $H(t - 2)$ and $H(t - 10)$ are the same after time $t = 10$.

15 Draw the ramp function that solves $y' = H(t - T)$ with $y(0) = 2$.

Solution The solution is a ramp with $y(t) = y(0) = 2$ up to time T and then $y(t) = 2 + t - T$ beyond time T.

16 Find $y_n(t)$ and $y_p(t)$ as in equation (10), with step function inputs starting at $T = 4$.

(a) $y' - 5y = 3H(t - 4)$ (b) $y' + y = 7H(t - 4)$ (*What is* y_∞ ?)

Solution (a) $y_p(t) = \frac{3}{5}(e^{5(t-4)} - 1)$ for $t \geq 4$ with no steady state.

(b) $y_p(t) = \frac{7}{-1}(e^{-(t-4)} - 1)$ for $t \geq 4$ with $a = -1$ and $y_\infty = 7$.

17 Suppose the step function turns on at $T = 4$ and off at $T = 6$. Then $q(t) = H(t - 4) - H(t - 6)$. Starting from $y(0) = 0$, solve $y' + 2y = q(t)$. What is y_∞ ?

Solution The solution has 3 parts. First $y(t) = y(0) = 0$ up to $t = 4$. Then $H(t - 4)$ turns on and $y(t) = \frac{1}{-2}(e^{-2(t-4)} - 1)$. This reaches $y(6) = -\frac{1}{2}(e^{-4} - 1)$ at time $t = 6$. After $t = 6$, the source is turned off and the solution decays to zero: $y(t) = y(6)e^{-2(t-6)}$.

Method 2: We use the same steps as in equations (8) - (10), noting that $y(0) = 0$.

$$(e^{2t}y)' = e^{2t}H(t - 4) - e^{2t}H(t - 6)$$

$$e^{2t}y(t) - e^{2t}y(0) = \int_4^t e^{2x}\,dx - \int_6^t e^{2x}\,dx$$

$$e^{2t}y(t) = -\tfrac{1}{2}(e^{2\cdot 4} - e^{2t})H(t - 4) + \tfrac{1}{2}(e^{2\cdot 6 - e^{2t}})H(t - 6)$$

$$y(t) = -\tfrac{1}{2}(e^{8-2t} - 1)H(t - 4) + \tfrac{1}{2}(e^{12-2t} - 1)H(t - 6)$$

.

For $t \to \infty$, we have:

$$y_\infty = \frac{1}{2}(e^{8-2\cdot\infty} - 1)H(t-4) + \frac{1}{2}(e^{12-2\cdot\infty} - 1)H(t-6) = \mathbf{0}.$$

18 Suppose $y' = H(t-1) + H(t-2) + H(t-3)$, starting at $y(0) = 0$. Find $y(t)$.

Solution We integrate both sides of the equation.

$$\begin{aligned} \int_0^t y'(t)dt &= \int_0^t (H(t-1) + H(t-2) + H(t-3))dt \\ y(t) - y(0) &= R(t-1) + R(t-2) + R(t-3) \\ y(t) &= R(t-1) + R(t-2) + R(t-3) \end{aligned}$$

$R(t)$ is the unit ramp function $= \max(0, t)$.

Problems 19–25 are about delta functions and solutions to $y' - ay = q\,\delta(t-T)$.

19 For all $t > 0$ find these integrals $a(t), b(t), c(t)$ of point sources and graph $b(t)$:

(a) $\displaystyle\int_0^t \delta(T-2)\,dT$ (b) $\displaystyle\int_0^t (\delta(T-2) - \delta(T-3))\,dT$ (c) $\displaystyle\int_0^t \delta(T-2)\delta(T-3)dT$

Solution For $t < 2$, the spike in $\delta(t-2)$ does not appear in the integral from 0 to t:

$$\text{(a)} \quad \int_0^t \delta(T-2)dT = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

The integral (b) equals $\mathbf{1}$ **for** $\mathbf{2 \leq t < 3}$. This is the difference $H(t-2) - H(t-3)$. The integral (c) is **zero** because $\delta(T-2)\delta(T-3)$ is everywhere zero.

20 Why are these answers reasonable? (They are all correct.)

(a) $\displaystyle\int_{-\infty}^{\infty} e^t\delta(t)dt = 1$ (b) $\displaystyle\int_{-\infty}^{\infty} (\delta(t))^2 dt = \infty$ (c) $\displaystyle\int_{-\infty}^{\infty} e^T\delta(t-T)dT = e^t$

Solution (a) The difference $e^t\delta(t) - \delta(t)$ is everywhere zero (notice it is zero at $t = 0$). So $e^t\delta(t)$ and $\delta(t)$ have the same integral (from $-\infty$ to ∞ that integral is 1). This reasoning can be made more precise.

(b) This is the difference between the step functions $H(t-2)$ and $H(t-3)$. So it equals 1 for $2 \leq t \leq 3$ and otherwise zero.

(c) As in part (a), the difference between $e^T\delta(t-T)$ and $e^t\delta(t-T)$ is zero at $t = T$ (and also zero at every other t). So

$$\int_{-\infty}^{\infty} e^T\delta(t-T)dT = e^t \int_{-\infty}^{\infty} \delta(t-T)dT = e^t.$$

21 *The solution to* $y' = 2y + \delta(t-3)$ *jumps up by* 1 *at* $t = 3$. Before and after $t = 3$, the delta function is zero and y grows like e^{2t}. Draw the graph of $y(t)$ when (a) $y(0) = 0$ and (b) $y(0) = 1$. Write formulas for $y(t)$ before and after $t = 3$.

Solution (a) $y(0) = 0$ gives $y(t) = 0$ until $t = 3$. Then $y(3) = 1$ from the jump. After the jump we are solving $y' = 2y$ and y grows exponentially from $y(3) = 1$. So $y(t) = \boldsymbol{e^{2(t-3)}}$.

(b) $y(0) = 1$ gives $y(t) = e^{2t}$ until $t = 3$. The jump produces $y(3) = e^6 + 1$. Then exponential growth gives $y(t) = e^{2(t-3)}(e^6 + 1) = \boldsymbol{e^{2t} + e^{2(t-3)}}$. One part grows from $t = 0$, one part grows from $t = 3$ as before.

22 Solve these differential equations starting at $y(0) = 2$:

(a) $y' - y = \delta(t-2)$ (b) $y' + y = \delta(t-2)$. (*What is* y_∞ ?)

Solution (a) $y' - y = \delta(t-2)$ starts with $y(t) = y(0)e^t = 2e^t$ up to the jump at $t = 2$. The jump brings another term into $y(t) = \boldsymbol{2e^t + e^{t-2}}$ for $t \geq 2$. Note the jump of $e^{t-2} = 1$ at $t = 2$.

(b) $y' + y = \delta(t-2)$ starts with $y(t) = y(0)e^{-t} = 2e^{-t}$ up to $t = 2$. The jump of 1 at $t = 2$ starts another exponential $e^{-(t-2)}$ (decaying because $a = -1$). Then $y(t) = \boldsymbol{2e^{-t} + e^{-(t-2)}}$.

23 Solve $dy/dt = H(t-1) + \delta(t-1)$ starting from $y(0) = 0$: jump and ramp.

Solution Nothing happens and $y(t) = 0$ until $t = 1$. Then $H(t-1)$ starts a ramp in $y(t)$ and there is a jump from $\delta(t-1)$. So $y(t) = \textbf{ramp} + \textbf{constant} = \max(0, t-1) + 1$.

24 (My small favorite) What is the steady state y_∞ for $y' = -y + \delta(t-1) + H(t-3)$?

Solution $dy/dt = 0$ at the steady state y_{ss}. Then $-y + \delta(t-1) + H(t-3)$ is $-y_\infty + 0 + 1$ and $\boldsymbol{y_\infty = 1}$.

25 Which q and $y(0)$ in $y' - 3y = q(t)$ produce the step solution $y(t) = H(t-1)$?

Solution We simply substitute the particular solution $y(t) = H(t-1)$ into the original differential equation with $y(0) = 0)$:

$$\delta(t-1) - 3H(t-1) = q(t)$$

Notice how $\delta(t-1)$ in $q(t)$ produces the jump $H(t-1)$ in y, and then $-3H(t-1)$ in $q(t)$ cancels the $-3y$ and keeps $dy/dt = 0$ after $t = 1$.

Problems 26–31 are about exponential sources $\boldsymbol{q(t) = Qe^{ct}}$ and resonance.

26 Solve these equations $y' - ay = Qe^{ct}$ as in (19), starting from y(0) = 2:
(a) $y' - y = 8e^{3t}$ (b) $y' + y = 8e^{-3t}$ (*What is* y_∞ ?)

Solution

(a) $a = \mathbf{1}, c = \mathbf{3}$ and $y(0) = \mathbf{2}$

$$y(t) = y(0)e^{at} + 8\frac{e^{ct} - e^{at}}{c-a}$$
$$y(t) = 2e^t + 8\frac{e^{3t} - e^t}{3-1}$$
$$y(t) = 2e^t + 4(e^{3t} - e^t)$$
$$y(t) = 4e^{3t} - 2e^t$$

y goes to ∞ as $t \to \infty$

(b) $a = \mathbf{-1}, c = \mathbf{-3}$ and $y(0) = \mathbf{2}$

$$y(t) = y(0)e^{at} + 8\frac{e^{-3t} - e^{-t}}{c-a}$$
$$y(t) = 2e^{-t} + 8\frac{e^{-3t} - e^{-t}}{-3-(-1)}$$
$$y(t) = 2e^{-t} - 4(e^{-3t} - e^{-t})$$
$$y(t) = -4e^{-3t} + 2e^{-t}$$

y goes to 0 as $t \to \infty$

.

27 When $c = 2.01$ is very close to $a = 2$, solve $y' - 2y = e^{ct}$ starting from $y(0) = 1$. By hand or by computer, draw the graph of $y(t)$: near resonance.

Solution We substitute the values $a = 2, c = 2.01$ and $y(0) = 1$ into equation (18):

$$y(t) = y(0)e^{at} + \frac{e^{ct} - e^{at}}{c - a}$$
$$y(t) = 2e^{at} + \frac{e^{2t} - e^{2.01t}}{2.01 - 2}$$
$$y(t) = 2e^{2t} + 100(e^{2t} - e^{2.01t})$$
$$y(t) = 101e^{2t} - 100e^{2.01t}$$

The graph of this function shows the "near resonance" when $c \approx a$.

28 When $c = 2$ is exactly equal to $a = 2$, solve $y' - 2y = e^{2t}$ starting from $y(0) = 1$. This is resonance as in equation (20). By hand or computer, draw the graph of $y(t)$.

Solution We substitute $a = 2, c = 2$ (resonance) and $y(0) = 1$ into equation (19):

$$y(t) = y(0)e^{at} + te^{at} = e^{2t} + te^{2t}.$$

29 Solve $y' + 4y = 8e^{-4t} + 20$ starting from $y(0) = 0$. What is y_∞ ?

Solution We have $a = -4, c = -4$ and $y(0) = 0$. Equation (19) with resonance leads to $8te^{-4t}$. The constant source 20 leads to $20(e^{-4t} - 1)$. By linearity $y(t) = 8te^{-4t} + 20(e^{-4t} - 1)$. The steady state is $y_\infty = -20$.

30 The solution to $y' - ay = e^{ct}$ didn't come from the main formula (4), but it could. Integrate $e^{-as}e^{cs}$ in (4) to reach the very particular solution $(e^{ct} - e^{at})/(c - a)$.

Solution

$$\begin{aligned}
y(t) &= e^{at}y(0) + e^{at}\int_0^t e^{-aT}q(T)dT \\
&= e^{at}y(0) + e^{at}\int_0^t e^{-aT}e^{cT}dT \\
&= e^{at}y(0) + e^{at}\int_0^t e^{(c-a)T}dT \\
&= e^{at}y(0) + e^{at}\left(\frac{e^{(c-a)t} - e^0}{c - a}\right) \\
&= e^{at}y(0) + \frac{e^{ct} - e^{at}}{c - a} = y_n + y_{vp}
\end{aligned}$$

.

31 *The easiest possible equation $y' = 1$ has resonance*! The solution $y = t$ shows the factor t. What number is the growth rate a and also the exponent c in the source ?

Solution The growth rate in $y' = 1$ or $dy/dt = e^{0t}$ is $\boldsymbol{a = 0}$. The source is e^{ct} with $\boldsymbol{c = 0}$. **Resonance** $\boldsymbol{a = c}$. The resonant solution $y(t) = te^{at}$ is $\boldsymbol{y = t}$, certainly correct for the equation $dy/dt = 1$.

32 Suppose you know two solutions y_1 and y_2 to the equation $y' - a(t)y = q(t)$.

(a) Find a null solution to $y' - a(t)y = 0$.
(b) Find all null solutions y_n. Find all particular solutions y_p.

Solution (a) $y = y_1 - y_2$ will be a null solution by linearity.

(b) $y = C(y_1 - y_2)$ will give all null solutions. Then $y = C(y_1 - y_2) + y_1$ will give all particular solutions. (Also $y = c(y_1 - y_2) + y_2$ will also give all particular solutions.)

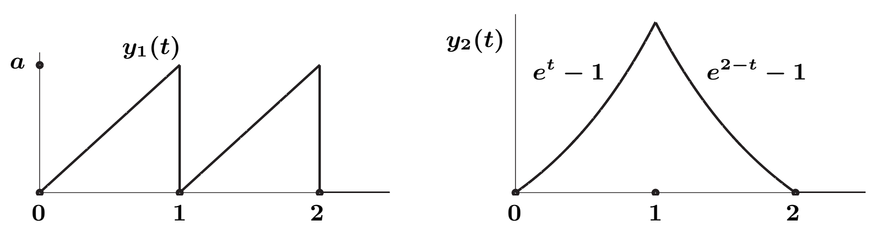
33 Turn back to the first page of this Section 1.4. Without looking, can you write down a solution to $y' - ay = q(t)$ for all four source functions $\boldsymbol{q}, \boldsymbol{H(t)}, \boldsymbol{\delta(t)}, \boldsymbol{e^{ct}}$?

Solution Equations (5), (7), (14), (19).

34 Three of those sources in Problem 33 are actually the same, if you choose the right values for q and c and $y(0)$. What are those values ?

Solution The sources $q = 1$ and $q = H(t)$ and $q = e^{0t}$ are all the same for $t \geq 0$.

35 What differential equations $y' = ay + q(t)$ would be solved by $y_1(t)$ and $y_2(t)$? Jumps, ramps, corners—maybe harder than expected (`math.mit.edu/dela/Pset1.4`).



Solution (a) $\dfrac{dy_1}{dt} = 1 - \delta(t-1) - \delta(t-2)$ with $a = 0$.

(b) $\dfrac{dy_2}{dt} = y_2 + 1$ up to $t = 1$. Add in $-2e\,\delta(t-1)$ to drop the slope from e to $-e$ at $t = 1$. After $t = 1$ we need $dy_2/dt = -y_2 - 1$ to keep $y_2 = e^{2-t} - 1$.

Problem Set 1.5, page 37

Problems 1-6 are about the sinusoidal identity (9). It is stated again in Problem 1.

1 These steps lead again to the sinusoidal identity. This approach doesn't start with the usual formula $\cos(\omega t - \phi) = \cos\omega t \ \cos\phi + \sin\omega t \ \sin\phi$ from trigonometry. The identity says:

$$\textbf{If } \boldsymbol{A + iB = Re^{i\phi}} \textbf{ then } \boldsymbol{A} \cos \boldsymbol{\omega t} + \boldsymbol{B} \sin \boldsymbol{\omega t} = \boldsymbol{R} \cos(\boldsymbol{\omega t - \phi}).$$

Here are the four steps to find that real part of $Re^{i(\omega t-\phi)}$. Explain Step 3 where $R\,e^{-i\phi}$ equals $A - iB$:

$$\boldsymbol{R}\ \cos(\boldsymbol{\omega t - \phi}) = \text{Re}\left[R\,e^{i(\omega t-\phi)}\right] = \text{Re}\left[e^{i\omega t}(R\,e^{-i\phi})\right] = \ (\textit{what is } R\,e^{-i\phi}\,?)$$
$$= \text{Re}\left[(\cos\omega t + i\ \sin\omega t)\ (A - iB)\right] = \boldsymbol{A} \cos \boldsymbol{\omega t} + \boldsymbol{B} \sin \boldsymbol{\omega t}.$$

Solution The key point is that if $A + iB = Re^{i\phi}$ then $A - iB = Re^{-i\phi}$ (the complex conjugate).

2 To express $\sin 5t + \cos 5t$ as $R\ \cos(\omega t - \phi)$, what are R and ϕ?

Solution The sinusoidal identity has $A = 1, B = 1,$ and $\omega = 5$. Therefore:
$R^2 = A^2+B^2 = 2 \to R = \sqrt{2}$ and $\tan\phi = \dfrac{1}{1} \to \phi = \dfrac{\pi}{4}$. Answer $\sqrt{2}\cos\left(5t - \dfrac{\pi}{4}\right)$.

3 To express $6\ \cos 2t + 8\ \sin 2t$ as $R\ \cos(2t - \phi)$, what are R and $\tan\phi$ and ϕ?

Solution Use the Sinusoidal Identity with $A = 6, B = 8$ and $\omega = 2$.

$$R^2 = A^2 + B^2 = 6^2 + 8^2 = 100 \text{ and } R = 10$$

$$\tan\phi = \tfrac{B}{A} = \tfrac{8}{6} = \tfrac{4}{3} \text{ and } \phi \text{ is in the positive quadrant } 0 \textit{ to } \tfrac{\pi}{2} \left(\textit{not } \pi \textit{ to } \tfrac{3\pi}{2}\right)$$

$$6\cos(2t) + 8\sin(2t) = 10\cos\left(2t - \arctan\left(\frac{4}{3}\right)\right)$$

4 Integrate $\cos\omega t$ to find $(\sin\omega t)/\omega$ in this complex way.

(i) $dy_{\text{real}}/dt = \cos\omega t$ is the real part of $dy_{\text{complex}}/dt = e^{i\omega t}$.

(ii) Take the real part of the complex solution.

Solution (i) The complex equation $y' = e^{i\omega t}$ leads to $y = \dfrac{e^{i\omega t}}{i\omega}$.

(ii) Take the real part of that solution (since the real part of the right side is $\cos\omega t$).

$$\text{Re}\frac{e^{i\omega t}}{i\omega} = \text{Re}\left[\frac{\cos\omega t}{i\omega} + \frac{\sin\omega t}{\omega}\right] = \frac{\sin\omega t}{\omega}.$$

5 The sinusoidal identity for $A = 0$ and $B = -1$ says that $-\sin\omega t = R\cos(\omega t - \phi)$. Find R and ϕ.

Solution $R^2 = A^2 + B^2 = 0^2 + 1^2 = 1 \to R = 1$

$\tan\phi = \frac{1}{0} = \infty \to \phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$: Here it is $\frac{3\pi}{2}$, since $A + iB = -i$

Therefore we have

SOLUTION: $-\sin\omega t = \cos(\omega t - \frac{3\pi}{2})$

CHECK: $t = 0$ gives $0 = 0, \omega t = \frac{\pi}{2}$ gives $-1 = -1$.

6 Why is the sinusoidal identity useless for the source $q(t) = \cos\, t + \sin\, 2t$?

Solution The sinusoidal identity needs the same ω in all terms. But the first term has $\omega = 1$ while the second term has $\omega = 2$.

7 Write $2+3i$ as $re^{i\phi}$, so that $\frac{1}{2+3i} = \frac{1}{r}e^{-i\phi}$. Then write $y = e^{i\omega t}/(2+3i)$ in polar form. Then find the real and imaginary parts of y. And also find those real and imaginary parts directly from $(2-3i)e^{i\omega t}/(2-3i)(2+3i)$.

Solution

$$r = \sqrt{2^2+3^2} = \sqrt{13} \text{ and } \phi = \arctan(3/2)$$

$$2+3i = \sqrt{13}\, e^{i\,\arctan(3/2)}$$

$$y = e^{i\omega t}/(2+3i) = \sqrt{13}\, e^{i\,\arctan(3/2)+i\omega t}$$

Writing this in cartesian (rectangular) form gives

$$\text{real part} = \sqrt{13}\cos(\arctan(3/2)+\omega t) = 2\cos(\omega t) - 3\sin(\omega t)$$

$$\text{imag part} = \sqrt{13}\sin(\arctan(3/2)+\omega t) = 3\cos(\omega t) + 2\sin(\omega t)$$

We can also find the real and imaginary parts from:

$$\frac{(2-3i)e^{i\omega t}}{(2-3i)(2+3i)} = \frac{2-3i}{13}e^{i\omega t} = \frac{2-3i}{13}(\cos(\omega t) + i\sin(\omega t)).$$

8 Write these functions $A\cos\omega t + B\sin\omega t$ in the form $R\cos(\omega t - \phi)$: Right triangle with sides A, B, R and angle ϕ.

(1) $\cos 3t - \sin 3t$ (2) $\sqrt{3}\cos\pi t - \sin\pi t$ (3) $3\cos(t-\phi) + 4\sin(t-\phi)$

Solution (1) $\cos 3t - \sin 3t = \sqrt{2}\cos(3t - \frac{7\pi}{4}) = \sqrt{2}\cos(3t + \frac{\pi}{4})$.

Check $t = 0 : 1 = \sqrt{2}\cos(-\frac{7\pi}{4}) = \sqrt{2}\cos(\frac{\pi}{4})$.

(2) $\sqrt{3}\cos\pi t - \sin\pi t = 2\cos(\pi t + \frac{\pi}{6})$.

Check: $(\sqrt{3})^2 + (-1)^2 = 2^2$ At $t = 0 : \sqrt{3} = 2\cos 30\,°$.

(3) $3\cos(t-\phi) + 4\sin(t-\phi) = 5\cos(t - \phi - \tan^{-1}\frac{4}{3})$.

Problems 9-15 solve real equations using the real formula (3) for M and N.

9 Solve $dy/dt = 2y + 3\cos t + 4\sin t$ after recognizing a and ω. Null solutions Ce^{2t}.

Solution $\frac{dy}{dt} = 2y + 3\cos t + 4\sin t = 2y + 5\cos(t-\phi)$ with $\tan\phi = \frac{4}{3}$.

Method 1: Look for $y = M\cos t + N\sin t$.

Method 2: Solve $\frac{dY}{dt} = 2Y + 5e^{i(t-\phi)}$ and then $y =$ real part of Y.

$Y = \frac{5}{i-2}e^{i(t-\phi)} = \frac{5}{5}(-i-2)e^{i(t-\phi)}$ and $y = -2\cos(t-\phi) + \sin(t-\phi)$.

10 Find a particular solution to $dy/dt = -y - \cos 2t$.

Solution Substitute $y = M\cos t + N\sin t$ into the equation to find M and N

$$-M\sin t + N\cos t = -M\cos t - N\sin t - \cos 2t$$

Match coefficients of $\cos t$ and $\sin t$ separately to find M and N.

$$N = -M - 1 \text{ and } -M = -N \text{ give } M = N = -\frac{1}{2}$$

Note: This is called the "method of undetermined coefficients" in Section 2.6.

11 What equation $y' - ay = A\cos\omega t + B\sin\omega t$ is solved by $y = 3\cos 2t + 4\sin 2t$?

Solution Clearly $\omega = 2$. Substitute y into the equation:

$$-6\sin 2t + 8\cos 2t - 3a\cos 2t - 4a\sin 2t = A\cos 2t + B\sin 2t.$$

Match separately the coefficients of $\cos 2t$ and $\sin 2t$:

$$A = 8 - 3a \text{ and } B = -6 - 4a$$

12 The particular solution to $y' = y + \cos t$ in Section 4 is $y_p = e^t \int e^{-s}\cos s\,ds$. Look this up or integrate by parts, from $s = 0$ to t. Compare this y_p to formula (3).

Solution That integral goes from 0 to t, and it leads to $y_p = \frac{1}{2}(\sin t - \cos t + e^t)$

If we use formula (3) with $a = 1, \omega = 1, A = 1, B = 0$ we get

$$M = -\frac{aA + \omega B}{\omega^2 + a^2} = \frac{-1}{2} \qquad N = \frac{\omega A - aB}{\omega^2 + a^2} = \frac{1}{2}$$

This solution $y = M\cos t + N\sin t = \dfrac{-\cos t + \sin t}{2}$ is a different particular solution (not starting from $y(0) = 0$). The difference is a null solution $\frac{1}{2}e^t$.

13 Find a solution $y = M\ \cos\ \omega t + N\ \sin\ \omega t$ to $y' - 4y = \cos\ 3t + \sin\ 3t$.

Solution Formula (3) with $a = 4, \omega = 3, A = B = 1$ gives

$$M = -\frac{4+3}{9+16} = -\frac{7}{25} \qquad N = \frac{3-4}{9+16} = -\frac{1}{25}.$$

14 Find the solution to $y' - ay = A\ \cos\ \omega t + B\ \sin\ \omega t$ **starting from $\boldsymbol{y(0) = 0}$**.

Solution One particular solution $M\cos\omega t + N\sin\omega t$ comes from formula (3). But this starts from $y_p(0) = M$. So subtract off the null solution $y_n = Me^{at}$ to get the very particular solution $y_{vp} = y_p - y_n$ that starts from $y_{vp}(0) = 0$.

15 If $a = 0$ show that M and N in equation (3) still solve $y' = A\ \cos\ \omega t + B\ \sin\ \omega t$.

Solution Formula (3) still applies with $a = 0$ and it gives

$$M = -\frac{\omega B}{\omega^2} \qquad N = \frac{\omega A}{\omega^2} \qquad y = -\frac{B}{\omega}\cos\omega t + \frac{A}{\omega}\sin\omega t.$$

This is the correct integral of $A\cos\omega t + B\sin\omega t$ in the differential equation.

Problems 16-20 solve the complex equation $y' - ay = Re^{i(\omega t - \phi)}$.

16 Write down complex solutions $y_p = Ye^{i\omega t}$ to these three equations:
(a) $y' - 3y = 5e^{2it}$ (b) $y' = Re^{i(\omega t-\phi)}$ (c) $y' = 2y - e^{it}$

Solution (a) $y' - 3y = 5e^{2it}$ has $i\omega Ye^{i\omega t} - 3Ye^{i\omega t} = 5e^{2it}$.
So $\omega = 2$ and $Y = \frac{5}{2i-3}$.

(b) $y' = Re^{i(\omega t-\phi)}$ has $i\omega Ye^{i\omega t} = Re^{i(\omega t-\phi)}$. So $Y = \frac{R}{i\omega}e^{-i\phi}$ and the solution is $y = Ye^{i\omega t} = \frac{R}{i\omega}e^{i(\omega t-\phi)}$.

(c) $y' = 2y - e^{it}$ has $\omega = 1$ and $iYe^{it} = 2Ye^{it} - e^{-it}$.
Then $Y = \frac{-1}{i-2} = \frac{1}{2-i} = \frac{2+i}{5}$ and $y = Ye^{it}$.

17 Find complex solutions $z_p = Ze^{i\omega t}$ to these complex equations:
(a) $z' + 4z = e^{8it}$ (b) $z' + 4iz = e^{8it}$ (c) $z' + 4iz = e^{8t}$

Solution (a) $z' + 4z = e^{8it}$ has $z = Ze^{8it}$ with $8iZ + 4Z = 1$ and $Z = \frac{1}{4+8i} = \frac{4-8i}{16+64} = \frac{1}{20}(1-2i)$.

(b) $z' + 4iz = e^{8it}$ is like part (a) but 4 changes to $4i$. Then $Z = \frac{1}{4i+8i} = \frac{1}{12i} = -\frac{i}{12}$.

(c) $z' + 4iz = e^{8t}$ has $z = Ze^{8t}$. Then $8Ze^{8t} + 4iZe^{8t}$ gives $Z = \frac{1}{8+4i} = \frac{8-4i}{8^2+4^2}$.

18 Start with the real equation $y' - ay = R\cos(\omega t - \phi)$. Change to the complex equation $z' - az = Re^{i(\omega t-\phi)}$. Solve for $z(t)$. Then take its real part $y_p = \text{Re}\, z$.

Solution Put $z = Ze^{i(\omega t-\phi)}$ in the complex equation to find Z:

$$i\omega Z - aZ = R \text{ gives } Z = \frac{R}{-a+i\omega} = \frac{R(-a-i\omega)}{a^2+\omega^2}.$$

The real part of $z = Z(\cos(\omega t-\phi) + i\sin(\omega t-\phi))$ is
$\frac{R}{a^2+\omega^2}(-a\cos(\omega t-\phi) + \omega\sin(\omega t-\phi))$.

19 What is the initial value $y_p(0)$ of the particular solution y_p from Problem 18? If the desired initial value is $y(0)$, how much of the null solution $y_n = Ce^{at}$ would you add to y_p?

Solution That solution to 18 starts from $y_p(0) = \frac{R}{a^2+\omega^2}(-a\cos(-\phi) + \omega\sin(-\phi))$ at $t = 0$. So subtract that number times e^{at} to get the very particular solution that starts from $y_{vp}(0) = 0$.

20 Find the real solution to $y' - 2y = \cos\omega t$ starting from $y(0) = 0$, in three steps: Solve the complex equation $z' - 2z = e^{i\omega t}$, take $y_p = \text{Re}\, z$, and add the null solution $y_n = Ce^{2t}$ with the right C.

Solution Step 1. $z' - 2Z = e^{i\omega t}$ is solved by $z = Ze^{i\omega t}$ with $i\omega Z - 2Z = 1$ and $Z = \frac{1}{-2+i\omega} = \frac{-2-i\omega}{4+\omega^2}$.

Step 2. The real part of $Ze^{i\omega t}$ is $y_p = \frac{1}{4+\omega^2}(-2\cos\omega t + \omega\sin\omega t)$.

Step 3. $y_p(0) = \frac{-2}{4+\omega^2}$ so $y_{vp} = y_p + \frac{2}{4+\omega^2}e^{2t}$ includes the right $y_n = Ce^{2t}$ for $y_{vp}(0) = 0$.

Problems 21-27 solve real equations by making them complex. First a note on α.

Example 4 was $y' - y = \cos t - \sin t$, with growth rate $a = 1$ and frequency $\omega = 1$. The magnitude of $i\omega - a$ is $\sqrt{2}$ and the polar angle has $\tan\alpha = -\omega/a = -1$. Notice: *Both* $\alpha = 3\pi/4$ *and* $\alpha = -\pi/4$ *have that tangent*! How to choose the correct angle α?

The complex number $i\omega - a = i - 1$ is in the *second quadrant*. Its angle is $\alpha = 3\pi/4$. **We had to look at the actual number and not just the tangent of its angle**.

21 Find r and α to write each $i\omega - a$ as $re^{i\alpha}$. Then write $1/re^{i\alpha}$ as $Ge^{-i\alpha}$.
(a) $\sqrt{3}\,i + 1$ (b) $\sqrt{3}\,i - 1$ (c) $i - \sqrt{3}$

Solution (a) $\sqrt{3}i + 1$ is in the first quadrant (positive quarter $0 \le \theta \le \pi/2$) of the complex plane. The angle with tangent $\sqrt{3}/1$ is $60^\circ = \pi/3$. The magnitude of $\sqrt{3}i + 1$ is $r = 2$. Then $\sqrt{3}i + 1 = \boldsymbol{2e^{i\pi/3}}$.

(b) $\sqrt{3}i - 1$ is in the second quadrant $\pi/2 \le \theta \le \pi$. The tangent is $-\sqrt{3}$, the angle is $\theta = 2\pi/3$, the number is $\boldsymbol{2e^{2\pi i/3}}$.

(c) $i - \sqrt{3}$ is also in the second quadrant (left from zero and up). Now the tangent is $-1/\sqrt{3}$, the angle is $\theta = 150^\circ = 5\pi/6$. The magnitude is still 2, the number is $2e^{5\pi i/6}$.

22 Use G and α from Problem 21 to solve (a)-(b)-(c). Then take the real part of each equation and the real part of each solution.
(a) $y' + y = e^{i\sqrt{3}t}$ (b) $y' - y = e^{i\sqrt{3}t}$ (c) $y' - \sqrt{3}y = e^{it}$

Solution (a) $y' + y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y + Y = 1$. Then $Y = \frac{1}{\sqrt{3}i+1} = \frac{1}{2}e^{-i\pi/3}$ from Problem 21(a). The real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - \pi/3)$ of $Ye^{i\sqrt{3}t}$ solves the real equation $y'_{\text{real}} + y_{\text{real}} = \cos(\sqrt{3}t)$.

(b) $y' - y = e^{i\sqrt{3}t}$ is solved by $y = Ye^{i\sqrt{3}t}$ when $i\sqrt{3}Y - Y = 1$. Then $Y = \frac{1}{2}e^{-2\pi i/3}$ from Problem 21(b). the real part $y_{\text{real}} = \frac{1}{2}\cos(\sqrt{3}t - 2\pi/3)$ solves the real equation $y'_{\text{real}} - y_{\text{real}} = \cos(\sqrt{3}t)$.

(c) $y' - \sqrt{3}y = e^{it}$ is solved by $y = Ye^{it}$ when $iY - \sqrt{3}Y = 1$. Then $Y = \frac{1}{2}e^{-5\pi i/6}$ from Problem 21(c). The real part $y_{\text{real}} = \frac{1}{2}\cos(t - 5\pi/6)$ of Ye^{it} solves $y_{\text{real}} - \sqrt{3}y_{\text{real}} = \cos t$.

23 Solve $y' - y = \cos\omega t + \sin\omega t$ in three steps: real to complex, solve complex, take real part. This is an important example.

Solution **Note: I intended to choose $\boldsymbol{\omega = 1}$.** Then $y' - y = \cos t + \sin t$ has the simple solution $y = -\sin t$. I will apply the 3 steps to this case and then to the harder problem for any ω.

(1) Find R and ϕ in the sinusoidal identity to write $\cos\omega t + \sin\omega t$ as the real part of $Re^{i(\omega t - \phi)}$. This is easy for any ω.

$$\left[\tan\phi = \frac{1}{1} \;\text{ so }\; \phi = \frac{\pi}{4}\right] \qquad \boldsymbol{\cos\omega t + \sin\omega t = \sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)}$$

(2) Solve $y' - y = e^{i\omega t}$ by $y = Ge^{-i\alpha}e^{i\omega t}$. Multiply by $Re^{-i\phi}$ to solve $z' - z = Re^{i(\omega t - \phi)}$.

$\omega = 1$ $\;y' - y = e^{it}$ has $y = Ye^{it}$ with $iY - Y = 1$. Then $Y = \frac{1}{i-1} = \frac{1}{\sqrt{2}}e^{3\pi i/4} = Ge^{-i\alpha}$.

$z = \left(\sqrt{2}e^{i(t-\pi/4)}\right)\left(\frac{1}{\sqrt{2}}e^{3\pi i/4}\right) = e^{it}e^{\pi i/2} = ie^{it}$. The real part of z is $\boldsymbol{y = -\sin t}$.

Any ω $\;y' - y = e^{i\omega t}$ leads to $i\omega Y - Y = 1$ and $Y = \dfrac{1}{i\omega - 1} = \dfrac{\mathbf{1}}{\boldsymbol{\sqrt{1+\omega^2}}}\boldsymbol{e^{-i\alpha}}$

with $\tan\alpha = \omega$. Then $z(t) = \left(\frac{1}{1+\omega^2}e^{-i\alpha}\right)\left(\sqrt{2}e^{i(\omega t-\pi/4)}\right)$.

(3) Take the real part $y(t) = \text{Re}\, z(t)$. Check that $y' - y = \cos\omega t + \sin\omega t$.

$y(t) = \text{Re}\, z(t) = \frac{\sqrt{2}}{1+\omega^2}\cos(\omega t - \alpha - \frac{\pi}{4})$. Now we need $\tan\alpha = \omega, \cos\alpha = \frac{1}{\sqrt{1+\omega^2}}$, $\sin\alpha = \frac{\omega}{\sqrt{1+\omega^2}}$. Finally $y = \frac{\sqrt{2}}{1+\omega^2}[\cos(\omega t - \frac{\pi}{4})\cos\alpha + \sin(\omega t - \frac{\pi}{4})\sin\alpha]$.

24 Solve $y' - \sqrt{3}y = \cos\, t + \sin\, t$ by the same three steps with $a = \sqrt{3}$ and $\omega = 1$.

Solution (1) $\cos t + \sin t = \sqrt{2}\cos(t - \frac{\pi}{4})$.

(2) $y = Ye^{it}$ with $iY - \sqrt{3}Y = 1$ and $Y = \frac{1}{i-\sqrt{3}} = \frac{1}{2}e^{-5\pi i/6}$ from 1.5.21(c).

Then $z(t) = (\sqrt{2}e^{i(t-\pi/4)})(\frac{1}{2}e^{-5\pi i/6})$.

(3) The real part of $z(t)$ is $y(t) = \frac{1}{\sqrt{2}}\cos(t - \frac{13\pi}{12})$.

25 (**Challenge**) Solve $y' - ay = A\cos\omega t + B\sin\omega t$ in two ways. First, find R and ϕ on the right side and G and α on the left. Show that the final real solution $RG\cos(\omega t - \phi - \alpha)$ agrees with $M\cos\omega t + N\sin\omega t$ in equation (3).

Solution The first way has $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$ from the sinusoidal identity. On the left side $1/(i\omega - a) = Ge^{-i\alpha}$ from equation (8) with $G = 1/\sqrt{\omega^2 + a^2}$ and $\tan\alpha = -\omega/a$. Combining, the real solution is $y = RG\cos(\omega t - \phi - \alpha)$.

This agrees with $y = M\cos\omega t + N\sin\omega t$ (equation (3) gives M and N).

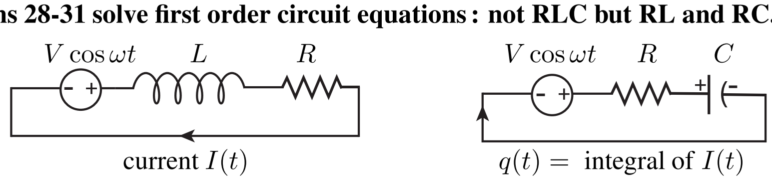
26 We don't have resonance for $y' - ay = Re^{i\omega t}$ when a and $\omega \neq 0$ are real. *Why not*? (Resonance appears when $y_n = Ce^{at}$ and $y_p = Ye^{ct}$ share the exponent $a = c$.)

Solution Resonance requires the exponents a and $i\omega$ to be equal. For real a this only happens if $a = \omega = 0$.

27 If you took the imaginary part $y = \text{Im}\, z$ of the complex solution to $z' - az = Re^{i(\omega t-\phi)}$, what equation would $y(t)$ solve? Answer first with $\phi = 0$.

Solution Assuming a is real, the imaginary part of $z' - az = Re^{i(\omega t-\phi)}$ is the equation $y' - ay = R\sin(\omega t - \phi)$. With $\phi = 0$ this is $y' - ay = R\sin\omega t$.

Problems 28-31 solve first order circuit equations: not RLC but RL and RC.



28 Solve $\boldsymbol{L\,dI/dt + RI(t) = V\ \cos\ \omega t}$ for the current $I(t) = I_n + I_p$ in the RL loop.

Solution Divide the equation by L to produce $dI/dt - aI = X\cos\omega t$ with $a = -R/L$ and $X = V/L$. In this standard form, equation (3) gives the real solution:

$$I = M\cos\omega t + N\sin\omega t \;\text{ with }\; M = -\frac{aX}{\omega^2 + a^2} \;\text{ and }\; N = \frac{\omega X}{\omega^2 + a^2}.$$

29 With $L = 0$ and $\omega = 0$, that equation is Ohm's Law $V = IR$ for direct current. The **complex impedance** $Z = R + i\omega L$ replaces R when $L \neq 0$ and $I(t) = Ie^{i\omega t}$.

$$L\,dI/dt + RI(t) = (\boldsymbol{i\omega L + R})\boldsymbol{Ie^{i\omega t}} = \boldsymbol{Ve^{i\omega t}} \quad \text{gives} \quad \boldsymbol{Z\,I = V}.$$

What is the magnitude $|Z| = |R + i\omega L|$? What is the phase angle in $Z = |Z|e^{i\theta}$? Is the current $|I|$ larger or smaller because of L?

Solution $|Z| = \sqrt{R^2 + \omega^2 L^2}$ and $\tan\theta = \frac{\omega L}{R}$.

Since $|Z|$ increases with L, the current $|I|$ must decrease.

30 Solve $\boldsymbol{R\dfrac{dq}{dt} + \dfrac{1}{C}q(t) = V\ \cos\ \omega t}$ for the charge $q(t) = q_n + q_p$ in the RC loop.

Solution Dividing by R produces $\frac{dq}{dt} - aq = X\cos\omega t$ with $a = -\frac{1}{RC}$ and $X = \frac{V}{R}$. As in Problem 28, equation (3) gives M and N from ω and this a.

31 Why is the complex impedance now $Z = R + \frac{1}{i\omega C}$? Find its magnitude $|Z|$. **Note that mathematics prefers $\boldsymbol{i = \sqrt{-1}}$, we are not conceding yet to $\boldsymbol{j = \sqrt{-1}}$!**

Solution The physical RC equation for the current $I = \frac{dq}{dt}$ is $RI + \frac{1}{C}\int I dt = V\cos\omega t = \text{Re}\,(Ve^{i\omega t})$.

The solution I has the same frequency factor $Xe^{i\omega t}$, and the integral has the factor $e^{i\omega t}/i\omega$. Substitute into the equation and match coefficients of $e^{i\omega t}$:

$RX + \frac{1}{i\omega C}X = V$ is $ZX = V$ with impedance $Z = R + \frac{1}{i\omega C}$.

Problem Set 1.6, page 50

1 Solve the equation $dy/dt = y + 1$ up to time t, starting from $y(0) = 4$.

Solution We use the formula $y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1)$ with $a = 1$ and $s = 1$ and $y(0) = 4$:

$$y(t) = 4e^t + e^t - 1 = 5e^t - 1$$

2 You have \$1000 to invest at rate $a = 1 = 100\,\%$. Compare after one year the result of depositing $y(0) = 1000$ immediately with no source ($s = 0$), or choosing $y(0) = 0$ and $s = 1000$/year to deposit continually during the year. In both cases $dy/dt = y + q$.

Solution We substitute the values for the different scenarios into the solution formula:

$$y(t) = 1000e^t \qquad = 1000e \;\text{ at one year}$$

$$y(t) = 1000e^t - 1000 = 1000(e - 1) \;\text{ at one year}$$

You get more for depositing immediately rather than during the year.

3 If $dy/dt = y - 1$, when does your original deposit $y(0) = \frac{1}{2}$ drop to zero?

Solution Again we use the equation $y(t) = y(0)e^{at} + \frac{s}{a}(e^{at} - 1)$ with $a = 1$ and $s = -1$. We set $y(t) = 0$ and find the time t:

$$y(t) = y(0)e^t - e^t + 1 = e^t(y(0) - 1) + 1 = 0$$

$$e^t = \frac{1}{1 - y(0)} = 2 \text{ and } t = \ln 2.$$

Notice! If $y(0) > 1$, the balance never drops to zero. Interest exceeds spending.

4 Solve $\dfrac{dy}{dt} = y + t^2$ from $y(0) = 1$ with increasing source term t^2.

Solution Solution formula (12) with $a = 1$ and $y(0) = 1$ gives

$$y(t) = e^t + \int_0^t e^{t-s}s^2 ds = e^t - t(t+2) + 2e^t - 2 = 3e^t - t(t+2) - 2$$

$$\text{Check: } \frac{dy}{dt} = 3e^t + 2t - 2 \text{ equals } y + t^2.$$

5 Solve $\dfrac{dy}{dt} = y + e^t$ (resonance $a = c$!) from $y(0) = 1$ with exponential source e^t.

Solution The solution formula with $a = 1$ and source e^t (resonance!) gives:

$$y(t) = e^t + \int_0^t e^{t-s}e^s ds = e^t + \int_0^t e^t ds = e^t(1 + t)$$

$$\text{Check by the product rule: } \frac{dy}{dt} = e^t(1+t) + e^t = y + e^t.$$

6 Solve $\dfrac{dy}{dt} = y - t^2$ from an initial deposit $y(0) = 1$. The spending $q(t) = -t^2$ is growing. When (if ever) does $y(t)$ drop to zero ?

Solution

$y(t) = e^t - \int_0^t e^{t-s}s^2 ds = e^t + t(t+2) - 2e^t + 2 = -e^t + t(t+2)$. This definitely drops to zero (I regret there is no nice formula for that time t).

$$\text{Check: } \frac{dy}{dt} = -e^t + 2t + 2 = y - t^2.$$

7 Solve $\dfrac{dy}{dt} = y - e^t$ from an initial deposit $y(0) = 1$. This spending term $-e^t$ grows at the same e^t rate as the initial deposit (resonance). When (if ever) does y drop to zero ?

Solution $y(t) = e^t - \int_0^t e^{t-s}e^s ds = e^t - \int_0^t e^t ds = e^t(1 - t)$ (**this is zero at $t = 1$**)

Check by the product rule: $\frac{dy}{dt} = e^t(1 - t) - e^t = y - e^t$.

8 Solve $\dfrac{dy}{dt} = y - e^{2t}$ from $y(0) = 1$. At what time T is $y(T) = 0$?

Solution $y(t) = e^t - \int_0^t e^{t-s} e^{2s} ds = e^t - \int_0^t e^{t+s} ds = e^t + e^t(1 - e^t) = 2e^t - e^{2t}$

This solution is zero when $2e^t = e^{2t}$ and $2 = e^t$ and $t = \ln 2$.

Check that $y = 2e^t - e^{2t}$ solves the equation: $\frac{dy}{dt} = 2e^t - 2e^{2t} = y - e^{2t}$.

9 Which solution (y or Y) is eventually larger if $y(0) = 0$ and $Y(0) = 0$?

$$\frac{dy}{dt} = y + 2t \quad \text{or} \quad \frac{dY}{dt} = 2Y + t.$$

Solution

$$\frac{dy}{dt} = y + 2t \qquad\qquad \frac{dY}{dt} = 2Y + t$$

$$y(t) = \int_0^t e^{t-s} \cdot 2s ds \qquad\qquad Y(t) = \int_0^t e^{2t-2s} \cdot s ds$$

$$y(t) = 2(-t + e^t - 1) \qquad\qquad Y(t) = \frac{e^{2t} - 1}{2}$$

In the long run $Y(t)$ is larger than $y(t)$, since the exponent $2t$ is larger than t.

10 Compare the linear equation $y' = y$ to the separable equation $y' = y^2$ starting from $y(0) = 1$. Which solution $y(t)$ must grow faster ? It grows so fast that it blows up to $y(T) = \infty$ at what time T ?

Solution

$$\frac{dy}{dt} = y \qquad\qquad \frac{dy}{dt} = y^2$$

$$\frac{dy}{y} = dt \qquad\qquad \frac{dy}{y^2} = dt$$

$$\int_{y(0)}^{y(t)} \frac{du}{u} = \int_0^t dt \qquad\qquad \int_{y(0)}^{y(t)} \frac{du}{u^2} = \int_0^t dt$$

$$\ln(y(t)) - \ln(y(0)) = t \qquad\qquad -\frac{1}{y(t)} + \frac{1}{y(0)} = t$$

$$\frac{y(t)}{y(0)} = e^t \qquad\qquad y(t) = \frac{1}{\frac{1}{y(0)} - t} = \mathbf{\frac{1}{1 - t}}$$

$$y(t) = y(0)e^t = \boldsymbol{e^t}$$

The second solution grows much faster, and reaches a vertical asymptote at $T = 1$.

11 $Y' = 2Y$ has a larger growth factor (because $a = 2$) than $y' = y + q(t)$. What source $q(t)$ would be needed to keep $y(t) = Y(t)$ for all time ?

Solution $\frac{dY}{dt} = 2Y + 1$ with for example $Y(0) = y(0) = 0$

$$Y(t) = \int_0^t e^{2t-2s} ds = \frac{e^{2t} - 1}{2}$$

Put this solution into $\frac{dy}{dt} = y + q(t)$:

$$e^{2t} = \frac{e^{2t}-1}{2} + q(t)$$
$$\frac{e^{2t}+1}{2} = q(t)$$

12 Starting from $y(0) = Y(0) = 1$, does $y(t)$ or $Y(t)$ eventually become larger ?

$$\frac{dy}{dt} = 2y + e^t \qquad \frac{dY}{dt} = Y + e^{2t}.$$

Solution $\frac{dy}{dt} = 2y + e^t$

$$y(t) = e^{2t} + \int_0^t e^{2t-2s}e^s ds = e^{2t} + e^{2t} - e^t = 2e^{2t} - e^t$$

Solving the second equation:

$$\frac{dY}{dt} = Y + e^{2t}$$

$$Y(t) = e^t + \int_0^t e^{t-s}e^{2s} ds = e^t + e^{2t} - e^t = e^{2t} \text{ is always smaller than } y(t).$$

Questions 13-18 are about the growth factor $G(s, t)$ from time s to time t.

13 What is the factor $G(s, s)$ in zero time ? Find $G(s, \infty)$ if $a = -1$ and if $a = 1$.

Solution The solution doesn't change in zero time so $G(s, s) = 1$. (Note that the integral of $a(t)$ from $t = s$ to $t = s$ is zero. Then $G(s, s) = e^0 = 1$. We are talking about change in the null solution, with $y' = a(t)y$. A source term with a delta function does produce instant change.)

If $a = -1$, the solution drops to zero at $t = \infty$. So $G(s, \infty) = 0$.

If $a = 1$, the solution grows infinitely large as $t \to \infty$. So $G(s, \infty) = \infty$.

14 Explain the important statement after equation (13): *The growth factor $G(s, t)$ is the solution to $y' = a(t)y + \delta(t - s)$.* The source $\delta(t - s)$ deposits $\$1$ at time s.

Solution When the source term $\delta(t - s)$ deposits $\$1$ at time s, that deposit will grow or decay to $y(t) = G(s, t)$ at time $t > s$. This is consistent with the main solution formula (13).

15 Now explain this meaning of $G(s, t)$ when t *is less than* s. We go backwards in time. *For $t < s$, $G(s, t)$ is the value at time t that will grow to equal 1 at time s.*

When $t = 0$, $G(s, 0)$ is the "present value" of a promise to pay $\$1$ at time s. If the interest rate is $a = 0.1 = 10\%$ per year, what is the present value $G(s, 0)$ of a million dollar inheritance promised in $s = 10$ years ?

Solution In fact $G(t, s) = 1/G(s, t)$. In the simplest case $y' = y$ of exponential growth, $G(s, t)$ is the growth factor e^{t-s} from s to t. Then $G(t, s)$ is $e^{s-t} = 1/e^{t-s}$.

That number $G(t, s)$ would be the "present value" at the earlier time t of a promise to pay $\$1$ at the later time s. You wouldn't need to deposit the full $\$1$ because your deposit will grow by the factor $G(s, t)$. All you need to have at the earlier time is $1/G(s, t)$, which then grows to 1.

16 (a) What is the growth factor $G(s,t)$ for the equation $y' = (\sin\, t)y + Q\,\sin\, t$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = (\sin\, t)y$ when $y(0) = 1$?

(c) What is the particular solution $y_p = \int_0^t G(s,t)\; Q\,\sin\, s\, ds$?

Solution (a) Growth factor: $G(s,t) = \exp\left(\int_s^t \sin T\, dT\right) = \exp(\cos s - \cos t)$.

(b) Null solution: $y_n = G(0,t)\, y(0) = e^{1-\cos t}$.

(c) Particular solution: $y_p = \int_0^t e^{\cos s - \cos t} Q \sin s\, ds$

$= Qe^{-\cos t}\left[-e^{\cos s}\right]_0^t = Q\left(e^{1-\cos t} - 1\right)$. Check $y_p(0) = Q(e^0 - 1) = 0$.

17 (a) What is the growth factor $G(s,t)$ for the equation $y' = y/(t+1) + 10$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = y/(t+1)$ with $y(0) = 1$?

(c) What is the particular solution $y_p = 10\int_0^t G(s,t)\, ds$?

Solution (a) $G(s,t) = \exp\left[\int_s^t \frac{dT}{T+1}\right] = \exp\left[\ln(t+1) - \ln(s+1)\right] = \frac{t+1}{s+1}$.

Null solution $y_n = G(0,t)\, y(0) = \exp\left[\boldsymbol{\ln(t+1)}\right] \boldsymbol{= t+1}$ since $\ln(0+1) = 0$.

Particular solution $y_p = 10\int_0^t \exp\left[\ln(t+1) - \ln(s+1)\right] ds = 10(t+1)\int_0^t \frac{ds}{s+1} =$ $10(t+1)\ln(t+1)$.

18 Why is $G(t,s) = 1/G(s,t)$? Why is $G(s,t) = G(s,S)G(S,t)$?

Solution Multiplying $G(s,t)\, G(t,s)$ gives the growth factor $G(s,s)$ from going up to time t and back to time s. This factor is $G(s,s) = 1$. So $G(t,s) = 1/G(s,t)$. Multiplying $G(s,S)\, G(S,t)$ gives the growth factor $G(s,t)$ from going up from s to S and continuing from S to t. In the example $y' = y$, this is $e^{S-s}e^{t-S} = e^{t-s} = G(s,t)$.

Problems 19–22 are about the "units" or "dimensions" in differential equations.

19 (recommended) If $dy/dt = ay + qe^{i\omega t}$, with t in seconds and y in meters, what are the units for a and q and ω ?

Solution a is in "inverse seconds"—for example $a = .01$ per second.

q is in meters.

ω is in "inverse seconds" or 1/seconds—for example $\omega = 2\pi$ radians per second.

20 The logistic equation $dy/dt = ay - by^2$ often measures the time t in years (and y counts people). What are the units of a and b ?

Solution a is in "inverse years"—for example $a = 1$ percent per year.

b is in "inverse people-years" as in $b = 1$ percent per person per year.

21 Newton's Law is $m\, d^2y/dt^2 + ky = F$. If the mass m is in grams, y is in meters, and t is in seconds, what are the units of the stiffness k and the force F ?

Solution ky has the same units as $m\, d^2y/dt^2$ so k is in grams per (second)^2.

F is in gram-meters per (second)^2—the units of force.

22 Why is our favorite example $y' = y + 1$ very unsatisfactory dimensionally ? Solve it anyway starting from $y(0) = -1$ and from $y(0) = 0$.

The three terms in $y' = y + 1$ seem to have different units. The rate $a = 1$ is hidden (with its units of 1/time). Also hidden are the units of the source term 1.

Solution $y(t) = y(0)e^t + \frac{1}{1}(e^t - 1)$. This is $e^t - 1$ if $y(0) = 0$. The solution stays at steady state if $y(0) = -1$.

23 The difference equation $Y_{n+1} = cY_n + Q_n$ produces $Y_1 = cY_0 + Q_0$. Show that the next step produces $Y_2 = c^2Y_0 + cQ_0 + Q_1$. After N steps, the solution formula for Y_N is like the solution formula for $y' = ay + q(t)$. Exponentials of a change to powers of c, the null solution $e^{at}y(0)$ becomes c^NY_0. The particular solution

$$Y_N = c^{N-1}Q_0 + \cdots + Q_{N-1} \quad \text{is like} \quad y(t) = \int_0^t e^{a(t-s)}q(s)ds.$$

Solution $Y_2 = cY_1 + Q_1 = c(cY_0 + Q_0) + Q_1 = c^2Y_0 + cQ_0 + Q_1$.

The particular solution $cQ_0 + Q_1$ agrees with the general formula when $N = 2$. The null solution c^2Y_0 is Step 2 in $Y_0, cY_0, c^2Y_0, c^3Y_0, \ldots$ like $e^{at}y(0)$.

24 Suppose a fungus doubles in size every day, and it weighs a pound after 10 days. If another fungus was twice as large at the start, would it weigh a pound in 5 days ?

Solution This is an ancient puzzle and the answer is 9 days. Starting twice as large cuts off 1 day.

Problem Set 1.7, page 61

1 If $y(0) = a/2b$, the halfway point on the S-curve is at $t = 0$. Show that $d = b$ and $y(t) = \dfrac{a}{d\,e^{-at} + b} = \dfrac{a}{b}\dfrac{1}{e^{-at} + 1}$. Sketch the classic S-curve — graph of $1(e^{-at} + 1)$ from $y_{-\infty} = 0$ to $y_\infty = \dfrac{a}{b}$. Mark the inflection point.

Solution $d = \dfrac{a}{y(0)} - b$ and $y(0) = \dfrac{a}{2b}$ lead to $d = \dfrac{a}{\frac{a}{2b}} - b = 2b - b = b$

$$\text{Therefore } y(t) = \frac{a}{d\,e^{-at} + b} = \frac{a}{b\,e^{-at} + b} = \frac{a}{b}\frac{1}{e^{-at} + 1}$$

2 If the carrying capacity of the Earth is $K = a/b = 14$ billion people, what will be the population at the inflection point? What is dy/dt at that point? The actual population was 7.14 billion on January $1, 2014$.

Solution The inflection point comes where $y = a/2b = 7$ million. The slope dy/dt is

$$\frac{dy}{dt} = ay - by^2 = a\frac{a}{2b} - b\left(\frac{a}{2b}\right)^2 = \frac{a^2}{4b}. \text{ This is } \ b\left(\frac{a}{2b}\right)^2 = 49b.$$

3 Equation (18) must give the same formula for the solution $y(t)$ as equation (16). If the right side of (18) is called R, we can solve that equation for y:

$$y = R\left(1 - \frac{b}{a}y\right) \quad \to \quad \left(1 + R\frac{b}{a}\right)y = R \quad \to \quad y = \frac{R}{\left(1 + R\frac{b}{a}\right)}.$$

Simplify that answer by algebra to recover equation (16) for $y(t)$.

Solution This problem asks us to complete the partial fractions method which integrated $dy/(y - \frac{b}{a}y^2) = a dt$. The result in equation (18) can be solved for $y(t)$. The right side of (18) is called R:

$$R = e^{at}\frac{y(0)}{1 - \frac{b}{a}y(0)} = e^{at}a\frac{y(0)}{a - by(0)} = e^{at}\frac{a}{d}.$$

Then the algebra in the problem statement gives

$$y = \frac{R}{1 + R\frac{b}{a}} = \frac{e^{at}\frac{a}{d}}{1 + e^{at}\frac{b}{d}} = \text{ multiply by } \frac{de^{-at}}{de^{-at}} = \frac{a}{de^{-at} + b}.$$

4 Change the logistic equation to $y' = y + y^2$. Now the nonlinear term is positive, and *cooperation of* y *with* y promotes growth. Use $z = 1/y$ to find and solve a linear equation for z, starting from $z(0) = y(0) = 1$. Show that $y(T) = \infty$ when $e^{-T} = 1/2$. Cooperation looks bad, the population will explode at $t = T$.

Solution Put $y = 1/z$ and the chain rule $\frac{dy}{dt} = \frac{-1}{z^2}\frac{dz}{dt}$ into the cooperation equation $y' = y + y^2$:

$$-\frac{1}{z^2}\frac{dz}{dt} = \frac{1}{z} + \frac{1}{z^2} \text{ gives } \frac{dz}{dt} = -z - 1.$$

The solution starting from $z(0) = 1$ is $z(t) = 2e^{-t} - 1$. This is zero when $2e^{-T} = 1$ or $e^T = 2$ or $\boldsymbol{T = \ln 2}$.

At that time $z(T) = 0$ means $y(T) = 1/z(T)$ is infinite: blow-up at time $T = \ln 2$.

5 The US population grew from $313,873,685$ in 2012 to $316,128,839$ in 2014. If it were following a logistic S-curve, what equations would give you a, b, d in the formula (4)? Is the logistic equation reasonable and how to account for immigration?

Solution We need a third data point to find all three numbers a, b, d. **See Problem (23)**. There seems to be no simple formula for those numbers. Certainly the logistic equation is too simple for serious science. Immigration would give a negative value for h in the harvesting equation $y' = ay - by^2 - h$.

6 The **Bernoulli equation** $y' = ay - by^n$ has competition term by^n. Introduce $z = y^{1-n}$ which matches the logistic case when $n = 2$. Follow equation (4) to show that $z' = (n-1)(-az+b)$. Write $z(t)$ as in (5)-(6). Then you have $y(t)$.

Solution We make the suggested transformation:

$$\begin{aligned} z &= y^{1-n} \\ z' &= (1-n)y^{-n}y' \\ \tfrac{dz}{dt} &= (1-n)y^{-n}(ay - by^n) = (1-n)(ay^{1-n} - b) \\ \tfrac{dz}{dt} &= (1-n)(az - b) \\ z(t) &= e^{(1-n)at}z(0) - \frac{b}{a}(e^{(1-n)at} - 1) = \frac{de^{(1-n)at} + b}{a} \\ d &= az(0) - b = \frac{a}{y(0)} - b \\ y(t) &= \frac{a}{de^{(1-n)at} + b} \end{aligned}$$

Problems 7–13 develop better pictures of the logistic and harvesting equations.

7 $y' = y - y^2$ is solved by $y(t) = 1/(de^{-t} + 1)$. This is an S-curve when $y(0) = 1/2$ and $d = 1$. But show that $y(t)$ is very different if $y(0) > 1$ or if $y(0) < 0$.

If $y(0) = 2$ then $d = \frac{1}{2} - 1 = -\frac{1}{2}$. Show that $y(t) \to 1$ from above.
If $y(0) = -1$ then $d = \frac{1}{-1} - 1 = -2$. At what time T is $y(T) = -\infty$?

Solution First, $y(0) = 2$ is *above* the steady-state value $y_\infty = a/b = 1/1$. Then $d = -\frac{1}{2}$ and $y(t) = 1/(1 - \frac{1}{2}e^{-t})$ is larger than 1 and approaches $y(\infty) = 1/1$ from above as e^{-t} goes to zero.

Second, $y(0) = -1$ is below the $S-$curve growing from $y(-\infty) = 0$ to $y(\infty) = 1$. The value $d = -2$ gives $y(t) = 1/(-2e^{-t} + 1)$. When e^{-t} equals $\frac{1}{2}$ this is $y(t) = 1/0$ and the solution blows up. That blowup time is $\boldsymbol{t = \ln 2}$.

8 (recommended) Show those 3 solutions to $y' = y - y^2$ in one graph ! They start from $y(0) = 1/2$ and 2 and -1. The S-curve climbs from $\frac{1}{2}$ to 1. Above that, $y(t)$ descends from 2 to 1. Below the S-curve, $y(t)$ drops from -1 to $-\infty$.

Can you see 3 regions in the picture ? **Dropin curves above $\boldsymbol{y = 1}$ and $\boldsymbol{S}$-curves sandwiched between 0 and 1 and dropoff curves below $\boldsymbol{y = 0}$.**

Solution The three curves are drawn in Figure 3.3 on page 157. The upper curves and middle curves approach $y_\infty = a/b$. The lowest curves reach $y = -\infty$ in finite time: blow-up.

9 Graph $f(y) = y - y^2$ to see the unstable steady state $Y = 0$ and the stable $Y = 1$. Then graph $f(y) = y - y^2 - 2/9$ with harvesting $h = 2/9$. What are the steady states Y_1 and Y_2 ? The 3 regions in Problem 8 now have Z-curves above $y = 2/3$, S-curves sandwiched between $1/3$ and $2/3$, dropoff curves below $y = 1/3$.

Solution The steady states are the points where $Y - Y^2 = 0$ (logistic) and $Y - Y^2 - \frac{2}{9} = 0$ (harvesting). That second equation factors into $(Y - \frac{1}{3})(Y - \frac{2}{3})$ to show the steady states $\frac{1}{3}$ and $\frac{2}{3}$.

10 What equation produces an S-curve climbing to $y_\infty = K$ from $y_{-\infty} = L$?

Solution We can choose $y' = ay - by^2 - h$ with steady states K and L. Then $aK - bK^2 - h = 0$ and $aL - bL^2 - h = 0$. If we divide by h, these two linear equations give

$$\frac{a}{h} = \frac{K+L}{KL} = \frac{\mathbf{1}}{\boldsymbol{K}} + \frac{\mathbf{1}}{\boldsymbol{L}} \quad\text{and}\quad \frac{b}{h} = \frac{\mathbf{1}}{\boldsymbol{KL}}$$

$$\text{Check}: \frac{a}{h}K - \frac{b}{h}K^2 - 1 = \frac{K}{L} - \frac{K}{L} = \mathbf{0} \quad\text{and}\quad \frac{a}{h}L - \frac{b}{h}L^2 - 1 = \frac{L}{K} - \frac{L}{K} = 0$$

11 $y' = y - y^2 - \frac{1}{4} = -(y - \frac{1}{2})^2$ shows *critical harvesting* with a double steady state at $y = Y = \frac{1}{2}$. The layer of S-curves shrinks to that single line. Sketch a dropin curve that starts above $y(0) = \frac{1}{2}$ and a dropoff curve that starts below $y(0) = \frac{1}{2}$.

Solution The solution to $y' = -(y - \frac{1}{2})^2$ comes from integrating $-dy/(y - \frac{1}{2})^2 = dt$ to get $1/(y - \frac{1}{2}) = t + C$. Then $\boldsymbol{y(t) = \frac{1}{2} + \frac{1}{t+C}}$. If $y(0) > \frac{1}{2}$ then $C > 0$ and this curve approaches $y(\infty) = \frac{1}{2}$; it is a hyperbola coming down toward that horizontal line. If $y(0) < \frac{1}{2}$ then C is negative and the above solution $y = \frac{1}{2} + \frac{1}{t+C}$ blows up (or blows down! since y is negative) at the positive time $t = -C$. This is a dropoff curve below the horizontal line $y = \frac{1}{2}$. (If $y(0) = \frac{1}{2}$ the equation is $dy/dt = 0$ and the solution stays at that steady state.)

12 Solve the equation $y' = -(y - \frac{1}{2})^2$ by substituting $v = y - \frac{1}{2}$ and solving $v' = -v^2$.

Solution This approach uses the solutions we know to $dv/dt = -v^2$. Those solutions are $v(t) = \frac{1}{t+C}$. Then $v = y - \frac{1}{2}$ gives the same $y = \frac{1}{2} + \frac{1}{t+C}$ as in Problem 11.

13 With overharvesting, every curve $y(t)$ drops to $-\infty$. There are no steady states. Solve $Y - Y^2 - h = 0$ (quadratic formula) to find only complex roots if $4h > 1$.

The solutions for $h = \frac{5}{4}$ are $y(t) = \frac{1}{2} - \tan(t + C)$. Sketch that dropoff if $C = 0$. Animal populations don't normally collapse like this from overharvesting.

Solution Overharvesting is $y' = y - y^2 - h$ with h larger than $\frac{1}{4}$ (Problems 11 and 12 had $h = \frac{1}{4}$ and critical harvesting). The fixed points come from $Y - Y^2 - h = 0$. The quadratic formula gives $Y = \frac{1}{2}(1 \pm \sqrt{1 - 4h})$. These roots are complex for $h > \frac{1}{4}$: **No fixed points**.

For $h = \frac{5}{4}$ the equation is $y' = y - y^2 - \frac{5}{4} = -(y - \frac{1}{2})^2 - 1$. Then $v = y - \frac{1}{2}$ has $v' = -v^2 - 1$. Integrating $dv/(1 + v^2) = -dt$ **gives** $\tan^{-1} v = -t - C$ or $v = -\tan(t + C)$. $y = v + \frac{1}{2} = \frac{1}{2} - \tan(t + C)$. The graph of $-\tan t$ starts at zero and drops to $-\infty$ at $t = \pi/2$.

14 With **two partial fractions**, this is my preferred way to find $A = \dfrac{1}{r - s}$, $B = \dfrac{1}{s - r}$

PF2 $$\frac{\mathbf{1}}{\boldsymbol{(y-r)(y-s)}} = \frac{\mathbf{1}}{\boldsymbol{(y-r)(r-s)}} + \frac{\mathbf{1}}{\boldsymbol{(y-s)(s-r)}}$$

Check that equation: The common denominator on the right is $\boldsymbol{(y - r)(y - s)(r - s)}$. The numerator should cancel the $\boldsymbol{r - s}$ when you combine the two fractions.

Separate $\dfrac{1}{y^2-1}$ and $\dfrac{1}{y^2-y}$ into two fractions $\dfrac{A}{y-r}+\dfrac{B}{y-s}$.

Note When y approaches r, the left side of **PF2** has a blowup factor $1/(y-r)$. The other factor $1/(y-s)$ correctly approaches $A = 1/(r-s)$. So the right side **of PF2** needs the same blowup at $y = r$. The first term $A/(y-r)$ fits the bill.

Solution
$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1}+\frac{B}{y+1} = \frac{1/2}{y-1}-\frac{1/2}{y+1}$$

The constants are $A = \dfrac{1}{r-s} = \dfrac{1}{1-(-1)} = -\dfrac{1}{2} = -B$

$$\frac{1}{y^2-y} = \frac{1}{(y-1)y} = \frac{A}{y-1}+\frac{B}{y} = \frac{1}{y-1}-\frac{1}{y}, \quad A = \frac{1}{r-s} = \frac{1}{1-0} = -B$$

15 The **threshold equation** is the logistic equation backward in time :

$$-\frac{dy}{dt} = ay - by^2 \quad \text{is the same as} \quad \frac{dy}{dt} = -ay + by^2.$$

Now $Y = 0$ is the stable steady state. $Y = a/b$ is the unstable state (why ?). If $y(0)$ is below the threshold a/b then $y(t) \to 0$ and the species will die out.

Graph $y(t)$ with $y(0) < a/b$ (reverse S-curve). Then graph $y(t)$ with $y(0) > a/b$.

Solution The steady states of $dy/dt = -ay + by^2$ come from $-aY + bY^2 = 0$ so again $Y = 0$ or $Y = a/b$. The stability is controlled by the **sign of $\boldsymbol{df/dy}$ at $\boldsymbol{y = Y}$** :

$f = -ay + by^2$ tells how y grows $\qquad \dfrac{df}{dy} = -a + 2by$ tells how Δy grows

$Y = 0$ has $\dfrac{df}{dy} = -a$ (STABLE) $\qquad Y = \dfrac{a}{b}$ has $\dfrac{df}{dy} = -a+2b\left(\dfrac{a}{b}\right) = a$ (UNSTABLE)

The S-curves go downward from $Y = a/b$ toward the line $Y = 0$ (never touch).

16 (Cubic nonlinearity) The equation $y' = y(1-y)(2-y)$ has **three steady states**: $Y = 0, 1, 2$. By computing the derivative df/dy at $y = 0, 1, 2$, decide whether each of these states is stable or unstable.

Draw the *stability line* for this equation, to show $y(t)$ leaving the unstable Y's.

Sketch a graph that shows $y(t)$ starting from $y(0) = \frac{1}{2}$ and $\frac{3}{2}$ and $\frac{5}{2}$.

Solution $y' = f(y) = y(1-y)(2-y) = 2y - 3y^2 + y^3$ has slope $\frac{df}{dy} = 2 - 6y + 3y^2$.

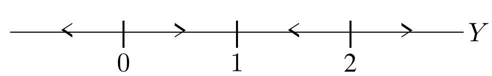
$Y = 0$ has $\frac{df}{dy} = 2$ (unstable)

$Y = 1$ has $\frac{df}{dy} = -1$ (stable)

$Y = 2$ has $\frac{df}{dy} = 2$ (unstable)

$S-$curves go up from $Y = 0$ toward $Y = 1$

$S-$curves from $Y = 2$ go down toward $Y = 1$



17 (a) Find the steady states of the **Gompertz equation** $dy/dt = y(1 - \ln y)$.

Solution (a) $Y(1 - \ln Y) = 0$ at steady states $Y = 0$ and $Y = e$.

(b) Show that $z = \ln y$ satisfies the linear equation $dz/dt = 1 - z$.

Solution (b) $z = \ln y$ has $\frac{dz}{dt} = \frac{1}{y}\frac{dy}{dt} = y(1 - \ln y)/y = 1 - \ln y = 1 - z$.

(c) The solution $z(t) = 1 + e^{-t}(z(0) - 1)$ gives what formula for $y(t)$ from $y(0)$?

Solution (c) $z' = 1/z$ gives that $z(t)$. Then set $y(t) = 1/z(t)$:

$$y(t) = \left[1 + e^{-t}(z(0) - 1)\right]^{-1} = \left[1 + e^{-t}\left(\frac{1}{y(0)} - 1\right)\right]^{-1}.$$

18 Decide stability or instability for the steady states of

(a) $dy/dt = 2(1 - y)(1 - e^y)$ (b) $dy/dt = (1 - y^2)(4 - y^2)$

Solution (a) $f(y) = 2(1 - y)(1 - e^y) = 0$ at $Y = 1$ and $Y = 0$

$\frac{df}{dy} = -2e^y(1 - y) - 2(1 - e^Y)$

At $Y = 1$ $\frac{df}{dy} = -2(1 - e) > 0$ (UNSTABLE) At $Y = 0$ $\frac{df}{dy} = -2$ (STABLE)

(b) $f(y) = (1 - y^2)(4 - y^2) = 0$ at $Y = 1, -1, 2, -2$ $\frac{df}{dy} = -10y + 4y^3$

$Y = 1$ gives $\frac{df}{dy} = -6$ (STABLE) $Y = -1$ gives $\frac{df}{dy} = 6$ (UNSTABLE)

$Y = 2$ gives $\frac{df}{dy} = 12$ (UNSTABLE) $Y = -2$ gives $\frac{df}{dy} = -12$ (STABLE)

19 Stefan's Law of Radiation is $dy/dt = K(M^4 - y^4)$. It is unusual to see fourth powers. Find all real steady states and their stability. Starting from $y(0) = M/2$, sketch a graph of $y(t)$.

Solution $f(Y) = K(M^4 - Y^4)$ equals 0 at $Y = M$ and $Y = -M$ (also $Y = \pm iM$).

$\frac{df}{dy} = -4KY^3 = -4KM^3(Y = M$ is STABLE) $\frac{df}{dy} = 4KM^3(Y = -M$ is UNSTABLE)

The graph starting at $y(0) = M/2$ must go upwards to approach $y(\infty) = M$.

20 $dy/dt = ay - y^3$ has how many steady states Y for $a < 0$ and then $a > 0$? Graph those values $Y(a)$ to see a *pitchfork bifurcation*—new steady states suddenly appear as a passes zero. The graph of $Y(a)$ looks like a pitchfork.

Solution $f(Y) = aY - Y^3 = Y(a - Y^2)$ has 3 steady states $Y = 0, \sqrt{a}, -\sqrt{a}$.

$\frac{df}{dy} = a - 3y^2$ equals a at $Y = 0$, $\frac{df}{dy} = -2a$ at $Y = \sqrt{a}$ and $Y = -\sqrt{a}$.

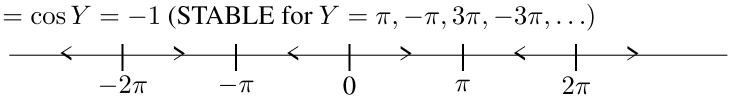
Then $Y = 0$ is UNSTABLE and $Y = \pm\sqrt{a}$ are STABLE.

21 (Recommended) The equation $dy/dt = \sin y$ has **infinitely many steady states**. What are they and which ones are stable? Draw the stability line to show whether $y(t)$ increases or decreases when $y(0)$ is between two of the steady states.

Solution $f(Y) = \sin Y$ is zero at every steady state $Y = n\pi$ $(0, \pi, -\pi, 2\pi, -2\pi, \ldots)$

$$\frac{df}{dy} = \cos Y = 1 \text{ (UNSTABLE for } Y = 0, 2\pi, -2\pi, 4\pi, \ldots)$$
$$= \cos Y = -1 \text{ (STABLE for } Y = \pi, -\pi, 3\pi, -3\pi, \ldots)$$

Stability line



22 Change Problem 21 to $dy/dt = (\sin\, y)^2$. The steady states are the same, but now the derivative of $f(y) = (\sin\, y)^2$ is zero at all those states (because $\sin\, y$ is zero). What will the solution actually do if $y(0)$ is between two steady states ?

Solution $f(y) = (\sin y)^2$ has $\frac{\delta f}{\delta y} = 2\sin y \cos y = \sin 2y$.

Now $\frac{df}{dy} = 0$ at ALL THE STEADY STATES $Y = n\pi$.

Since $\frac{dy}{dt} = (\sin y)^2$ is always positive, the solution $y(t)$ will always increase toward the next larger steady state.

We have an infinite stack of $S-$curves.

23 (*Research project*) Find actual data on the US population in the years 1950, 1980, and 2010. What values of a, b, d in the solution formula (7) will fit these values ? Is the formula accurate at 2000, and what population does it predict for 2020 and 2100 ?

You could reset $t = 0$ to the year 1950 and rescale time so that $t = 3$ is 1980.

Solution Resetting time gives $T = c(t - 1950)$. Rescaling gives $c(1980 - 1950) = 3$ so $c = \frac{1}{10}$. Then a, b, d depend on your data.

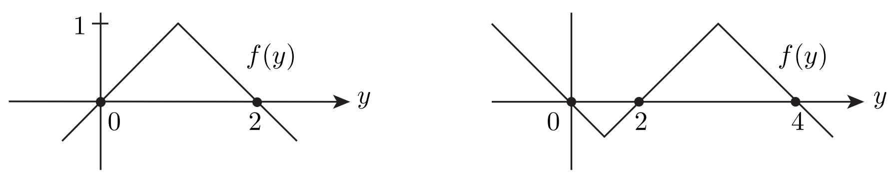
The graphs from $t = 1950$ to 1980 will show $T = \frac{1}{10}(t - 1950)$ from $T = 0$ to 3.

24 If $dy/dt = f(y)$, what is the limit $y(\infty)$ starting from each point $y(0)$?

Solution

$$\frac{dy}{dt} = \begin{cases} y & \text{for } y \le 1 \\ 2 - y & \text{for } y \ge 1 \end{cases} \quad \text{has fixed points } Y = \mathbf{0} \text{ and } \mathbf{2}$$

Slope $\frac{df}{dy} = 1$ at $Y = 0$ (UNSTABLE). Slope $\frac{df}{dy} = -1$ at $Y = 2$ (STABLE), $\boldsymbol{y(\infty) = 2}$.



Fixed points $Y = \mathbf{0}, \mathbf{2}, \mathbf{4}$. Slopes $\frac{df}{dy} = -1, 1, -1$.

$0, 2, 4 =$ STABLE, UNSTABLE, STABLE $\quad y(\infty) = 0$ if $y(0) < 2$ and $y(\infty) = 4$ if $y(0) > 2$.

25 (a) Draw a function $f(y)$ so that $y(t)$ approaches $y(\infty) = 3$ from every $y(0)$.

Solution The right side $f(y)$ must be zero only at $Y = 3$ which is STABLE.

Example: $\frac{dy}{dt} = f(y) = \mathbf{3 - y}$ has solutions $y = 3 + Ce^{-t}$.

(b) Draw $f(y)$ so that $y(\infty) = 4$ if $y(0) > 0$ and $y(\infty) = -2$ if $y(0) < 0$.

Solution This requires $Y = 4, -2$ to be stable and $Y = 0$ to be unstable.

Example: $\frac{dy}{dt} = f(y) = \boldsymbol{-y(y - 4)(y + 2)}$ Notice $\frac{df}{dy} = 8$ at $Y = 0$.

26 Which exponents n in $dy/dt = y^n$ produce blowup $y(T) = \infty$ in a finite time ? You could separate the equation into $dy/y^n = dt$ and integrate from $y(0) = 1$.

Solution $\int \frac{dy}{y^n} = \int dt$ gives $\frac{y^{1-n}}{1-n} = t + C$. The right side is zero at a finite time $t = -C$. Then y blows up at that time **if $n > 1$**.

If $n = 1$ the integrals give $\ln y = t + C$ and $y = e^{t+C}$: NO BLOWUP in finite time.

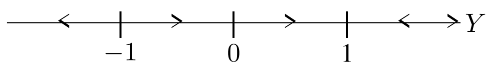
27 Find the steady states of $dy/dt = y^2 - y^4$ and decide whether they are stable, unstable, or one-sided stable. Draw a stability line to show the final value $y(\infty)$ from each initial value $y(0)$.

Solution $f(y) = y^2 - y^4 = 0$ at $Y = 0, 1, -1$

$$\frac{df}{dy} = 2y - 4y^3 = \begin{array}{rl} 0 & \text{at } Y = \ \ 0 \ \text{(Double root of } f) \\ -2 & \text{at } Y = \ \ 1 \ \text{(STABLE)} \\ 2 & \text{at } Y = -1 \ \text{(UNSTABLE)} \end{array}$$

Since $Y = -1$ is unstable, $y(t)$ must go toward $Y = 0$ if $-1 < y(0) < 0$.

Since $Y = 1$ is stable, $y(t)$ must go toward $Y = 1$ if $0 < y(0) < 1$.



28 For an autonomous equation $y' = f(y)$, why is it impossible for $y(t)$ to be increasing at one time t_1 and decreasing at another time t_2 ?

Solution Reason: The stability line shows a movement of y **in one direction**, away from one (unstable) steady state Y and toward another (stable) steady state. "One direction" means that $y(t)$ is steadily increasing or steadily decreasing.

Problem Set 1.8, page 69

1 Finally we can solve the example $dy/dt = y^2$ in Section 1.1 of this book.

Start from $y(0) = 1$. Then $\int_1^y \frac{dy}{y^2} = \int_0^t dt$. Notice the limits on y and t. Find $y(t)$.

Solution With those limits, integration gives $-\frac{1}{y} + 1 = t$. Then $\frac{1}{y} = 1 - t$ and $y(t) = \boldsymbol{\frac{1}{1-t}}$.

2 Start the same equation $dy/dt = y^2$ from any value $y(0)$. At what time t does the solution blow up ? For which starting values $y(0)$ does it never blow up ?

Solution $$-\frac{1}{y} + \frac{1}{y(0)} = t \text{ gives } \frac{1}{y} = \frac{1}{y(0)} - t \text{ and } \boldsymbol{y = \frac{y(0)}{1 - ty(0)}}.$$

If $y(0)$ is negative, then $1 - ty(0)$ never touches zero for $t > 0$: No blowup.

3 Solve $dy/dt = a(t)y$ as a separable equation starting from $y(0) = 1$, by choosing $f(y) = 1/y$. This equation gave the growth factor $G(0, t)$ in Section 1.6.

Solution
$$\int_{y(0)}^{y} \frac{dy}{y} = \int_0^t a(t)dt \text{ gives } \ln y(t) - \ln y(0) = \int_0^t a(t)dt$$

$$y(t) = y(0)\exp\left(\int_0^t a(t)dt\right) = \boldsymbol{G(0,t)\,y(0)}$$

4 Solve these separable equations starting from $y(0) = 0$:

(a) $\dfrac{dy}{dt} = ty$ (b) $\dfrac{dy}{dt} = t^m\, y^n$

Solution (a) $\displaystyle\int_{y(0)}^{y} \frac{dy}{y} = \int_0^t t\,dt$ and $\ln y - \ln y(0) = t^2/2$: Then $y(t) = y(0)\exp(t^2/2)$.

(b) $\frac{dy}{dt} = t^m y^n$ has $\displaystyle\int \frac{dy}{y^n} = \int t^m\,dt$ and $\frac{y^{1-n}}{1-n} = \frac{t^{m+1}}{m+1}$. Then $y = \left(\frac{1-n}{m+1}t^{m+1}\right)^{1/(1-n)}$ for $n \neq 1$.

5 Solve $\dfrac{dy}{dt} = a(t)y^2 = \dfrac{a(t)}{1/y^2}$ as a separable equation starting from $y(0) = 1$.

Solution

$$\frac{dy}{dt} = a(t)y^2$$

$$\int_1^y \frac{du}{u^2} = \int_0^t a(x)\,dx \quad (u \text{ and } x \text{ are just integration variables})$$

$$-\frac{1}{y} + 1 = \int_0^t a(x)\,dx \text{ gives } y = \frac{1}{1 - \displaystyle\int_0^t a(x)\,dx}$$

6 The equation $\dfrac{dy}{dt} = y + t$ is not separable or exact. But it is linear and $y =$ _____.

Solution We solve the equation by taking advantage of its linearity:

Given $a = 1$, the growth factor is e^t. The source term is t. Therefore using equation (14) gives:

$$y(t) = e^t y(0) + \int_0^t e^{t-s} s\,ds = e^t y(0) - t + e^t - 1.$$

$$\text{Check: } dy/dt = e^t y(0) - 1 + e^t \text{ does equal } y + t.$$

7 The equation $\dfrac{dy}{dt} = \dfrac{y}{t}$ has the solution $y = At$ for every constant A. Find this solution by separating $f = 1/y$ from $g = 1/t$. Then integrate $dy/y = dt/t$. Where does the constant A come from ?

Solution We use separation of variables to find the constant A

$$\frac{dy}{y} = \frac{dt}{t}$$

$$\int_{y(1)}^{t} \frac{du}{u} = \int_{1}^{t} \frac{dx}{x}$$

$$\ln(y) - \ln(y(1)) = \ln t$$

$$\frac{y}{y(\mathbf{1})} = t$$

$$\boldsymbol{y = y(1)\, t}$$

Therefore we find that the constant A is equal to $y(1)$, the initial value.

8 For which number A is $\dfrac{dy}{dt} = \dfrac{ct - ay}{At + by}$ an exact equation ? For this A, solve the equation by finding a suitable function $F(y,t) + C(t)$.

Solution $f(y,t) = At + by$ and $g(y,t) = ct - ay$

The equation is exact if: $\frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y}$ and $\boldsymbol{A = a}$.

We follow the three solution steps for exact equations.

1 Integrate f with respect to y:

$$\int f(y,t)\, dy = \int (At + by)\, dy = Aty + \frac{1}{2}by^2 = F(y,t)$$

2 Choose $C(t)$ so that $\frac{\partial}{\partial t}(F(y,t) + C(t)) = -g(y,t)$

$$\frac{\partial}{\partial t}(Aty + \frac{1}{2}by^2 + C(t)) = Ay + C'(t) = -ct + ay$$

$$C'(t) = -ct \text{ and } C(t) = -\frac{1}{2}ct^2$$

3 We therefore have that:

$$\frac{dy}{dt} = \frac{g(y,t)}{f(y,t)} \text{ is solved by } F(y,t) + C(t) = \text{ constant}$$

$$Aty + \frac{1}{2}by^2 - \frac{1}{2}ct^2 = \text{ constant}$$

9 Find a function $y(t)$ different from $y = t$ that has $dy/dt = y^2/t^2$.

Solution Using separation of variables:

$$dy/dt = y^2/t^2$$

$$dy/y^2 = dt/t^2$$

$$\int_{y(t_0)}^{y} \frac{du}{u^2} = \int_{t_0}^{t} \frac{dx}{x^2}$$

$$-\tfrac{1}{y(t)} + \tfrac{1}{y(t_0)} = -\tfrac{1}{t} + \tfrac{1}{t_0}$$

$t_0 = 1$ and $y(t_0) = 2$ give $-\frac{1}{y(t)} + \frac{1}{2} = -\frac{1}{t} + 1$ and $y(t) = \left(\frac{1}{t} - \frac{1}{2}\right)^{-1} = \boldsymbol{\frac{2t}{2-t}}$

10 These equations are separable after factoring the right hand sides :

$$\text{Solve} \quad \frac{dy}{dt} = e^{y+t} \quad \text{and} \quad \frac{dy}{dt} = yt + y + t + 1.$$

Solution (a) $\dfrac{dy}{dt} = e^y e^t$ and $\displaystyle\int_{y_0}^{y} e^{-y} dy = \int_{t_0}^{t} e^t dt$

$$-e^{-y} + e^{-y_0} = e^t - e^{t_0}$$
$$e^{-y} = e^{-y_0} - e^t + e^{t_0}$$
$$y = -\ln\left[e^{-y_0} - e^t + e^{t_0}\right]$$

(b) $dy/dt = (y+1)(t+1)$

$$\int_{y_0}^{y} \frac{dy}{y+1} = \int_{t_0}^{t} (t+1)\, dt$$
$$\ln(y+1) - \ln(y_0+1) = \frac{1}{2}(t^2 - t_0^2) + (t - t_0) = G$$
$$y + 1 = (y_0 + 1)\, e^G$$

11 These equations are linear and separable : Solve $\dfrac{dy}{dt} = (y+4)\cos t$ and $\dfrac{dy}{dt} = ye^t$.

Solution (a) $\displaystyle\int_{y_0}^{y} \frac{dy}{y+4} = \int_{t_0}^{t} \cos t\, dt$

$\ln(y+4) - \ln(y_0+4) = \sin t - \sin t_0$

$y + 4 = (y_0 + 4)\exp(\sin t - \sin t_0)$

(b) $\displaystyle\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} e^t\, dt$

$\ln y - \ln y_0 = e^t - e^{t_0}$

$y = y_0 \exp(e^t - e^{t_0})$

12 Solve these three separable equations starting from $y(0) = 1$:

Solution (a) $\dfrac{dy}{dt} = -4ty$ has $\displaystyle\int_{1}^{y} \frac{dy}{y} = \int_{0}^{t} -4t\, dt$

$$\ln y = -2t^2 \text{ and } y = \exp(-2t^2)$$

(b) $\dfrac{dy}{dt} = ty^3$ has $\displaystyle\int_{1}^{y} \frac{dy}{y^3} = \int_{0}^{t} t\, dt$ and $-\dfrac{1}{2y^2} + \dfrac{1}{2y_0^2} = \dfrac{1}{2}t^2$

$$\frac{1}{y^2} = \frac{1}{y_0^2} - t^2$$
$$y = \left(\frac{1}{y_0^2} - t^2\right)^{-1/2} = y_0\left(1 - t^2 y_0^2\right)^{-1/2}$$

(c) $(1+t)\dfrac{dy}{dt} = 4y$ has $\displaystyle\int_1^y \frac{dy}{y} = \int_0^t \frac{4\,dt}{1+t}$

$$\ln y = 4\ln(1+t) - 4\ln(1) = 4\ln(1+t)$$
$$y = \boldsymbol{(1+t)^4}$$

Check $(1+t)\frac{dy}{dt} = 4(1+t)(1+t)^3 = 4y$

Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$ and solve Problems 13-14.

13 Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$.

Solution (a) $g = -3t^2 - 2y^2$ has $\partial g/\partial y = -4y$

$f = 4ty + by^2$ has $-\partial f/\partial y = -4y$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(4ty + 6y^2\right)\,dy = 2ty^2 + 2y^3 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(2ty^2 + 2y^3 + C(t)\right) = 2y^2 + C'(t)$.

This equals $-g$ when $C'(t) = 3t^2$ and $C(t) = t^3$.

Step 3 : Solution $\boldsymbol{2ty^2 + 2y^3 + t^3 =}$ constant

Solution (b) $g = -1 - ye^{ty}$ has $\partial g/\partial y = -yte^{ty} - e^{ty}$

$f = 2y + te^{ty}$ has $-\partial f/\partial t = -yte^{ty} - e^{ty}$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(2y + te^{ty}\right)\,dy = y^2 + e^{ty} + C(t) = F(y,t)$

Step 2 : $\frac{\partial}{\partial t}\left(y^2 + e^{ty} + C(t)\right) = ye^{ty} + C'(t) = -g$ where $C'(t) = 1$

Step 3 : $C'(t) = 1$ gives $C(t) = t$ and the solution is

$F(y,t) + C(t) = \boldsymbol{-yte^{ty} - e^{ty} + t} =$ constant

14 Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$.

Solution (a) $g = 4t - y$ and $f = t - 6y$ have $\frac{\partial g}{\partial y} = -1 = \frac{\partial f}{\partial t}$: EXACT

Step 1 : $\displaystyle\int f\,dy = ty - 3y^2 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(ty - 3y^2 + C(t)\right) = y + C'(t) = -g = y - 4t$ when $C(t) = -2t^2$

Step 3 : Solution $\boldsymbol{ty - 3y^2 - 2t^2 =}$ **constant**

Solution (b) $g = -3t^2 - 2y^2$ and $f = 4ty + 6y^2$ have $\frac{\partial g}{\partial y} = -4y = -\frac{\partial f}{\partial t}$: EXACT

Step 1 : $\displaystyle\int f\,dy = \int \left(4ty + 6y^2\right)dy = 2ty^2 + 2y^3 + C(t)$

Step 2 : $\frac{\partial}{\partial t}\left(2ty^2+2y^3+C(t)\right) = 2y^2+C'(t) = -g = 3t^2+2y^2$ when $C' = 3t^2$ and $C = t^3$

Step 3 : Solution $\mathbf{2ty^2+2y^3+t^3 = }$ **constant**

15 Show that $\dfrac{dy}{dt} = -\dfrac{y^2}{2ty}$ is exact but the same equation $\dfrac{dy}{dt} = -\dfrac{y}{2t}$ is not exact. Solve both equations. (This problem suggests that many equations become exact when multiplied by an integrating factor.)

Solution $g=-y^2$ and $f=2ty$ have $\frac{\partial g}{\partial y} = -2y = -\frac{\partial f}{\partial t}$: EXACT

$g=-y$ and $f=2t$ have $\frac{\partial g}{\partial y}$ NOT EQUAL TO $-\frac{\partial f}{\partial t}$

Solve the second form which is SEPARABLE

$$\int \frac{dy}{y} = \int -\frac{dt}{2t} \text{ gives } \ln y = -\frac{1}{2}\ln t + C$$

Then $y = e^C t^{-1/2}$ is the same as $y = ct^{-1/2}$.

The same solution must come from Steps 1, 2, 3 using the exact form.

16 Exactness is really the condition to solve two equations with the same function $H(t,y)$:

$$\frac{\partial H}{\partial y} = f(t,y) \text{ and } \frac{\partial H}{\partial t} = -g(t,y) \text{ can be solved if } \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y}.$$

Take the t derivative of $\partial H/\partial y$ and the y derivative of $\partial H/\partial t$ to show that exactness is *necessary*. It is also *sufficient* to guarantee that a solution H will exist.

Solution The point is to see the underlying idea of exactness.

$$\text{If } \frac{\partial H}{\partial y} = f(t,y) \quad \text{then} \quad \frac{\partial^2 H}{\partial t\,\partial y} = \frac{\partial f}{\partial t}$$

$$\text{If } \frac{\partial H}{\partial t} = -g(t,y) \quad \text{then} \quad \frac{\partial^2 H}{\partial y\,\partial t} = -\frac{\partial g}{\partial y}$$

The cross derivatives of H are always equal. **IF** a function H solves both equations then $\frac{\partial f}{\partial t}$ *must equal* $-\frac{\partial g}{\partial y}$. So behind every exact equation is a function H : exactness is a necessary and also sufficient to find H with $\partial H/\partial y = f$ and $\partial H/\partial t = -g$.

17 The linear equation $\dfrac{dy}{dt} = aty + q$ is not exact or separable. Multiply by the integrating factor $e^{-\int at\,dt}$ and solve the equation starting from $y(0)$.

Solution This problem just recalls the idea of an integrating factor :

For $\dfrac{dy}{dt} = aty+q$ the factor is $P = \exp\left(-\int at\,dt\right) = \exp\left(-\dfrac{1}{2}at^2\right)$.

Then $P\left(\frac{dy}{dt} - aty\right)$ agrees with $(Py)' = P\frac{dy}{dt} + \frac{dP}{dt}y$

So the original equation multiplied by P is $\frac{d}{dt}(Py) = Pq$.

Integrate both sides $P(t)y(t) - P(0)y(0) = \displaystyle\int_0^t P(t)q\,dt$. Divide by $P(t)$ to find $y(t)$.

Second order equations $F(t,y,y',y'')=0$ involve the second derivative y''. This reduces to a first order equation for y' (not y) in two important cases:

I. When y is missing in F, set $y'=v$ and $y''=v'$. Then $\boldsymbol{F(t,v,v')=0}$.
II. When t is missing in F, set $y''=\dfrac{dv}{dt}=\dfrac{dv}{dy}\dfrac{dy}{dt}=v\dfrac{dv}{dy}$. Then $\boldsymbol{F\left(y,v,v\dfrac{dv}{dy}\right)=0}$.

See the website for **reduction of order** when one solution $y(t)$ is known.

18 (y is missing) Solve these differential equations for $v=y'$ with $v(0)=1$. Then solve for y with $y(0)=0$.

Solution (a) $y''+y'=0$. Set $y'=v$. Then $\boldsymbol{v'+v=0}$ gives $v(t)=Ce^{-t}$.

Now solve $y'=v=Ce^{-t}$ to find $y=\boldsymbol{-Ce^{-t}+D}$.

Solution (b) $2ty''-y'=0$. Set $y'=v$. Then $\boldsymbol{2tv'-v=0}$ is solved by

$\displaystyle\int\frac{dv}{v}=\int\frac{dt}{2t}$ and $\ln v=\ln\sqrt{t}+C$ and $v=c\sqrt{t}$. Now solve $y'=v=c\sqrt{t}$ to find $y=c_1t^{3/2}+c_2$.

19 Both y and t are missing in $\boldsymbol{y''=(y')^2}$. Set $v=y'$ and go two ways:

I. Solve $\dfrac{dv}{dt}=v^2$ to find $v=\dfrac{1}{1-t}$ as in Section 1.1.

Then solve $\dfrac{dy}{dt}=v=\dfrac{1}{1-t}$ to find $y=-\dfrac{(1-t)^{-2}}{2}+\dfrac{1}{2}$ with $y(0)=0$.

II. Solve $v\dfrac{dv}{dy}=v^2$ or $\dfrac{dv}{dy}=v$ to find $v=e^y$.

Then $\dfrac{dy}{dt}=v(y)=e^y$ gives $\displaystyle\int e^{-y}\,dy=\int dt$ satisfying $v(0)=1,y(0)=0$

and $-e^{-y}=t-1$: not the same solution as part I (??)

20 An **autonomous equation $\boldsymbol{y'=f(y)}$** has no terms that contain t (t is missing).

Explain why every autonomous equation is separable. A non-autonomous equation could be separable or not. For a linear equation we usually say LTI (**linear time-invariant**) when it is autonomous: coefficients are constant, not varying with t.

Solution Every autonomous equation separates into $\displaystyle\int\frac{dy}{f(y)}=\int dt.$

Linear equations can be $\frac{dy}{dt}=a(t)y$: Non-autonomous

LTI equations are $\frac{dy}{dt}=ay$ (linear and also a is time-invariant $\Rightarrow$ autonomous).

21 $my''+ky=0$ is a highly important LTI equation. Two solutions are $\cos\omega t$ and $\sin\omega t$ when $\omega^2=k/m$. Solve differently by reducing to a first order equation for $y'=dy/dt=v$ with $y''=v\,dv/dy$ as above:

$$mv\frac{dv}{dy}+ky=0 \text{ integrates to } \frac{1}{2}mv^2+\frac{1}{2}ky^2=\text{ constant } E.$$

For a mass on a spring, kinetic energy $\frac{1}{2}mv^2$ plus potential energy $\frac{1}{2}ky^2$ is a constant energy E. What is E when $y = \cos\omega t$? What integral solves the separable $m(y')^2 = 2E - ky^2$? I would not solve the linear oscillation equation this way.

Solution With $y' = v$ and $y'' = v\frac{dv}{dy}$, the equation $my'' + ky = 0$ becomes

$mv\frac{dv}{dy} + ky = 0$. This is *nonlinear* but *separable*. Integrate $mv\,dv = -ky\,dy$ to get

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{ constant } E \text{ [Conservation of Energy].}$$

If $y = \cos(\omega t)$ then $v = y' = -\omega\sin(\omega t)$ and E is $\frac{1}{2}m\cos^2(\omega t) + \frac{1}{2}K\omega^2\sin^2(\omega t)$.

The separable equation $m(y')^2 = 2E - ky^2$ could be solved by $\left(\dfrac{m}{2E - Ky^2}\right)^{1/2} dy = dt$. The integral could lead to $\cos^{-1} y = \omega t$ and $y = \cos\omega t$.

22 $my'' + k\sin y = 0$ is the *nonlinear* oscillation equation: not so simple. Reduce to a first order equation as in Problem 21:

$$mv\frac{dv}{dy} + k\sin y = 0 \text{ integrates to } \frac{1}{2}mv^2 - k\cos y = \text{ constant } E.$$

With $v = dy/dt$ what impossible integral is needed for this first order separable equation? Actually that integral gives the period of a nonlinear pendulum—this integral is extremely important and well studied even if impossible.

Solution Take square roots in $\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = K\cos y + E$.

Then separate into $\left[\dfrac{m/2}{K\cos y + E}\right]^{1/2} dy = dt$.

An unpleasant integral but important for nonlinear oscillation. Chapter 1 is ending with an example that shows the reality of nonlinear differential equations: Numerical solutions possible, elementary formulas are often impossible.

Problem Set 2.1, page 79

1 Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice):

Simple harmonic motion $\quad y(t) = C\ \cos\,\omega t + D\ \sin\,\omega t.$ What is ω?

If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero?

Solution Letting $y(t) = C\cos(\omega t) + D\sin(\omega t)$:

$$\frac{d^2y}{dt^2} + 9y = -\omega^2 C\cos(\omega t) + 9C\cos(\omega t) - \omega^2\sin(\omega t) + 9\sin(\omega t) = 0$$

$$\boldsymbol{\omega = 3}$$

Differentiating $y(t)$ and equating to zero at time $t = 0$ gives us:

$$y'(t) = -C\omega\sin(\omega t) + D\omega\cos(\omega t) = 0$$

At $t = 0 : D\omega = 0 \rightarrow D = 0$

2 In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

Solution $y(0) = C\cos(\omega 0) + D\sin(\omega 0) = 0$ gives $C = 0$

Differentiating $y(t)$ and equating to 1 at time $t = 0$ gives us:

$$y'(0) = D\omega = 1 \ \text{ and } \ D = \frac{1}{\omega} = \frac{1}{3}$$

3 Draw Figure 2.3 to show simple harmonic motion $y = A\ \cos\,(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.

Solution Notice that A is the maximum height $y_{\max}$. At $t = 0$ we see $y = A\cos(-\alpha) = A\cos\alpha$.

4 Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\ \cos\,(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x axis?

Solution $f = \omega/2\pi = 60$ Hertz is equivalent to $\omega = 120\pi$ radians per second. With magnitude $A = 3$ and $\alpha = -45\,° = -\pi/4$ radians, $A\cos(\omega t - \alpha)$ becomes $3\cos(120\pi t + \pi/4)$. The point going around the circle hits the x-axis when that angle is a multiple of π. The first hit occurs at $120\pi t + \pi/4 = \pi$ and $120\,t = 3/4$ and $t = 3/480 = 1/160$.

5 If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit? Your circular frequency is $f =$ _____ and your angular frequency is $\omega =$ _____ (with what units?). The period is T.

Solution The distance around a circle of radius $R = 3$ miles is $2\pi R = 6\pi$ miles. The time T for a complete circuit at 60 miles per hour is $T = 6\pi/60 = \boldsymbol{\pi/10}$ hours. From $T = 1/f = 2\pi/\omega$ the circular frequency is $f = 10/\pi$ cycles per hour and $\omega = 2\pi f = 2\pi/T = 20$ radians per hour.

6 The total energy E in the oscillating spring-mass system is

$$E = \textbf{kinetic} \text{ energy in mass} + \textbf{potential} \text{ energy in spring} = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2.$$

Compute E when $y = C\ \cos\,\omega t + D\ \sin\,\omega t$. The energy is constant!

Solution $y = C\cos\omega t + D\sin\omega t$ has $dy/dt = -\omega C\sin\omega t + \omega D\cos\omega t$.

The total energy is $E = \frac{1}{2}m\omega^2(C^2\sin^2\omega t - 2CD\sin\omega t\cos\omega t + D^2\cos^2\omega t)$
$+\ \frac{1}{2}k(C^2\cos^2\omega t + 2CD\sin\omega t\cos\omega t + D^2\sin^2\omega t)$.

When $\omega = \sqrt{k/m}$ and $m\omega^2 = k$, use $\sin^2\omega t + \cos^2\omega t = 1$ to find

$$E = \frac{1}{2}k\left(C^2 + D^2\right)(\sin^2\omega t + \cos^2\omega t) = \frac{1}{2}k(C^2 + D^2) = \textbf{constant}.$$

7 Another way to show that the total energy E is constant:

Multiply $\boldsymbol{my'' + ky = 0}$ by $\boldsymbol{y'}$. Then integrate $my'y''$ and kyy'.

Solution $(my'' + ky)\,y' = 0$ is the same as $\frac{d}{dt}(\frac{1}{2}my'^2 + \frac{1}{2}ky^2) = 0$.

This says that $E = \frac{1}{2}my'^2 + \frac{1}{2}ky^2$ is constant.

8 A **forced oscillation** has another term in the equation and $A\cos\omega t$ in the solution:

$$\frac{d^2y}{dt^2} + 4y = F\ \cos\,\omega t \quad \text{has} \quad y = C\ \cos\,2t + D\ \sin\,2t + A\ \cos\,\omega t.$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n). Find the forced amplitude A in the particular solution $y_p = A\ \cos\,\omega t$.

(b) In case $\omega = 2$ (forcing frequency = natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

Solution (a) The frequency $\omega = 2$ gives the null solutions $y = C\cos 2t + D\sin 2t$: $\boldsymbol{y_n'' + 4y_n = 0}$.

The choice of A gives a particular solution $y_p = A\cos\omega t$. Substitute this y_p:

$$y_p'' + 4y_p = (-\omega^2 + 4)A\cos\omega t = F\cos\omega t \text{ and } A = \boldsymbol{\frac{F}{4-\omega^2}}.$$

(b) $\omega = 2$ leads to $A = \infty$ and that solution y_p breaks down: **resonance**. (The correct y_p will include a factor t)

9 Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F\ \cos\,\omega t \text{ with } \omega \neq \omega_n = \sqrt{k/m} \text{ (no resonance).}$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A *is fixed* by F).

Solution $y = y_n + y_p = C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{A}{k-m\omega^2}\cos\omega t$.

10 Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:

$$my'' = +ky \quad \text{including} \quad y'' = 4y.$$

Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

Solution The exponents in $y_n = Ce^{t\sqrt{k/m}} + De^{-t\sqrt{k/m}}$ are now real. Those numbers $\pm\sqrt{k/m}$ come from substituting $y = e^{st}$ into the differential equation:

$$my'' - ky = (ms^2 - k)e^{st} = 0 \text{ when } s = \sqrt{\boldsymbol{k/m}} \text{ and } s = -\sqrt{\boldsymbol{k/m}}.$$

11 Here is a *fourth order* equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

Solution Substitute $y = e^{st}$ in the differential equation to find $s^4 = 16$. This has four solutions: $s = 2, -2, 2i, -2i$. The constants in $y = c_1e^{2t} + c_2e^{-2t} + c_3e^{2it} + c_4e^{-2it}$ are determined by the initial values $y(0), y'(0), y''(0), y'''(0)$.

12 To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

Solution Substitute $y_p = Ye^{ct}$ to find $(c^2 + 9)Ye^{ct} = e^{ct}$ and $Y = 1/(c^2 + 9)$. This is called the "exponential response function" in Section 2.4. The resonant case $Y = \infty$ occurrs when $c^2 + 9 = 0$ or $c = \pm 3i$. Then a new formula for $y(t)$ involves $\boldsymbol{te^{ct}}$ as well as e^{ct}.

13 In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency $\omega =$ what natural frequency?

Solution Substitute $y_p = Ae^{i\omega t}$ to find $i^2\omega^2 Ae^{i\omega t} + 9Ae^{i\omega t} = e^{i\omega t}$. With $i^2 = -1$ this gives $A = 1/(9 - \omega^2)$. This blows up when $9 - \omega^2 = 0$ at the natural frequency $\omega_n = 3$.

14 If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0, 0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

Solution If $y(0) > 0$ and $y'(0) < 0$ then α falls between $3\pi/2$ and 2π. This occurs because the vector from $(0, 0)$ to $(y(0), y'(0)/\omega)$ is in the fourth quadrant.

15 Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping *UP* and bending *UP*.

Solution For $\frac{\pi}{2} < t < \pi$ ($90°$ to $180°$), $y(t) = \sin t > 0$ but $y'(t) < 0$ and $y''(t) < 0$.

Note that for $\frac{3\pi}{2} < t < 2\pi$, $y(t) < 0$ but $y'(t) > 0$ and $y''(t) > 0$. The point is below the x-axis but the bold sine curve is sloping upwards and bending upwards.

16 (a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is $\boldsymbol{y_n}$.**)

(b) Solve $y'' + 100y = \cos\omega t$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be $\boldsymbol{y_p}$.**)

Solution (a) Substitute $y = e^{ct}$

$$\begin{aligned} y'' + 100y &= 0 \\ c^2e^{ct} + 100e^{ct} &= 0 \\ c^2 &= -100 \\ c &= \pm 10i \\ y &= ce^{10it} + de^{-10it} \end{aligned}$$

This can be rewritten in terms of sines and cosines of $10t$. Introducing the initial conditions we have:

$$\begin{aligned} y(t) &= A\cos(10t) + B\sin(10t) \\ y(0) &= A = 1 \\ y'(0) &= 10B = 10 \to B = 1 \\ y(t) &= \sin(10t) + \cos(10t) \end{aligned}$$

(b) As in equation (11) we assume the particular solution is

$$y(t) = \frac{1}{100-\omega^2}\cos(\omega t)$$

Adding in the null solution and substituting in the initial conditions gives:

$$\begin{aligned} y(t) &= B\sin(10t) + A\cos(10t) + \frac{1}{100-\omega^2}\cos(\omega t) \\ y(0) &= B\sin(0) + A\cos(0) + \frac{1}{100-\omega^2}\cos(0) = 0 \\ A &= \frac{1}{\omega^2-100} \\ y'(0) &= 10B\cos(0) - 10A\sin(0) - \frac{\omega}{100-\omega^2}\sin(0) \\ &= 10B = 0 \to B = 0 \end{aligned}$$

Therefore the solution is:

$$y(t) = \frac{1}{100-\omega^2}(\cos(\omega t) - \cos(10t))$$

17 Find a particular solution $y_p = R\cos(\omega t - \alpha)$ to $y'' + 100y = \cos\omega t - \sin\omega t$.

Solution

$$\begin{aligned} \text{Right side } : \cos\omega t - \sin\omega t &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ \text{Diff. Eqn } : -\omega^2 R\cos(\omega t - \alpha) + 100R\cos(\omega t - \alpha) &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ (100-\omega^2)R\cos(\omega t - \alpha) &= \sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right) \\ \text{Then } \alpha = -\frac{\pi}{4} \text{ and } R &= \frac{\sqrt{2}}{100-\omega^2} \end{aligned}$$

18 Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A \cos \omega t$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

Solution The equation describing Simple Harmonic Motion is:

$$x(t) = A\cos(\omega t - \phi)$$

If the period is $T = 1$ second, the frequency is $f = 1$ Hertz or $\omega = 2\pi$ radians per second.

19 If the phase lag is α, what is the time lag in graphing $\cos(\omega t - \alpha)$?

Solution

$$\cos(\omega t - \alpha) = \cos\left(\omega\left(t - \frac{\alpha}{\omega}\right)\right)$$

Therefore the time lag is α/ω.

20 What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

Solution Similar to equation (15) we have

$$y_p(t) = \frac{\sin(\omega_n(t - T))}{m\omega_n}$$

The conditions at time T are:

$$y_p(T) = 0 \text{ and } y_p'(T) = \frac{1}{m}$$

Note that y_p starts from time $t = T$. We have $y_p = 0$.

21 (Good challenge) Show that $y = \int_0^t g(t - s)f(s)\,ds$ has $my'' + ky = f(t)$.

1 Why is $y' = \int_0^t g'(t - s)f(s)\,ds + g(0)f(t)$? Notice the two t's in y.

Solution 1 The variable t appears twice in the formula for y, so the derivative dy/dt has **two terms** (called the Leibniz rule). One term is the value of $g(t - s)f(s)$ at the upper limit $s = t$; this is from the Fundamental Theorem of Calculus. Since t also appears in the quantity $g(t - s)f(s)$, its derivative $g'(t - s)f(s)$ also appears in y'.

2 Using $g(0) = 0$, explain why $y'' = \int_0^t g''(t - s)f(s)\,ds + g'(0)f(t)$.

Solution 2 Since $g(0) = 0$, part 1 produced $y' = \int_0^t g'(t - s)f(s)ds$. Using the Leibniz rule again (now on y'), we get the two terms in y''.

3 Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

Solution 3 $my''+ky = m\left(\int_0^t g''(t - s)f(s)\,ds + g'(0)f(t)\right)+k\left(\int_0^t g(t - s)f(s)\,ds\right) = m(1/m)f(t)$. The integrals cancelled because $mg'' + kg = 0$.

22 With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y :

$$\textbf{Step response} \quad y(t) = \int_0^t \frac{\sin\omega_n(t - s)}{m\omega_n}\,1\,ds = y_p + y_n \text{ equals } \frac{\mathbf{1}}{\boldsymbol{k}} - \frac{\mathbf{1}}{\boldsymbol{k}}\,\mathbf{cos}\,\boldsymbol{\omega_n t}.$$

Solution This $y(t)$ certainly solves $my'' + ky = 1$. *Comment*: That formula for $y(t)$ fits with the usual $\int g(t - s)f(s)\,ds$ when $f = 1$ and the impulse response is $g(t) = (\sin\omega_n t)/m\omega_n$ in equation (15). And always this **step response should be the integral of the impulse response**. The natural frequency is $\omega_n = k/m$:

$$y(t) = \int_0^t \frac{\sin(\omega_n(t-s))}{m\omega_n}\,ds = -\frac{\cos(\omega_n(t-s))}{m\omega_n^2}\Bigg]_0^t = \frac{1}{k} - \frac{\cos(\omega_n t)}{k}.$$

Notice that without damping resistance, the step response oscillates forever—not approaching the steady state $y_\infty = 1/k$.

23 (Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

Solution

$$\frac{d^2y}{dt^2} = 0 \text{ gives } y_n = A + Bt.$$

We get the fundamental solution $\boldsymbol{g(t) = t}$ for $t \geq 0$ by starting the null solution with $g(0) = 0$ and $g'(0) = 1$. Then $g(t) = t$ and $g(t-s) = t - s$. This gives the particular solution for $d^2y/dt^2 = f(t)$ using formula (16) :

$$y(t) = \int_0^t (t-s)f(s)\,ds.$$

24 For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

Solution Substitute $y = Ye^{i\omega t}$:

$$\begin{aligned} -Y(\omega)\omega^2 e^{i\omega t} &= e^{i\omega t} \\ Y(\omega) &= -1/\omega^2 \\ y_p(t)_p &= e^{i\omega t}/\omega^2 \end{aligned}$$

The null solution to $y'' = 0$ is $y(t)_n = At + B$.

When $A = 0$ and $B = 1$, we get $y_n = 1$. This causes resonance at $\omega = 0$, the solution formula $y_p = e^{i\omega t}/\omega^2$ breaks down.

25 Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite ?

Solution

$$\begin{aligned} &\text{Substitute } y(t) = Ye^{i\omega t} \text{ in } my'' - ky = e^{i\omega t} \\ &\text{Then } -Ym\omega^2 e^{i\omega t} - kYe^{i\omega t} = e^{i\omega t} \\ &-Ym\omega^2 - Yk = 1 \\ &Y(\omega) = \frac{1}{k + m\omega^2} \end{aligned}$$

Y is infinite for $\omega = i\sqrt{\frac{k}{m}}$. No resonance at real frequencies ω, because the equation has $-ky$ instead of ky.

Problem Set 2.2, page 87

1 Mark the numbers $s_1 = 2+i$ and $s_2 = 1-2i$ as points in the complex plane. (The plane has a real axis and an imaginary axis.) Then mark the sum $s_1 + s_2$ and the difference $s_1 - s_2$.

Solution The sum is $s_1 + s_2 = 3 - i$. The difference is $s_1 - s_2 = 1 + 3i$.

2 Multiply $s_1 = 2 + i$ times $s_2 = 1 - 2i$. Check absolute values: $|s_1||s_2| = |s_1 s_2|$.

Solution The product $(2 + i)(1 - 2i)$ is $2 + i - 4i - 2i^2 = 4 - 3i$. The absolute values of $2 + i$ and $1 - 2i$ are $\sqrt{2^2 + 1^2} = \sqrt{5}$. The product $4 - 3i$ has absolute value $\sqrt{4^2 + 3^2} = 5$, agreeing with $(\sqrt{5})(\sqrt{5})$.

3 Find the real and imaginary parts of $1/(2 + i)$. Multiply by $(2 - i)/(2 - i)$:

$$\frac{1}{2+i} \quad \frac{2-i}{2-i} = \frac{2-i}{|2+i|^2} = ?$$

Solution $\dfrac{1}{2+i} \quad \dfrac{2-i}{2-i} = \dfrac{2-i}{5}$ In general $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$ because $z\overline{z} = |z|^2$.

4 *Triple angles* Multiply equation (2.10) by another $e^{i\theta} = \cos\theta + i\sin\theta$ to find formulas for $\cos 3\theta$ and $\sin 3\theta$.

Solution Equation (10) is $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$. Multiply by another $\cos\theta + i\sin\theta$:

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos\theta\cos 2\theta + i\sin\theta\cos 2\theta + i\cos\theta\sin 2\theta - \sin\theta\sin 2\theta \\ &= \cos(\theta + 2\theta) + i\sin(\theta + 2\theta) \ \text{ by sum formulas} \\ &= \cos 3\theta + i\sin 3\theta\end{aligned}$$

Real part $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **Imaginary part** $\sin 3\theta = 3\cos^2\theta\sin\theta - sin^3\theta$.

5 *Addition formulas* Multiply $e^{i\theta} = \cos\theta + i\sin\theta$ times $e^{i\phi} = \cos\phi + i\sin\phi$ to get $e^{i(\theta+\phi)}$. Its real part is $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. What is its imaginary part $\sin(\theta + \phi)$?

Solution The imaginary part of $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$ is the coeffieient of i:

$\sin\theta\cos\phi + \cos\theta\sin\phi$ must equal $\sin(\theta + \phi)$.

6 Find the real part and the imaginary part of each cube root of 1. Show directly that the three roots add to zero, as equation (2.11) predicts.

Solution The cube roots of 1 are at angles $0, 2\pi/3, 4\pi/3$ (or $0°, 120°, 240°$). They are equally spaced on the unit circle (absolute value 1). The three roots are 1 and

$e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \boldsymbol{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \boldsymbol{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

The sum $1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}$ equals **zero**. Always: n roots of $2^n = 1$ add to zero.

7 The three cube roots of 1 are z and z^2 and 1, when $z = e^{2\pi i/3}$. What are the three cube roots of 8 and the three cube roots of i? (The angle for i is 90° or $\pi/2$, so the angle for one of its cube roots will be ____. The roots are spaced by 120°.)

Solution The three cube roots of 8 are 2 and $2e^{2\pi i/3} = -1 + \sqrt{3}i$ and $2e^{4\pi i/3} = -1 - \sqrt{3}i$. (They also add to zero.)

The three cube roots of $i = e^{\pi i/2}$ are $e^{\pi i/6}$ and $e^{5\pi i/6}$ and $e^{9\pi i/6}$ still add to zero.

8 (a) The number i is equal to $e^{\pi i/2}$. Then its i^{th} power $\boldsymbol{i^i}$ comes out equal to a real number, using the fact that $(e^s)^t = e^{st}$. What is that real number $\boldsymbol{i^i}$?

(b) $e^{i\pi/2}$ is also equal to $e^{5\pi i/2}$. Increasing the angle by 2π does not change $e^{i\theta}$ — it comes around a full circle and back to i. Then i^i has another real value $(e^{5\pi i/2})^i = e^{-5\pi/2}$. What are all the possible values of i^i?

Solution (a) The i^{th} power of $i = e^{\pi i/2}$ is $i^i = (e^{\pi i/2})^i = e^{-\pi/2}$ by the ordinary rule for exponents. Surprising that i^i is a real number.

(b) i also equals $e^{5\pi i/2}$ since $\frac{5\pi}{2}$ is a full rotation from $\frac{\pi}{2}$. So i^i also equals $(e^{5\pi i/2})^i = e^{-5\pi/2}$—and infinitely many other possibilities $e^{-(2\pi+1)\pi/2}$ for every whole number n. We are on a "Riemann surface" with an infinity of layers.

9 The numbers $s = 3 + i$ and $\overline{s} = 3 - i$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$ and also $\overline{s}^2 + B\overline{s} + C = 0$. Those numbers s and $\overline{s}$ are the two roots of the quadratic equation $x^2 + Bx + C = 0$.

Solution $-B = s + \overline{s} = (3 + i) + (3 - i) = \mathbf{6}$. $C = (s)(\overline{s}) = (3 + i)(3 - i) = \mathbf{10}$.

Then s and $\overline{s}$ are the two roots of $x^2 - Bx + C = x^2 - 6x + 10 = 0$. The usual quadratic formula gives $\frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$.

10 The numbers $s = a + i\omega$ and $\overline{s} = a - i\omega$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$. The two solutions of $x^2 + Bx + C = 0$ are s and $\overline{s}$.

Solution $-B = (a + i\omega) + (a - i\omega) = \boldsymbol{2a}$ $\quad C = (a + i\omega)(a - i\omega) = \boldsymbol{a^2 + i\omega^2}$.

Then the roots of $x^2 - 2ax + a^2 + \omega^2 = 0$ are $x = \frac{2a \pm \sqrt{-4\omega^2}}{2} = a \pm i\omega$.

11 (a) Find the numbers $(1 + i)^4$ and $(1 + i)^8$.

(b) Find the polar form $re^{i\theta}$ of $(1 + i\sqrt{3})/(\sqrt{3} + i)$.

Solution (a) $(1 + i)^4 = (\sqrt{2}e^{i\pi/4})^4 = (\sqrt{2})^4 e^{i\pi} = \mathbf{-4}$

$(1 + i)^8 =$ square of $(1 + i)^4 =$ (square of -4) $= \mathbf{16}$.

(b) $(1 + i\sqrt{3})(\sqrt{3} + i) = \sqrt{3} + 3i + i - \sqrt{3} = \boldsymbol{4i}$. Dividing by $(2)(2) = 4$ this is

$(\cos\theta + i\sin\theta)(\sin\theta + i\cos\theta) = i(\cos^2\theta + \sin^2\theta) = \boldsymbol{i}$.

The unexpected part is $\boldsymbol{\sin\theta + i\cos\theta} = \cos(\frac{\pi}{2} - \theta) + i\sin(\frac{\pi}{2} - \theta) = \boldsymbol{e^{i(\pi/2-\theta)}}$.

Then the product of $e^{i\theta}$ and $e^{i(\pi/2-\theta)}$ is $e^{i\pi/2}$ which equals $\boldsymbol{i}$ as above.

12 The number $z = e^{2\pi i/n}$ solves $z^n = 1$. The number $Z = e^{2\pi i/2n}$ solves $Z^{2n} = 1$. How is z related to Z ? (This plays a big part in the Fast Fourier Transform.)

Solution If $Z = e^{2\pi i/2n}$ then $Z^2 = e^{2\pi i/n} = z$. The square of the $2n$ th root is the n th root. The angle for Z is half the angle for z.

The Fast Fourier Transform connects the transform at level $2n$ to the transform at level n (and on down to $n/2$ and $n/4$ and eventually to 1, if these numbers are powers of 2).

13 (a) If you know $e^{i\theta}$ and $e^{-i\theta}$, how can you find $\sin\theta$?

(b) Find all angles θ with $e^{i\theta} = -1$, and all angles ϕ with $e^{i\phi} = i$.

Solution (a) $\sin\theta = \frac{1}{2i}[(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)] = \boldsymbol{\frac{1}{2i}(e^{i\theta} - e^{-i\theta})}$.

(b) The angles with $e^{i\theta} = -1$ are $\theta = \pi +$ (any multiple of 2π) $= \boldsymbol{(2n+1)\pi}$.

The angles with $e^{i\phi} = 1$ are $\phi =$ any multiple of $2\pi = \boldsymbol{2n\pi}$.

14 Locate all these points on one complex plane :

(a) $2 + i$ (b) $(2 + i)^2$ (c) $\dfrac{1}{2+i}$ (d) $|2 + i|$

Solution $2 + i$ is in quadrant 1. $(2 + i)^2$ is in quadrant 2. $\frac{1}{2+i}$ is in quadrant 4. $|2 + i| = \sqrt{5}$ is on the positive real axis.

15 Find the absolute values $r = |z|$ of these four numbers. If θ is the angle for $6 + 8i$, what are the angles for these four numbers ?

(a) $6 - 8i$ (b) $(6 - 8i)^2$ (c) $\dfrac{1}{6 - 8i}$ (d) $8i + 6$

Solution The absolute values are 10 and 100 and $\frac{1}{10}$ and 10.

The angles are $2\pi - \theta$ (or just $-\theta$), $2\pi - 2\theta$ (or just -2θ), θ, and θ.

16 What are the real and imaginary parts of $e^{a + i\pi}$ and $e^{a + i\omega}$?

Solution $e^{a+i\pi} = e^a e^{i\pi} = -e^{-a}(real)$ $\quad e^{a+i\omega} = e^a \cos\omega + ie^a \sin\omega$

17 (a) If $|s| = 2$ and $|z| = 3$, what are the absolute values of sz and s/z ?

(b) Find upper and lower bounds in $L \le |s + z| \le U$. When does $|s + z| = U$?

Solution (a) $|sz| = |s|\,|z| = 6 \qquad |s/z| = |s|/|z| = 2/3$.

(b) The best bounds are $L = 1$ and $U = 5 : 1 \le |s + z| \le 5$.

That bound 5 is reached when s and z have the **same angle** θ.

18 (a) Where is the product $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta)$ in the complex plane ?

(b) Find the absolute value $|S|$ and the polar angle ϕ for $S = \boldsymbol{\sin\theta + i\cos\theta}$.

This is my favorite problem, because S combines $\cos\theta$ and $\sin\theta$ in a new way. To find ϕ, you could plot S or add angles in the multiplication of part (a).

Solution $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta) = \sin\theta\cos\theta + i(\sin^2\theta + \cos^2\theta) - \cos\theta\sin\theta = \boldsymbol{i}$. The product is imaginary. The angles must add to $90\,°$.

Since $\cos\theta + i\sin\theta$ is at angle θ and the product i is at angle $\pi/2$, the first factor $\sin\theta + i\cos\theta$ must be $e^{i\phi}$ at angle $\boldsymbol{\phi = \frac{\pi}{2} - \theta}$. The absolute value is $\mathbf{1}$. See also Problem 2.2.11.

19 Draw the spirals $e^{(1\,-\,i)t}$ and $e^{(2\,-\,2i)t}$. Do those follow the same curves? Do they go clockwise or anticlockwise? When the first one reaches the negative x-axis, what is the time T? What point has the second one reached at that time?

Solution The spiral $e^{(1-i)t} = e^t e^{-it}$ starts at 1 when $t = 0$. As t increases, it goes outward (absolute value e^t) and clockwise (the angle is $-t$). It reaches the negative X axis when $\boldsymbol{t = \pi}$. The second spiral $e^{(2-2i)t}$ is **the same curve** but traveled twice as fast. Its angle $-2t$ reaches $-\pi$ (the X-axis) at time $t = \pi/2$.

20 The solution to $d^2y/dt^2 = -y$ is $y = \cos\, t$ if the initial conditions are $y(0) =$ _____ and $y'(0) =$ _____. The solution is $y = \sin\, t$ when $y(0) =$ _____ and $y'(0) =$ _____. Write each of those solutions in the form $c_1\, e^{it} + c_2\, e^{-it}$, to see that real solutions can come from complex c_1 and c_2.

Solution $y = \cos t$ has $y(0) = 1$ and $y'(0) = 0$. $y = \sin t$ has $y(0) = 0$ and $y'(0) = 1$. Those solutions are $\cos t = (e^{it} + e^{-it})/2$ and $\sin t = (e^{it} - e^{-it})/2i$.

The complete solution to $y'' = -y$ is $y = C_1 \cos t + C_2 \sin t$. The same complete solution is $C_1(e^{it} + e^{-it})/2 + C_2(e^{it} - e^{-it})/2i = c_1 e^{it} + c_2 e^{-it}$ with $c_1 = (C_1 + C_2)/2$ and $c_2 = (C_1 - C_2)/2i$.

21 Suppose $y(t) = e^{-t}\, e^{it}$ solves $y'' + By' + Cy = 0$. What are B and C? If this equation is solved by $y = e^{3it}$, what are B and C?

Solution If $y = e^{st}$ solves $y'' + By' + Cy = 0$ then substituting e^{st} shows that $s^2 + Bs + C = 0$. This problem has $s = -1 + i$. Then the other root is the conjugate $\overline{s} = -1 - i$ (always assuming B and C are real numbers). The sum -2 is $-B$. The product $(s)(\overline{s}) = 2$ is C. So the underlying equation is $y'' + 2y' + 2y = 0$.

22 From the multiplication $e^{iA}\;\; e^{-iB} = e^{i(A\,-\,B)}$, find the "subtraction formulas" for $\cos\,(A - B)$ and $\sin\,(A - B)$.

Solution Start with the fact that $e^{iA}e^{-iB} = e^{i(A-B)}$. Use Euler's formula:

$$(\cos A + i \sin A)(\cos B - i \sin B) = \cos(A - B) + i \sin(A - B).$$

Compare real parts: $\cos A \cos B + \sin A \sin B = \cos(A - B)$.

Compare imaginary parts: $\sin A \cos B - \cos A \sin B = \sin(A - B)$.

23 (a) If r and R are the absolute values of s and S, show that rR is the absolute value of sS. (Hint: Polar form!)

(b) If $\overline{s}$ and $\overline{S}$ are the complex conjugates of s and S, show that $\overline{s}\overline{S}$ is the complex conjugate of sS. (Polar form!)

Solution (a) Given: $s = re^{i\theta}$ and $S = Re^{i\phi}$ for some angles θ and ϕ. Then $\boldsymbol{sS = rRe^{i(\theta+\phi)}}$. The absolute value of sS is $rR =$ (absolute value of s) (absolute value of S).

(b) Now $\overline{s} = re^{-i\theta}$ and $\overline{S} = Re^{-i\phi}$. Multiply to get $\overline{s}\overline{S} = rRe^{-i(\theta+\phi)}$. This is the complex conjugate of $sS = rRe^{i(\theta+\phi)}$ in part (a).

24 Suppose a complex number s solves a real equation $s^3 + As^2 + Bs + C = 0$ (with A, B, C real). Why does the complex conjugate $\overline{s}$ also solve this equation? "*Complex solutions to real equations come in conjugate pairs s and $\overline{s}$.*"

Solution The complex conjugate of $s^3 + As^2 + Bs + C = 0$ is $\overline{s}^3 + A\overline{s}^2 + B\overline{s} + C = 0$.

We took the conjugate of every term using the fact that A, B, C are real. (The conjugates of s^2 and s^3 are $\overline{s}^2$ and $\overline{s}^3$ by Problem 23).

For quadratic equations $x^2 + Bx + C = 0$, the formula $(-B \pm \sqrt{B^2 - 4C})/2$ is producing **complex conjugates from** $\pm$ when $B^2 - 4C$ is negative.

25 (a) If two complex numbers add to $s + S = 6$ and multiply to $sS = 10$, what are s and S ? (They are complex conjugates.)

(b) If two numbers add to $s + S = 6$ and multiply to $sS = -16$, what are s and S ? (Now they are real.)

Solution (a) s and S must have the same real part 3. They each have magnitude $\sqrt{10}$. So s and S are $3 + i$ and $3 - i$.

(b) If $s + S = 6$ and $sS = -16$ then s and S are the roots of $x^2 - 6x - 16 = 0$. Factor into $(x - 8)(x + 2) = 0$ to see that s and S are 8 and -2. (Not complex conjugates! In this example $B^2 - 4AC = 36 + 64 = 100$ and the quadratic has real roots 8 and -2.)

26 If two numbers s and S add to $s + S = -B$ and multiply to $sS = C$, show that s and S solve the quadratic equation $s^2 + Bs + C = 0$.

Solution Just check that $(x - s)(x - S) = x^2 + Bx + C$. The left side is $x^2 - (s + S)x + sS$. Then $s + S$ agrees with $-B$ and sS matches C.

27 Find three solutions to $s^3 = -8i$ and plot the three points in the complex plane. What is the sum of the three solutions ?

Solution The three solutions have the same absolute value 2. Their angles are separated by $120^\circ = 2\pi/3$ radians $= 4\pi/6$ radians. The first angle is $\theta = -30^\circ = -\pi/6$ radians (so that $3\theta = -90^\circ = -\pi/2$ radians matches $-i$).

The answers are $\boldsymbol{2e^{-\pi i/6}}$, $\boldsymbol{2e^{3\pi i/6}}$, $\boldsymbol{2e^{7\pi i/6}}$. They add to $\mathbf{0}$.

28 (a) For which complex numbers $s = a + i\omega$ does e^{st} approach 0 as $t \to \infty$? Those numbers s fill which "half–plane" in the complex plane ?

(b) For which complex numbers $s = a + i\omega$ does s^n approach 0 as $n \to \infty$? Those numbers s fill which part of the complex plane ? Not a half-plane !

Solution (a) If $s = a + i\omega$, the absolute value of e^{st} is e^{at}. This approaches 0 if a is **negative**. The numbers $s = a + i\omega$ with negative a fill the **left half-plane**.

(b) This part asks about the powers s^n instead of e^{st}. Powers of s approach zero if $|\boldsymbol{s}| < \mathbf{1}$. This is the same as $\boldsymbol{a^2 + \omega^2 < 1}$. These complex numbers fill the **inside of the unit circle**.

Problem Set 2.3, page 101

1 Substitute $y = e^{st}$ and solve the characteristic equation for s :

(a) $2y'' + 8y' + 6y = 0$ (b) $y'''' - 2y'' + y = 0$.

Solution (a) $2s^2 + 8s + 6$ factors into $2(s + 3)(s + 1)$ so the roots are $s = -3$ and $s = -1$. The null solutions are $y = e^{-3t}$ and $y = e^{-t}$ (and any combination).

(b) $s^4 - 2s^2 + 1$ factors into $(s^2 - 1)^2$ which is $(s - 1)^2(s + 1)^2$. The roots are $s = 1, 1, -1, -1$. The null solutions are $y = c_1e^t + c_2te^t + c_3e^{-t} + c_4te^{-t}$. (The factor t enters for double roots.)

2 Substitute $y = e^{st}$ and solve the characteristic equation for $s = a + i\omega$:

(a) $y'' + 2y' + 5y = 0$ (b) $y'''' + 2y'' + y = 0$

Solution (a) $s^2 + 2s + 5 = 0$ gives $s = (-2 \pm \sqrt{4 - 20})/2 = -1 \pm 2i = a + i\omega$. Then $y = e^{-t} \cos 2t$ and $y = e^{-t} \sin 2t$ solve the (null) equation.

(b) $s^4 + 2s^2 + 1 = 0$ factors into $(s^2 + 1)(s^2 + 1) = 0$. The roots are $i, i, -i, -i$. The solutions are $y = c_1 e^{it} + c_2 t e^{it} + c_3 e^{-it} + c_4 t e^{-it}$. They can also be written as $y = C_1 \cos t + C_2 t \cos t + C_3 \sin t + C_4 t \sin t$.

3 Which second order equation is solved by $y = c_1 e^{-2t} + c_2 e^{-4t}$? Or $y = te^{5t}$?

Solution If $s = -2$ and $s = 4$ are the exponents, the characteristic equation must be $s^2 + 6s + 8 = 0$ coming from $y'' + 6y' + 8y = 0$.

If $y = te^{5t}$ is a solution, then 5 is a **double root**. The characteristic equation must be $(s - 5)^2 = s^2 - 10s + 25 = 0$ coming from $y'' - 10y' + 25y = 0$.

4 Which second order equation has solutions $y = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t$?

Solution Those sine/cosine solutions combine to give $e^{-2t}e^{3it}$ and $e^{-2t}e^{-3it}$. Then $\boldsymbol{s = -2 \pm 3i}$. The sum is -4 and 4, the product is $2^2 + 3^2 = 13$.

The equation must be $\boldsymbol{y'' - 4y' + 13y = 0}$.

5 Which numbers B give (under) (critical) (over) damping in $4y'' + By' + 16y = 0$?

Solution The roots of $4s^2 + Bs + 16$ are $s = (-B \pm \sqrt{B^2 - 16^2})/2$. We have underdamping for $B^2 > 16^2$ (real roots); critical damping for $B^2 = 16^2$ (double root); overdamping for $B^2 < 16^2$ (complex roots).

6 If you want oscillation from $my'' + by' + ky = 0$, then b must stay below _____ .

Solution Oscillations mean underdamping. We need $\boldsymbol{b^2 < 4km}$.

Problems 7–16 are about the equation $\boldsymbol{As^2 + Bs + C = 0}$ and the roots $\boldsymbol{s_1, s_2}$.

7 The roots s_1 and s_2 satisfy $s_1 + s_2 = -2p = -B/2A$ and $s_1 s_2 = \omega_n^2 = C/A$. Show this two ways :

(a) Start from $As^2 + Bs + C = A(s - s_1)(s - s_2)$. Multiply to see $s_1 s_2$ and $s_1 + s_2$.

(b) Start from $s_1 = -p + i\omega_d, s_2 = -p - i\omega_d$

Solution (a) Match $As^2 + Bs + C$ to $A(s - s_1)(s - s_2) = As^2 - A(s_1 + s_2)s + As_1 s_2$. Then $-B = A(s_1 + s_2)$ and $C = As_1 s_2$. **Error in problem**: $s_1 + s_2$ equals $-B/A$ and not $-B/2A$.

(b) $s_1 + s_2 = (-p + i\omega_d) + (-p - i\omega_d) = -2p = \boldsymbol{-B/A}$. Then $p = B/2A$.

8 Find s and y at the bottom point of the graph of $y = As^2 + Bs + C$. At that minimum point $s = s_{\min}$ and $y = y_{\min}$, the slope is $dy/ds = 0$.

Solution The minimum of $As^2 + Bs + C$ is located by derivative $= 2As + B = 0$. Then $s = -B/2A$ (which is p). The value of $As^2 + Bs + C$ at that minimum point is $A(B^2/4A^2) - (B^2/2A) + C = -(B^2/4A) + C = (4AC - B^2)/4A$.

Notice: If $B^2 < 4AC$ the minimum is > 0. Then $As^2 + Bs + C \neq 0$ for real s.

9 The parabolas in Figure 2.10 show how the graph of $y = As^2 + Bs + C$ is raised by increasing B. Using Problem 8, show that the bottom point of the graph moves left (change in s_{min}) and down (change in y_{min}) when B is increased by ΔB.

Solution For the graph of $y = As^2+Bs+C$, the bottom point is $y = (4AC-B^2)/4A$ at $s = -B/2A$. When B is increased, s moves left and y moves down. (The convention is $A > 0$.)

10 (recommended) Draw a picture to show the paths of s_1 and s_2 when $s^2 + Bs + 1 = 0$ and the damping increases from $B = 0$ to $B = \infty$. At $B = 0$, the roots are on the _____ axis. As B increases, the roots travel on a circle (why ?). At $B = 2$, the roots meet on the real axis. For $B > 2$ the roots separate to approach 0 and $-\infty$. *Why is their product $s_1 s_2$ always equal to* 1 ?

Solution The roots of $s^2 + Bs + 1$ will move as B increases from 0 to ∞. At $B = 0$, the roots of $s^2 + 1 = 0$ are **imaginary**: $s = \pm i$. As B increases, the roots are complex conjugates always multiplying to $s_1 s_2 = 1$. They are on the **unit circle**. When B reaches 2, the roots of $s^2 + 2s + 1 = (s + 1)^2$ meet at $s = -1$. (Each root traveled a quarter-circle, from $\pm i$ to -1.) For larger B and overdamping $B^2 > 4AC = 4(1)(1)$, the roots $s_1 s_2$ are **real**. One root moves from -1 toward $s = 0$, the other moves from -1 toward $-\infty$. **At all times** $s_1 s_2 = C/A = 1/1$.

11 (this too if possible) Draw the paths of s_1 and s_2 when $s^2 + 2s + k = 0$ and the stiffness increases from $k = 0$ to $k = \infty$. When $k = 0$, the roots are _____ . At $k = 1$, the roots meet at $s =$ _____. For $k \to \infty$ the two roots travel up/down on a _____ in the complex plane. *Why is their sum $s_1 + s_2$ always equal to* -2 ?

Solution This problem changes k in $s^2 + 2s + k = 0$. So the **sum** $s_1 + s_2$ stays at -2, the **product** $s_1 s_2 = k/1$ increases from 0 to ∞.

When $k = 0$, the roots -2 and 0 are **real**. When $k = 1$, the roots are -1 and -1 (**repeated**). When $k \to \infty$, then $B^2 - 4AC = 4 - 4k$ is negative and the roots $s = -1 \pm i\omega$ are **complex conjugates**. They lie on the vertical line $x = \text{Re}\, s = -1$ in the complex plane.

12 If a polynomial $P(s)$ has a double root at $s = s_1$, then $(s - s_1)$ is a double factor and $P(s) = (s - s_1)^2 Q(s)$. Certainly $P = 0$ at $s = s_1$. Show that also $dP/ds = 0$ at $s = s_1$. Use the product rule to find dP/ds.

Solution $P = (s - s_1)^2 Q(s)$ has a double root $s = s_1$, together with the roots of $Q(s)$. The derivative is

$$\frac{dP}{ds} = (s - s_1)^2 \frac{dQ}{ds} + 2(s - s_1)Q(s). \text{ This is zero at } s = s_1.$$

13 Show that $y'' = 2ay' - (a^2 + \omega^2)y$ leads to $s = a \pm i\omega$. Solve $y'' - 2y' + 10y = 0$.

Solution Substitute $y = e^{st}$ in the differential equation. Cancel e^{st} from every term to leave $s^2 = 2as - (a^2 + \omega^2)$.

The roots are $a \pm i\omega$, their sum is $2a$, their product is $a^2 + \omega^2$.

For $y'' - 2y' + 10y = 0$ (negative damping!) the sum is $s_1 + s_2 = 2$ and the product is 10. The roots are $\boldsymbol{s = 1 \pm 3i}$. The solution $y(t)$ is $c_1 e^{(1+3i)t} + c_2 e^{(1-3i)t}$.

14 The undamped *natural frequency* is $\omega_n = \sqrt{k/m}$. The two roots of $ms^2 + k = 0$ are $s = \pm i\omega_n$ (pure imaginary). With $p = b/2m$, the roots of $ms^2 + bs + k = 0$ are $\boldsymbol{s_1, s_2 = -p \pm \sqrt{p^2 - \omega_n^2}}$. The coefficient $p = b/2m$ has the units of $1/\text{time}$.

Solve $s^2 + 0.1s + 1 = 0$ and $s^2 + 10s + 1 = 0$ with numbers correct to two decimals.

Solution $s^2 + 0.1s + 1 = 0$ gives $s = (-0.1 \pm \sqrt{0.01 - 4})/2 = (-0.1 \pm i\sqrt{3.99})/2$.

How to approximate that square root?

The square root of $4 - x$ is close to $2 - \frac{1}{4}x$. Computing $(2 - \frac{1}{4}x)^2 = 4 - x + x^2/16$ we see the small error $x^2/16$. Our problem has $4 - x = 3.99$ and $x = 1/100$. So the square root is close to $2 - \frac{1}{400}$. The roots are $s \approx \left(-0.1 \pm i\left(2 - \frac{1}{400}\right)\right)/2$. In other words $s = \boldsymbol{-0.05 + i(1 - 0.00125)}$.

For $s^2 + 10s + 1 = 0$, the roots are $s = (-10 \pm \sqrt{(100 - 4)})/2 = -5 \pm \sqrt{25 - 1}$. The square root of $25 - x$ is close to $5 - \frac{1}{10}x$, because squaring the approximation gives $25 - x + (x^2/100)$. Our example has $x = 1$ and $s \approx -5 \pm (5 - \frac{1}{10})$, which gives the two approximate roots $s = -\frac{1}{10}$ and $-10 + \frac{1}{10}$.

These add to -10 (correct) and multiply to .99 (almost correct).

15 With large overdamping $\boldsymbol{p >> \omega_n}$, the square root $\sqrt{p^2 - \omega_n^2}$ is close to $p - \omega_n^2/2p$. Show that the roots of $ms^2 + bs + k$ are $s_1 \approx \boldsymbol{-\omega_n^2/2p}$ = (small) and $s_2 \approx -2p = \boldsymbol{-b/m}$ (large).

Solution Use that approximate square root $p - \omega_n^2/2p$ in the quadratic formula:

$$s = -p \pm \sqrt{p^2 - \omega_n^2} \approx -p \pm \left(p - \frac{\omega_n^2}{2p}\right). \text{ Then } s = -\frac{\omega_n^2}{2p} \text{ and } -2p + \frac{\omega_n^2}{2p}.$$

When p is large and ω_n is small, a small root is near $-\omega_n^2/2p$ and a large root is near $-2p$. (Their product is the correct ω_n^2, their sum is close to the correct $-2p$.)

16 With small underdamping $\boldsymbol{p << \omega_n}$, the square root of $p^2 - \omega_n^2$ is approximately $i\omega_n - ip^2/2\omega_n$. Square that to come close to $p^2 - \omega_n^2$. Then the frequency for small underdamping is reduced to $\omega_d \approx \omega_n - p^2/2\omega_n$.

Solution Now p is much **smaller** than ω_n. So the roots $s = -p \pm \sqrt{p^2 - \omega_n^2}$ are complex. The damped frequency $\omega_d = \sqrt{\omega_n^2 - p^2}$ is close to ω_n and the correction term is $-p^2/2\omega_n$ from the approximation $\omega_n - p^2/2\omega_n$ to the square root. (Square that approximation to see $\omega_n^2 - p^2 + (p^4/4\omega_n^2)$.

17 Here is an 8th order equation with eight choices for solutions $y = e^{st}$:

$$\frac{d^8y}{dt^8} = y \quad \text{becomes} \quad s^8 e^{st} = e^{st} \quad \text{and} \quad \boldsymbol{s^8 = 1} \ : \quad \text{Eight roots in Figure 2.6.}$$

Find two solutions e^{st} that don't oscillate (s is real). Find two solutions that only oscillate (s is imaginary). Find two that spiral in to zero and two that spiral out.

Solution The equation $s^8 = 1$ has 8 roots. Two of them are $s = 1$ and $s = -1$ (**real**: no oscillation). Two are $s = i$ and $s = -i$ (**imaginary**: pure oscillation). Two are $s = e^{2\pi i/8}$ and $s = e^{-2\pi i/8}$ (positive real parts cos $\frac{\pi}{4}$: (oscillating growth, spiral out). Two are $s = e^{3\pi i/4}$ and $s = e^{-3\pi i/4}$ (negative real parts: oscillating decay, spiral in).

18 $A_n \dfrac{d^n y}{dt^n} + \cdots + A_1 \dfrac{dy}{dt} + A_0 y = 0$ leads to $\boldsymbol{A_n s^n + \cdots + A_1 s + A_0 = 0}$.

The n roots $s_1, \ldots, s_n$ produce n solutions $y(t) = e^{st}$ (if those roots are distinct). Write down n equations for the constants c_1 to c_n in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$ by matching the n initial conditions for $y(0), y'(0), \ldots, D^{n-1}y(0)$.

Solution The n roots give n solutions $y = e^{st}$ (when the roots s are all different). There are n constants in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$. These constants are found by matching the n initial conditions $y(0), y'(0), \ldots$ **Take derivatives of y and set $t = 0$:**

$$\begin{aligned} c_1 + c_2 + \cdots + c_n &= y(0) \\ c_1 s_1 + c_2 s_2 + \cdots + c_n s_n &= y'(0) \\ c_1 s_1^2 + c_2 s_2^2 + \cdots + c_n s_n^2 &= y''(0) \\ \cdots &= \cdots \end{aligned}$$

The n by n matrix A in those equations is the transpose of a **Vandermonde matrix**:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ \cdot & \cdot & \cdots & \cdot \end{bmatrix}$$

19 **Find two solutions to $d^{2015}y/dt^{2015} = dy/dt$.** Describe all solutions to $s^{2015} = s$.

Solution With $y = e^{st}$ we find $\boldsymbol{s^{2015} = s}$. One solution has $s = 1$ and $y = e^t$. The other 2014 solutions have $\boldsymbol{s^{2014} = 1}$ ($s = 1$ is double! Second solution $y = te^t$.) The 2014 values of s are equally spaced around the unit circle, separated by the angle $2\pi/2014$.

20 The solution to $y'' = 1$ starting from $y(0) = y'(0) = 0$ is $y(t) = t^2/2$. The fundamental solution to $g'' = \delta(t)$ is $g(t) = t$ by Example 5. Does the integral $\int g(t-s)f(s)ds = \int (t-s)ds$ from 0 to t give the correct solution $y = t^2/2$?

Solution The main formula for a particular solution is correct:

$$y_p(t) = \int_0^t g(t-s)f(s)\,ds = \int_0^t (t-s)\,ds = -\left.\frac{(t-s)^2}{2}\right]_{s=0}^{t} = \frac{t^2}{2}.$$

21 The solution to $y'' + y = 1$ starting from $y(0) = y'(0) = 0$ is $y = 1 - \cos t$. The solution to $g'' + g = \delta(t)$ is $\boldsymbol{g(t) = \sin t}$ by equation (13) with $\omega = 1$ and $A = 1$. Show that $1 - \cos t$ agrees with the integral $\int g(t-s)f(s)ds = \int \sin(t-s)ds$.

Solution The formula for a particular solution is again correct:

$$y_p(t) = \int_0^t g(t-s)f(s)\,ds = \int_0^t \sin(t-s)\,ds = \cos(t-s)]_{s=0}^{t} = 1 - \cos t.$$

Then $y_p'' + y_p = 1$.

22 The step function $H(t) = 1$ for $t \geq 0$ is the integral of the delta function. **So the step response $r(t)$ is the integral of the impulse response.** This fact must also come from our basic solution formula:

$$Ar'' + Br' + Cr = 1 \text{ with } r(0) = r'(0) = 0 \text{ has } \boldsymbol{r(t) = \int_0^t g(t-s)\,1\,ds}$$

Change $t - s$ to τ and change ds to $-d\tau$ to confirm that $r(t) = \int_0^t g(\tau)d\tau$.

Section 2.5 will find two good formulas for the step response $r(t)$.

Solution For any equation $Ar'' + Br' + Cr = 1$ with $f(t) = 1$, y_p comes from the integral formula:

$$y_p = \int_0^t g(t-s)f(s)\,ds = \int_0^t g(t-s)\,ds. \text{ Change to } t-s=\tau \text{ and } -ds = d\tau \text{ and}$$

$$-\int_t^0 g(\tau)d\tau = +\int_0^t \boldsymbol{g(\tau)d\tau} = \text{ step response}$$

Problem Set 2.4, page 114

Problems 1-4 use the exponential response $y_p = e^{ct}/P(c)$ to solve $P(D)y = e^{ct}$.

1 Solve these constant coefficient equations with exponential driving force:

(a) $y_p'' + 3y_p' + 5y_p = e^t$ (b) $2y_p'' + 4y_p = e^{it}$ (c) $y'''' = e^t$

Solution (a) Substitute $y = Ye^t$ to find Y:

$Ye^t + 3Ye^t + 5Ye^t = e^t$ gives $9Y = 1$ and $Y = 1/9$: $y = e^t/9$

(b) Substitute $y = Ye^{it}$: $2i^2Ye^{it} + 4Ye^{it} = e^{it}$: $2Y = 1$: $y = e^{it}/2$

(c) Substitute $y = Ye^t$ to find $Y = 1$ and $y = e^t$.

2 These equations $P(D)y = e^{ct}$ use the symbol D for d/dt. Solve for $y_p(t)$:

(a) $(D^2 + 1)y_p(t) = 10e^{-3t}$ (b) $(D^2 + 2D + 1)y_p(t) = e^{i\omega t}$

(c) $(D^4 + D^2 + 1)y_p(t) = e^{i\omega t}$

Solution (a) Substitute $y = Ye^{-3t}$ to find $9Y + Y = 10$: $Y = 1$ and $y = e^{-3t}$.

(b) Substitute $y = Ye^{i\omega t}$ to find $((i\omega)^2 + 2i\omega + 1)Y = 1$ and $Y = 1/(1 - \omega^2 + 2i\omega)$.

(c) Substitute $y = Ye^{i\omega t}$ to find $((i\omega)^4 + (i\omega)^2 + 1)Y = 1$ and $Y = 1/(1 - \omega^2 + \omega^4)$.

3 How could $y_p = e^{ct}/P(c)$ solve $y'' + y = e^t e^{it}$ and then $y'' + y = e^t \cos t$?

Solution First, $y''+y = e^{(1+i)t}$ has $c = 1+i$ and $y = Ye^{ct} = e^{(1+i)t}/((1+i)^2+1) = e^t e^{it}/(1+2i)$. The **real part** of that y solves the equation driven by $e^t \cos t$:

$$y = \text{Re}\left[e^t(\cos t + i\sin t)\left(\frac{1-2i}{1^2+2^2}\right)\right] = \frac{1}{5}e^t(\cos t + 2\sin t).$$

4 (a) What are the roots s_1 to s_3 and the null solutions to $y_n''' - y_n = 0$?

(b) Find particular solutions to $y_p''' - y_p = e^{it}$ and to $y_p''' - y_p = e^t - e^{i\omega t}$.

Solution (a) $y = e^{st}$ leads to $s^3 - 1 = 0$. The three roots $s = 1$, $s = e^{2\pi i/3} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, $s = e^{-2\pi i/3} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$ give three null solutions $y_n = e^t, e^{-t/2}\cos\frac{\sqrt{3}}{2}t, e^{-t/2}\sin\frac{\sqrt{3}}{2}t$.

(b) The particular solution with $f = e^{it}$ is $y_p = e^{it}/(i^3 - 1)$.

The particular solution with $f = e^t - e^{i\omega t}$ looks like $y = e^t/(1^3 - 1) - e^{i\omega t}((i\omega)^3 - 1)$. But the first part has $1^3 - 1 = 0$ and resonance: then $e^t/(1^3 - 1)$ changes by equation (19) to $te^t/3$: (The differential equation has $y''' - y = (D^3 - 1)y = P(D)y$ and is $P'(D) = 3D^2$ and $P'(c) = 3$ because e^t has $c = 1$.)

Problems 5-6 involve repeated roots s in y_n and resonance $P(c) = 0$ in y_p.

5 Which value of C gives resonance in $y'' + Cy = e^{i\omega t}$? Why do we never get resonance in $y'' + 5y' + Cy = e^{i\omega t}$?

Solution $y'' + Cy = e^{i\omega t}$ has resonance when $e^{i\omega t}$ solves the null equation, so $(i\omega)^2 + C = 0$ and $\boldsymbol{C = \omega^2}$. For this C the particular solution must change from $y_p = e^{i\omega t}/0$ to $y_p = \boldsymbol{te^{i\omega t}/2i\omega}$ (because the derivative of $P(D) = D^2 + C$ is $P'(D) = 2D$ and then $P'(i\omega) = 2i\omega$).

We never get resonance with $P(D) = D^2 + 5D + C$ because $P(i\omega) = (i\omega)^2 + 5i\omega + C$ is never zero and $y = e^{i\omega t}$ is never a null solution.

6 Suppose the third order equation $P(D)y_n = 0$ has solutions $y = c_1e^t + c_2e^{2t} + c_3e^{3t}$. What are the null solutions to the sixth order equation $P(D)P(D)y_n = 0$?

Solution The three roots of $P(s)$ must be $s = 1, 2, 3$. The sixth order equation $P(D)P(D)y = 0$ has those as **double roots** of $P(s)^2$. So the null solutions are

$$y = c_1e^t + c_2te^t + c_3e^{2t} + c_4te^{2t} + c_5e^{3t} + c_6te^{3t}$$

7 Complete this table with equations for s_1 and s_2 and y_n and y_p :

Undamped free	$my'' + ky = 0$	$\boldsymbol{y_n = c_1e^{i\omega_n t} + c_2e^{-i\omega_n t}}$
Undamped forced	$my'' + ky = e^{i\omega t}$	$\boldsymbol{y_p = e^{i\omega t}/m(\omega_n^2 - \omega^2)}$
Damped free	$my'' + by' + ky = 0$	$\boldsymbol{y_n = c_1e^{s_1t} + c_2e^{s_2t}}$
Damped forced	$my'' + by' + ky = e^{ct}$	$\boldsymbol{y_p = e^{ct}/(mc^2 + bc + k)}$

Here s_1 and s_2 are $-b/2m \pm \sqrt{b^2 - 4mk}/2m$.

8 Complete the same table when the coefficients are 1 and $2Z\omega_n$ and ω_n^2 with $Z < 1$.

Undamped free	$y'' + \omega_n^2 y = 0$	$\boldsymbol{y_n = c_1e^{i\omega_n t} + c_2e^{-i\omega_n t}}$
Undamped forced	$y'' + \omega_n^2 y = e^{i\omega t}$	$\boldsymbol{y_p = e^{i\omega t}/m(\omega_n^2 - \omega^2)}$
Underdamped free	$y'' + 2Z\omega_n y' + \omega_n^2 y = 0$	$\boldsymbol{y_n = c_1e^{s_1t} + c_2e^{s_2t}}$
Underdamped forced	$y'' + 2Z\omega_n y' + \omega_n^2 y = e^{ct}$	$\boldsymbol{y_p = e^{ct}/(c^2 + 2Z\omega_n c + \omega_n^2)}$

Those use equations (20) in 2.3 and (32-33) in 2.4.

9 What equations $y'' + By' + Cy = f$ have these solutions ? Hint: Find B and C from the exponents s in y_n : $s_1 + s_2 = -B$ and $s_1s_2 = C$. Find f by substituting y_p.

(a) $y = c_1\cos 2t + c_2\sin 2t + \cos 3t$ $\quad \boldsymbol{y'' + 4y = -5\cos 3t}$

(b) $y = c_1e^{-t}\cos 4t + c_2e^{-t}\sin 4t + \cos 5t$ $\quad \boldsymbol{y'' + 2y' + 17y = -8\cos 5t - 10\sin 5t}$

(c) $y = c_1e^{-t} + c_2te^{-t} + e^{i\omega t}$ $\quad \boldsymbol{y'' + 2y' + y = [(i\omega)^2 + 2i\omega + 1]e^{i\omega t}}$.

10 If $y_p = te^{-6t}\cos 7t$ solves a second order equation $Ay'' + By' + Cy = f$, what does that tell you about A, B, C, and f ?

Solution This particular y_p is showing **resonance** from the factor t. (If this was y_n, we would be seeing a double root of $As^2 + Bs + C = 0$.) The root is $s = -6 + 7i$ from the other factors of y_p.

So I believe that

$$As^2 + Bs + C = A(s + 6 - 7i)(s + 6 + 7i) = A(s^2 + 12s + 36 + 49)$$
$$f = Fe^{-6t}(A\cos 7t + B\sin 7t)$$

11 (a) Find the steady oscillation $y_p(t)$ that solves $y'' + 4y' + 3y = 5\cos\omega t$.

(b) Find the amplitude A of $y_p(t)$ and its phase lag α.

(c) Which frequency ω gives maximum amplitude (maximum gain) ?

Solution (a) y_p has $\sin\omega t$ as well as $\cos\omega t$. Use equations (22-23) for $y_p = M\cos\omega t + N\sin\omega t$:

$$D = (3-\omega^2)^2 + 16\omega^2 \qquad M = \frac{3-\omega^2}{D} \qquad N = \frac{4\omega}{D}$$

(b) From equation (26) and the page 112 table :

Amplitude $= G = \frac{1}{\sqrt{D}}$ and the angle α has tangent $= \frac{N}{M} = \frac{4\omega}{3-\omega^2}$.

(c) The maximum gain G and the minimum of $D = (3-\omega^2)^2 + 16\omega^2$ will occur when

$$\frac{dD}{d\omega} = -4\omega(3-\omega^2) + 32\omega = 0 \text{ and } 3-\omega^2 = 8 \text{ and } \omega = \pm\sqrt{5}.$$

This "practical resonance frequency" is computed at the end of section 2.5.

12 Solve $y'' + y = \sin\omega t$ starting from $y(0) = 0$ and $y'(0) = 0$. Find the limit of $y(t)$ as ω approaches 1, and the problem approaches resonance.

Solution The solution is $y = y_n + y_p = c_1\cos t + c_2\sin t + Y\sin\omega t$. Substituting into the equation gives $-\omega^2 Y\sin\omega t + Y\sin\omega t = \sin\omega t$ and $Y = \frac{1}{1-\omega^2}$.

$y(0) = 0$ gives $c_1 = 0$. And $y'(0) = c_2 + \omega Y = 0$ gives $c_2 = -\omega Y$:

$$y(t) = \frac{-\omega}{1-\omega^2}\sin t + \frac{1}{1-\omega^2}\sin\omega t = \frac{\sin\omega t - \omega\sin t}{1-\omega^2}.$$

As ω goes to 1, this goes to $0/0$. Then the l'Hopital Rule finds the ratio of ω-derivatives at $\omega = 1$:

$$\frac{t\cos\omega t - \sin t}{-2\omega} \quad\longrightarrow\quad \boldsymbol{\frac{t\cos t - \sin t}{-2}} = \textbf{Resonant solution}$$

13 Does critical damping and a double root $s = 1$ in $y'' + 2y' + y = e^{ct}$ produce an extra factor t in the null solution y_n or in the particular y_p (proportional to e^{ct}) ? What is y_n with constants c_1, c_2 ? What is $y_p = Ye^{ct}$?

Solution Critical damping is shown in the double root $s = -1, -1$ in $s^2 + 2s + 1 = 0$ and in the **null solutions** $y_n = c_1 e^{-t} + c_2 t e^{-t}$. (Resonance would come when c is also -1 in the right hand side.) The solution $y_p = Ye^{ct}$ has $y'' + 2y' + y = e^{ct}$ and $(c^2 Y + 2cY + Y) = 1$ and $Y = 1/(c^2 + 2c + 1)$.

14 If $c = i\omega$ in Problem 13, the solution y_p to $y'' + 2y' + y = e^{i\omega t}$ is _____. That fraction Y is the transfer function at $i\omega$. What are the magnitude and phase in $Y = Ge^{-i\alpha}$?

Solution Set $c = i\omega$ in the solution to Problem 13:

$$y_p + Ye^{ct} = e^{i\omega t}/(i^2\omega^2 + 2i\omega + 1) = Ge^{-i\alpha}e^{i\omega t}.$$

Then $G = 1/(1-\omega^2 + 2i\omega)$ has magnitude $|G| = 1/\sqrt{(1-\omega^2)^2 + 4\omega^2} = 1/\sqrt{D}$. The phase angle has $\tan\alpha = \frac{2\omega}{1-\omega^2}$.

By rescaling both t and y, we can reach $A = C = 1$. Then $\omega_n = 1$ and $B = 2Z$. The model problem is $y'' + 2Zy' + y = f(t)$.

15 What are the roots of $s^2 + 2Zs + 1 = 0$? Find two roots for $Z = 0, \frac{1}{2}, 1, 2$ and identify each type of damping. The natural frequency is now $\omega_n = 1$.

Solution The roots are $s = -Z \pm \sqrt{Z^2 - 1}$. (All factors 2 will cancel.)

$Z = 0 : s = \pm i$	No damping
$Z = \frac{1}{2} : s = (-1 \pm \sqrt{3}i)/2$	Underdamping
$Z = 1 : s = -1, -1$	Critical damping
$Z = 2 : s = -2 \pm \sqrt{3}$	Overdamping

16 Find two solutions to $y'' + 2Zy' + y = 0$ for every Z except $Z = 1$ and -1. Which solution $g(t)$ starts from $g(0) = 0$ and $g'(0) = 1$? What is different about $Z = 1$?

Solution If $Z^2 \neq 1$ the solutions are $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. The **impulse response** $g(t)$ on page 97 comes from $s = -Z \pm r$:

$$g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2} = e^{-Zt}(e^{rt} - e^{-rt})/2r \text{ with } r = \sqrt{Z^2 - 1} \text{ in formula (2.3.12).}$$

If $Z = 1$ (critical) then $s_1 = s_2$ and $r = 0$ and $g(t)$ changes to te^{-t} (formula 2.3.15).

17 The equation $my'' + ky = \cos \omega_n t$ is exactly at resonance. The driving frequency on the right side equals the natural frequency $\omega_n = \sqrt{k/m}$ on the left side. Substitute $y = Rt \sin(\sqrt{k/m}\, t)$ to find R. This resonant solution grows in time because of the factor t.

Solution $y' = R \sin \sqrt{\frac{k}{m}} t + R\sqrt{\frac{k}{m}} t \cos \sqrt{\frac{k}{m}} t$ and $y'' = 2R\sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}} t - R\frac{k}{m} t \sin \sqrt{\frac{k}{m}} t$.

Then $my'' + ky = 2R\sqrt{km} \cos \sqrt{\frac{k}{m}} t - Rkt \sin \sqrt{\frac{k}{m}} t + kRt \sin \sqrt{\frac{k}{m}} t = 2R\sqrt{km} \cos \sqrt{\frac{k}{m}} t$.

This agrees with $\cos \omega_n t$ on the right side of the differential equation if $\boldsymbol{R = 1/2\sqrt{km}}$.

18 Comparing the equations $Ay'' + By' + Cy = f(t)$ and $4Az'' + Bz' + (C/4)z = f(t)$, what is the difference in their solutions?

Correction The forcing term in the z-equation should be $f(\frac{t}{4})$.

Solution $\boldsymbol{z(t)}$ **will be** $\boldsymbol{4y(\frac{t}{4})}$. Then $z' = y'(\frac{t}{4})$ and $z'' = \frac{1}{4}y''(\frac{t}{4})$.

$4Az'' + Bz' + \frac{C}{4}z$ equals term by term to $Ay''(\frac{t}{4}) + By'(\frac{t}{4}) + Cy(\frac{t}{4}) = f(\frac{t}{4})$.

19 Find the fundamental solution to the equation $g'' - 3g' + 2g = \delta(t)$.

Solution The roots of $s^2 - 3s + 2 = 0$ are $s = 2$ and $s = 1$: **Real roots**. Use formula 2.3.12 on page 97 to find $g(t)$:

$$g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{A(s_2 - s_1)} = e^{2t} - e^t.$$

Notice that $g(0) = 0$ and $g'(0) = 1$ (and $A = 1$ in the differential equation).

20 (Challenge problem) Find the solution to $y'' + By' + y = \cos t$ that starts from $y(0) = 0$ and $y'(0) = 0$. Then let the damping constant B approach zero, to reach the resonant equation $y'' + y = \cos t$ in Problem 17, with $m = k = 1$.

Show that your solution $y(t)$ is approaching the resonant solution $\frac{1}{2}t \sin t$.

Solution The particular solution is $y_p = \frac{\sin t}{B}$. Then $y_p'' + y_p = 0$ and $By_p' = \cos t$. The roots of $s^2 + Bs + 1 = 0$ are $s = (-B \pm \sqrt{B^2 - 4})/2 = (-B \pm i\sqrt{4 - B^2})/2$.

Then $y = c_1e^{s_1t} + c_2e^{s_2t} + \frac{1}{B}\sin t$. At $t = 0$ we must have $c_1 + c_2 = 0$ and $s_1c_1 + s_2c_2 + \frac{1}{B} = 0$. Put $c_2 = -c_1$ to find $(s_1 - s_2)c_1 = i\sqrt{4-B^2}c_1 = -1/B$.

$$\text{Solution near } B = 0 \quad y = \frac{i}{B\sqrt{4-B^2}}(e^{s_1t} - e^{s_2t}) + \frac{1}{B}\sin t.$$

At $B = 0$ the roots are $s_1 = i$ and $s_2 = -i$, and $\sqrt{4-B^2} = 2$.

The solution $y(t)$ approaches $y = \frac{i}{2B}2i\sin t + \frac{1}{B}\sin t = \frac{0}{0}$ (sign of resonance).

l'Hopital asks for the ratio of the B-derivatives. Certainly B in the denominator has B-derivative equal to 1. And $\sqrt{4-B^2}$ approaches 2. So we want the **B-derivative of the numerator**, where s_1, s_2 depend on B. Then as $B \to 0, y$ approaches $\frac{d}{dB}\frac{i}{2}\left(e^{s_1t} - e^{s_2t}\right) = \frac{it}{2}\left[e^{s_1t}\frac{ds_1}{dB} - e^{s_2t}\frac{ds_2}{dB}\right] \to \frac{it}{2}\left(-\frac{1}{2}\right)e^{it} - \frac{it}{2}\left(-\frac{1}{2}\right)e^{-it} = \boldsymbol{\frac{1}{2}t\sin t}$. Wow!

21 Suppose you know three solutions $y_1,\ y_2,\ y_3$ to $y'' + B(t)y' + C(t)y = f(t)$. (Recommended) How could you find $B(t)$ and $C(t)$ and $f(t)$?

Solution The differences $u = y_1 - y_2$ and $v = y_1 - y_3$ are null solutions:

$$\begin{aligned} u'' + B(t)u' + C(t)u = 0 \\ v'' + B(t)v' + C(t)v = 0 \end{aligned}$$

Solve those two linear equations for the numbers $B(t)$ and $C(t)$ at each time t. Then y_1 is a particular solution so $y_1'' + B(t)y_1' + C(t)y_1$ gives $f(t)$.

Problem Set 2.5, page 127

1 (Resistors in parallel) Two parallel resistors R_1 and R_2 connect a node at voltage V to a node at voltage zero. The currents are V/R_1 and V/R_2. What is the total current I between the nodes? Writing R_{12} for the ratio V/I, what is R_{12} in terms of R_1 and R_2 ?

Solution Currents V/R_1 and V/R_2 in parallel give total current $I = V/R_1 + V/R_2$. Then the effective resistance in $I = V/R$ has

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1R_2} \quad \text{and} \quad R = \frac{R_1R_2}{R_1 + R_2}.$$

2 (Inductor and capacitor in parallel) Those elements connect a node at voltage $Ve^{i\omega t}$ to a node at voltage zero (grounded node). The currents are $(V/i\omega L)e^{i\omega t}$ and $V(i\omega C)e^{i\omega t}$. The total current $Ie^{i\omega t}$ between the nodes is their sum. Writing $Z_{12}e^{i\omega t}$ for the ratio $Ve^{i\omega t}/Ie^{i\omega t}$, what is Z_{12} in terms of $i\omega L$ and $i\omega C$?

Solution This is like Problem 1 with impedances $i\omega L$ and $1/i\omega C$ in parallel, instead of resistances R_1 and R_2. The effective impedance imitates that previous formula for $R = R_1R_2/(R_1 + R_2)$:

$$Z = \frac{Z_1Z_2}{Z_1 + Z_2} = \frac{i\omega L(1/i\omega C)}{i\omega L + (i\omega C)^{-1}} = \frac{i\omega L}{1 - \omega^2LC}.$$

3 The impedance of an RLC loop is $Z = i\omega L + R + 1/i\omega C$. This impedance Z is real when $\omega =$ _____. This impedance is pure imaginary when _____. This impedance is zero when _____.

Solution Z is real when $i\omega L$ cancels with $1/i\omega C = -i/\omega C$. Then $\omega L = 1/\omega C$ and $\boldsymbol{\omega^2 = 1/LC}$. Z is imaginary when $\boldsymbol{R = 0}$. The impedance is zero when both $R = 0$ and $\boldsymbol{\omega^2 = 1/LC}$.

4 What is the impedance Z of an RLC loop when $R = L = C = 1$? Draw a graph that shows the magnitude $|Z|$ as a function of ω.

Solution An RLC loop adds the impedances $R + i\omega L + i/(i\omega C)$. In case $R = L = C = 1$, the total impedance in series is $Z = \mathbf{1 + i\omega + 1/i\omega}$. The magnitude $|Z| = (1 + (\omega - 1/\omega)^2)^{1/2}$ will equal 1 at $\omega = 1$. For large ω, $|Z|$ is asymptotic to the line $|Z| = \omega$. For small ω, $|Z|$ is asymptotic to the curve $|Z| = 1/\omega$.

5 Why does an LC loop with no resistor produce a $90°$ phase shift between current and voltage ? Current goes around the loop from a battery of voltage V in the loop.

Solution The phase shift is the angle of the complex impedance Z. With no resistor, $R = 0$ and $Z = i\omega L + (1/i\omega C) = i(\omega L - (1/\omega C))$. This pure imaginary number has angle $\theta = \pm\pi/2 = \pm 90\,°$ in the complex plane.

6 The mechanical equivalent of zero resistance is zero damping : $my'' + ky = \cos\omega t$. Find c_1 and Y starting from $y(0) = 0$ and $y'(0) = 0$ with $\omega_n^2 = k/m$.

$$y(t) = c_1 \cos\omega_n t + Y\ \cos\omega t.$$

That answer can be written in two equivalent ways :

$$y = Y(\cos\omega t - \cos\omega_n t) = 2Y \sin\frac{(\omega_n - \omega)t}{2} \sin\frac{(\omega_n + \omega)t}{2}.$$

Solution The complete solution is $y = c_1 \cos\omega_n t + c_2 \sin\omega_n t + (\cos\omega t)/(k - m\omega^2)$. The initial conditions $y = y' = 0$ determine c_1 and c_2 :

$$y(0) = 0 \qquad c_1 = -1/(k - m\omega^2) \qquad y'(0) = 0 \qquad c_2 = 0.$$

Then $y(t) = \boldsymbol{(\cos\omega t - \cos\omega_n t)/(k - m\omega^2)}$. The identity $\cos\omega t - \cos\omega_n t = 2\sin\frac{(\omega-\omega_n)t}{2} \sin\frac{(\omega+\omega_n)t}{2}$ expresses y as the product of two oscillations.

7 Suppose the driving frequency ω is close to ω_n in Problem 2. A fast oscillation $\sin[(\omega_n + \omega)t/2]$ is multiplying a very slow oscillation $2Y\sin[(\omega_n - \omega)t/2]$. By hand or by computer, draw the graph of $y = (\sin t)(\sin 9t)$ from 0 to 2π.

You should see a fast sine curve inside a slow sine curve. This is a **beat**.

Solution When ω is close to ω_n, the first (bold) formula in Problem 6 is near $0/0$. The second formula is much better :

$$2\sin\frac{(\omega - \omega_n)t}{2} \approx (\omega - \omega_n)t \quad \sin\frac{(\omega + \omega_n)t}{2} \approx \sin\omega_n t \quad \boldsymbol{y \approx (\omega - \omega_n)t \sin\omega_n t}$$

This shows the typical t factor for resonance. The graph of $y = (\sin t)(\sin 9t)$ has $\omega = 10$ and $\omega_n = 8$, so that $(10 - 8)/2 = 1$ and $(10 + 8)/2 = 9$. The graph shows a fast "$\sin 9t$" curve inside a slow "$\sin t$" curve : good to draw by computer.

8 What m, b, k, F equation for a mass-dashpot-spring-force corresponds to Kirchhoff's Voltage Law around a loop ? What force balance equation on a mass corresponds to Kirchhoff's Current Law ?

Solution The Voltage Law says that **voltage drops add to zero** around a loop :

$$\text{Equation (5) is } L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt = Ve^{i\omega t}.$$

This corresponds to $my'' + by' + ky = f$. The Current Law says that "flow in equals flow out" at every node. The mechanical analog is that "**forces balance**" at every node.

In a static structure (no movement) we can have force balance equations in the x, y, and z direction. In a dynamic structure (with movement) the forces include the inertia term my'' and the friction term by'.

9 If you only know the natural frequency ω_n and the damping coefficient b for one mass and one spring, why is that *not enough* to find the damped frequency ω_d? If you know all of m, b, k what is ω_d?

Solution If we only know $\omega_n^2 = k/m$ and b, that does not determine the damping ratio $Z = b^2/4mk$ or the damped frequency $\omega_d = \sqrt{p^2 - \omega_n^2}$ with $p = B/2A = b/2m = \omega_n Z$ in equation (2.4.30). We need *three numbers* as in m, b, k or *two ratios* as in $\omega_n^2 = k/m$ and $2p = b/m$.

10 Varying the number a in a first order equation $y' - ay = 1$ changes the *speed* of the response. Varying B and C in a second order equation $y'' + By' + Cy = 1$ changes the *form* of the response. Explain the difference.

Solution The growth factor in a first order equation is e^{at}. The units of a are $1/\text{time}$ and this controls the speed. For a second-order equation $y'' + By' + Cy' = f$, the coefficients B and C control not only the frequency $\omega_n = \sqrt{C}$ but also the form of $y(t)$: damped oscillation if $B^2 < 4C$ and overdamping if $B^2 > 4C$.

11 Find the step response $r(t) = y_p + y_n$ for this overdamped system:

$$r'' + 2.5r' + r = 1 \quad \text{with} \quad r(0) = 0 \quad \text{and} \quad r'(0) = 0.$$

Solution The roots of $s^2 + 2.5s + 1 = (s+2)(s+\frac{1}{2})$ are $s_1 = -2$ and $s_2 = -\frac{1}{2}$. Then equation (18) for the step response gives

$$r(t) = 1 + \left(-\frac{1}{2}e^{-2t} + 2e^{-t/2}\right)/(-3/2) = 1 + \frac{1}{3}e^{-2t} - \frac{4}{3}e^{-t/2}.$$

Check that $r(0) = 0$ and $r'(0) = 0$ (and $r(\infty) = 1$).

12 Find the step response $r(t) = y_p + y_n$ for this critically damped system. The double root $s = -1$ produces what form for the null solution?

$$r'' + 2r' + r = 1 \quad \text{with} \quad r(0) = 0 \quad \text{and} \quad r'(0) = 0.$$

Solution The characteristic equation $s^2 + 2s + 1 = 0$ has a double root $s = -1$. The null solution is $\boldsymbol{y_n = c_1 e^{-t} + c_2 t e^{-t}}$. The particular solution with $f = 1$ is $y_p = 1$. The initial conditions give c_1 and c_2:

$$\begin{aligned} r(t) &= c_1 e^{-t} + c_2 t e^{-t} + 1 \\ r(0) &= c_1 + 1 = 0 && \boldsymbol{c_1 = -1} \\ r'(0) &= -c_1 + c_2 + 1 = 0 && \boldsymbol{c_2 = -2} \\ r(t) &= \mathbf{1 - (1 + 2}\boldsymbol{t})\boldsymbol{e^{-t}} \end{aligned}$$

13 Find the step response $r(t)$ for this underdamped system using equation (22):

$$r'' + r' + r = 1 \quad \text{with} \quad r(0) = 0 \quad \text{and} \quad r'(0) = 0.$$

Solution Equation (22) gives the step response for an underdamped system.

$$r(t) = 1 - \frac{\omega_n}{\omega_d} e^{-pt} \sin(\omega_d t + \phi).$$

Then $r'' + r' + r = 1$ has $m = b = k = 1$ and $b^2 < 4mk$ (underdamping).

$$p = \frac{b}{2m} = \mathbf{\frac{1}{2}} \quad \omega_n^2 = \frac{k}{m} = \mathbf{1} \quad \omega_d^2 = \omega_n^2 - p^2 = \mathbf{\frac{3}{4}} \quad \cos\phi = \frac{p}{\omega_n} = \frac{1}{2} \quad \phi = \frac{\pi}{3}.$$

Substituting in the formula gives $r(t) = 1 - \dfrac{2}{\sqrt{3}} e^{-t/2} \sin\left(\dfrac{\sqrt{3}}{2}t + \dfrac{\pi}{3}\right)$.

14 Find the step response $r(t)$ for this undamped system and compare with (22):

$$r'' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0.$$

Solution Now $r'' + r = 1$ has $m = k = 1$ and $b = 0$ (no damping):

In this case $\quad p = 0 \quad \omega_n^2 = 1 \quad \omega_d = \omega_n \quad \cos\phi = \frac{p}{\omega_n} = 0 \quad \phi = \frac{\pi}{2}$.

Substituting into (22) gives $r(t) = 1 - \sin\left(t + \frac{\pi}{2}\right) = \mathbf{1 - \cos t}$.

15 For $b^2 < 4mk$ (underdamping), what parameter decides the speed at which the step response $r(t)$ rises to $r(\infty) = 1$? Show that the **peak time** is $T = \pi/\omega_d$ when $r(t)$ reaches its maximum before settling back to $r = 1$. At peak time $r'(T) = 0$.

Solution With underdamping, formula (22) has the decay factor e^{-pt}. Then $p = B/2A = b/2m$ is the decay rate. The "peak time" is the time when r reaches its maximum (its peak). That time T has $dr/dt = 0$.

$$\frac{dr}{dt} = -\frac{\omega_n}{\omega_d}\left(-pe^{-pt}\sin(\omega_d t + \phi) + \omega_d e^{-pt}\cos(\omega_d t + \phi)\right) = 0 \text{ at } t = T \text{ (peak time).}$$

$$-p\sin(\omega_d T + \phi) + \omega_d \cos(\omega_d T + \phi) = 0$$

$$\tan(\omega_d T + \phi) = \omega_d / p \text{ which is } \tan\phi$$

Then $\omega_d T = \pi$ and $\boldsymbol{T = \pi/\omega_d}$. (Note: I seem to get $2\pi/\omega_d$.)

16 If the voltage source $V(t)$ in an RLC loop is a unit step function, what resistance R will produce an overshoot to $r_{\text{max}} = 1.2$ if $C = 10^{-6}$ Farads and $L = 1$ Henry? (Problem 15) found the peak time T when $r(T) = r_{\text{max}}$).

Sketch two graphs of $r(t)$ for $p_1 < p_2$. Sketch two graphs as ω_d increases.

Solution The peak time is $T = \pi/\omega_d$. Then $\omega_d T = \pi$ and we want $r = 1.2$:

$$r_{\max}(T) = 1 - \tfrac{\omega_n}{\omega_d} e^{-pT}\sin(\pi + \phi)$$

$$1.2 = 1 + \tfrac{\omega_n}{\omega_d} e^{-pT}\sin(\phi) = 1 + e^{-pT}$$

$$0.2 = e^{-p\pi/\omega_d}$$

$$p\pi/\omega_d = -\ln(0.2) = \ln 5$$

We substitute $p = B/2A = R/2\omega L$ and $\omega_d = \sqrt{\omega_n^2 - \omega^2} = \sqrt{(1/LC) - \omega^2}$. With known values of L and C **and** $\boldsymbol{\omega}$ we can find R.

17 What values of m, b, k will give the step response $r(t) = 1 - \sqrt{2}e^{-t}\sin(t + \frac{\pi}{4})$?

Solution This response $r(t)$ matches equation (22) when $\boldsymbol{\omega_n = \sqrt{2}\omega_d}$ and $\boldsymbol{p = 1}$ and $\boldsymbol{\phi = \pi/4}$. Then

$$\omega_d^2 = \omega_n^2 - p^2 = 2\omega_d^2 - 1 \text{ gives } \boldsymbol{\omega_d = 1} \text{ and } \boldsymbol{\omega_n = \sqrt{2}}.$$

Therefore $\omega_n^2 = k/m = 2$ and $p = b/2m = 1$. The numbers m, b, k are proportional to $\mathbf{1, 2, 2}$.

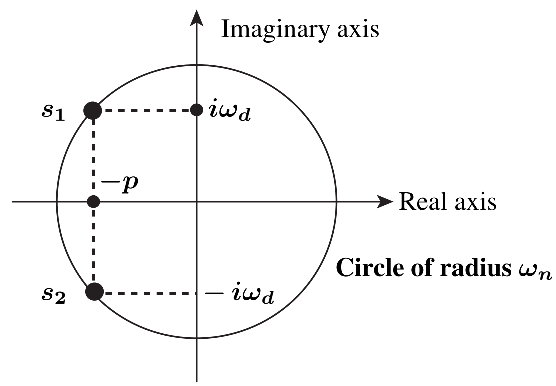
18 What happens to the $p - \omega_d - \omega_n$ right triangle as the damping ratio ω_n/p increases to 1 (critical damping)? At that point the damped frequency ω_d becomes _____. The step response becomes $r(t) =$ _____.

Solution Critical damping has equal roots $s_1 = s_2$ and $b^2 = 4mk$ and damping ratio $Z = 1$ and $\omega_d = \omega_n\sqrt{1-Z^2} = 0$. (The oscillation disappears and the damped frequency goes to $\omega_d = 0$ so that $\phi = 0$.) Then the step response is

$$r(t) = 1 - \frac{\omega_n t}{\omega_d t^e} - pt\sin(\omega_d t) \longrightarrow \mathbf{1 - \boldsymbol{\omega_n} te^{-pt}}.$$

19 The roots $s_1, s_2 = -p \pm i\omega_d$ are poles of the transfer function $1/(As^2 + Bs + C)$

Show directly that the product of the roots $s_1 = -p + i\omega_d$ and $s_2 = -p - i\omega_d$ is $s_1 s_2 = \omega_n^2$. The sum of the roots is $-2p$. The quadratic equation with those roots is $s^2 + 2ps + \omega_n^2 = 0$.



Solution Multiplying the complex conjugate number $s = -p \pm i\omega_d$ gives $|s|^2 = (-p + i\omega_d)(-p - i\omega_d) = p^2 + \omega_d^2 = \omega_n^2$.

For any quadratic $As^2 + Bs + C = A(s - s_1)(s - s_2), C$ matches $As_1 s_2$. Then $s_1 s_2 = C/A = \omega_n^2$. Complex roots **stay on the circle of radius ω_n**, as in the picture.

Adding $-p + i\omega$ to $-p - i\omega$ gives $s_1 + s_2 = -2p$. This always equals $-B/A$.

20 Suppose p is increased while ω_n is held constant. How do the roots s_1 and s_2 move?

Solution Increasing p will make both roots go along the circle in the direction of $-\omega_n$. Problem 19 showed that they stay on the circle of radius ω_n until they meet at $-\omega_n$. At that point $s_1 + s_2 = -2\omega_n = -2p$. Therefore that value of p is ω_n.

Increasing p beyond ω_n will give **two negative real roots** that add to $-2\omega_n$.

21 Suppose the mass m is increased while the coefficients b and k are unchanged. What happens to the roots s_1 and s_2?

Solution The key number $B^2 - 4AC = b^2 - 4mk$ will eventually go negative when m is increased. The roots will be complex (a conjugate pair). Further increasing the mass m will decrease both $p = b/2m$ and $\omega_n^2 = k/m$. The roots approach zero.

22 Ramp response How could you find $y(t)$ when $F = t$ is a ramp function?

$$y'' + 2py' + \omega_n^2 y = \omega_n^2 t \quad \text{starting from} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

A particular solution (straight line) is $y_p =$ _____. The null solution still has the form $y_n =$ _____. Find the coefficients c_1 and c_2 in the null solution from the two conditions at $t = 0$.

This ramp response $y(t)$ can also be seen as the integral of _____.

Solution A particular solution is $y_p = C + t$. Substitute into the equation:

$y'' + 2py' + \omega_n^2 y = 0 + 2p + \omega_n^2(C + t) = \omega_n^2 t$. Thus $C = -2p/\omega_n^2$.

The null solution is still $y_n = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. We find c_1 and c_2 at $\boldsymbol{t = 0}$:

$$y = c_1 e^{s_1 t} + c_2 e^{s_2 t} + C + t = c_1 + c_2 + C \qquad = 0$$
$$y' = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t} + 1 = c_1 s_1 + c_2 s_2 + 1 = 0$$

Solving those equations gives $c_1 = \frac{Cs_2 - 1}{s_1 - s_2}$ and $c_2 = \frac{1 - Cs_1}{s_1 - s_2}$ with $C = -2p/\omega_n^2$.

The ramp response is also the integral of the **step response**.

Problem Set 2.6, page 137

Find a particular solution by inspection (or the method of undetermined coefficients)

1 (a) $y'' + y = 4$ (b) $y'' + y' = 4$ (c) $y'' = 4$

Solution (a) $y_p = 4$ (b) $y_p = 4t$ (c) $y_p = 2t^2$

2 (a) $y'' + y' + y = e^t$ (b) $y'' + y' + y = e^{ct}$

Solution (a) $y_p = \frac{1}{3}e^t$ (b) $y_p = e^{ct}/(c^2 + c + 1)$

3 (a) $y'' - y = \cos\, t$ (b) $y'' + y = \cos\, 2t$ (c) $y'' + y = t + e^t$

Solution (a) $y_p = -\frac{1}{2}\cos t$ (b) $y_p = -\frac{1}{3}\cos 2t$ (c) $y_p = t + \frac{1}{2}e^t$

4 For these $f(t)$, predict the form of $y(t)$ with undetermined coefficients:

(a) $f(t) = t^3$ (b) $f(t) = \cos\, 2t$ (c) $f(t) = t\, \cos\, t$

Solution (a) $y_p = at^3 + bt^2 + ct + d$ (b) $y_p = a\cos 2t + b\sin 2t$
(c) $y_p = (At + B)\cos t + (Ct + D)\sin t$

5 Predict the form for $y(t)$ when the right hand side is

(a) $f(t) = e^{ct}$ (b) $f(t) = te^{ct}$ (c) $f(t) = e^t\, \cos\, t$

Solution (a) $y_p = Ye^{ct}$ (b) $y_p = (Yt + Z)e^{ct}$ (c) $y_p = ae^t\cos t + be^t\sin t$

6 For $f(t) = e^{ct}$ when is the prediction for $y(t)$ different from Ye^{ct} ?

Solution There will be a te^{ct} term in y_p when e^{ct} is a null solution. This is resonance:

$$Ac^2 + Bc + C = 0 \text{ and } c \text{ is } s_1 \text{ or } s_2.$$

Problems 7-11 : Use the method of undetermined coefficients to find a solution $y_p(t)$.

7 (a) $y'' + 9y = e^{2t}$ (b) $y'' + 9y = te^{2t}$

Solution (a) $y_p = Ye^{2t}$ with $4Ye^{2t} + 9Ye^{2t} = e^{2t}$ and $Y = \frac{1}{13}$

(b) $y_p = (Yt + Z)e^{2t}$ with $y' = (2Yt + Y + 2Z)e^{2t}$ and $y'' = (4Yt + 4Y + 4Z)e^{2t}$.

The equation $y'' + 9y = te^{2t}$ gives $(4Yt + 4Y + 4Z + 9Yt + 9Z)e^{2t} = te^{2t}$.

Then $13Yt = t$ and $4Y + 13Z = 0$ give $Y = \frac{1}{13}$ and $Z = -\frac{4}{13}Y$ and $y_p = \boldsymbol{\frac{1}{13}(t - \frac{4}{13})e^{2t}}$.

8 (a) $y'' + y' = t + 1$ (b) $y'' + y' = t^2 + 1$

Solution (a) $y_p = at^2 + bt$ and $y'' + y' = 2a + 2at + b = t + 1$.

Then $a = \frac{1}{2}$ and $b = 0$ and $\boldsymbol{y_p = \frac{1}{2}t^2}$.

*Notice that $y_p =$ constant is a null solution so we needed to assume $y_p = at^2 + bt$.

(b) $y_p = at^3 + bt^2 + ct$ (NOT $+d$) and $y'' + y' = (6at + 2b) + (3at^2 + 2bt + c) = t^2 + 1$.

Then $3a = 1$ and $6a + 2b = 0$ and $2b + c = 1$: $\boldsymbol{y_p = \frac{1}{3}t^3 - 1t^2 + 3t}$.

9 (a) $y'' + 3y = \cos\, t$ (b) $y'' + 3y = t\, \cos\, t$

Solution (a) $y_p = A\cos t + B\sin t$.

$y_p'' + 3y_p = -A\cos t - B\sin t + 3A\cos t + 3B\sin t = \cos t$.

Then $2A = 1$ and $2B = 0$ and $\boldsymbol{y_p = \frac{1}{2}\cos t}$.

(b) $y_p = (At + B)\cos t + (Ct + D)\sin t$.

$y_p' = (A + Ct + D)\cos t + (-At - B + C)\sin t$.

$y_p'' + 3y_p = C\cos t - A\sin t + (-A - Ct - D)\sin t + (-At - B + C)\cos t + 3(At + B)\cos t + 3(Ct + D)\sin t = t\cos t$.

Match $3At - At = t$ and $C - B + C + 3B = 0$ and $-Ct + 3Ct = 0$ and $-A - A - D + 3D = 0$.

Then $A = \frac{1}{2},\ C = 0,\ B = 0,\ D = A = \frac{1}{2}$ gives $\boldsymbol{y_p = \frac{1}{2}t\cos t + \frac{1}{2}\sin t}$.

10 (a) $y'' + y' + y = t^2$ (b) $y'' + y' + y = t^3$

Solution (a) $y_p = at^2 + bt + c$ give $y_p'' + y_p' + y = (2a) + (2at + b) + (at^2 + bt + c) = t^2$.

Then $a = 1$ and $2a + b = 0$ and $2a + b + c = 0$ give $a = 1,\ b = -2,\ c = 0$: $\boldsymbol{y_p = t^2 - 2t}$.

(b) Now $y_p = at^2 + bt + c + dt^3$. Added into part (a), the new dt^3 produces

$$y'' + y' + y = (2a) + (2at + b) + (at^2 + bt + c) + d(6t + 3t^2 + t^3) = t^3 + c = 0$$

Then $d = 1,\quad 3d + a = 0,\quad 6d + b + 2a = 0,\quad 2a + b + c = 0$ give $d = 1, a = -3, b = 0, c = 6$: $\boldsymbol{y_p = t^3 - 3t^2 + 6}$.

11 (a) $y'' + y' + y = \cos\, t$ (b) $y'' + y' + y = t\, \sin\, t$

Solution (a) $y_p = A\cos t + B\sin t$.

$y_p'' + y_p' + y_p = (-A + B + A)\cos t + (-B - A + B)\sin t = \cos t$.

Then $B = 1$ and $A = 0$ and $\boldsymbol{y_p = \sin t}$.

(b) The forms for y_p and y_p' and y_p'' are the same as in 2.6.9 (b). Then $y_p''+y_p'+y_p$ equals $C\cos t - A\sin t + (-A - Ct - D)\sin t + (-At - B + C)\cos t + (A + Ct + D)\cos t$ $+(-At - B + C)\sin t + (Ct + D)\sin t = t\sin t$.

Match coefficients of $t\cos t, t\sin t, \cos t, \sin t$:

$-A + C + A = 0 \qquad -C - A + C = 1 \qquad C - B + C + A + D + B = 0$
$-A - A - D - B + C + D = 0.$

Then $A = -1,\ C = 0,\ B = 2,\ D = 1$ give $\boldsymbol{y_p = -t\cos t + 2\cos t}$.

Problems 12–14 involve resonance. Multiply the usual form of y_p by t.

12 (a) $y'' + y = e^{it}$ (b) $y'' + y = \cos\, t$

Solution (a) Look for $y_p = Yte^{it}$. Then $y_p' = Y(it+1)e^{it}$.

$y_p'' + y_p = Y(i^2t + 2ie^{it}) + Yte^{it} = 2iYe^{it}$.

This matches e^{it} on the right side when $Y = 1/2i$ and $\boldsymbol{y_p = te^{it}/2i = -ite^{it}/2}$.

(b) Look for $y_p = At\cos t + Bt\sin t$. Then $y_p' = A\cos t - At\sin t + B\sin t + Bt\cos t$.

$y_p'' + y = -2A\sin t - At\cos t + 2B\cos t - Bt\sin t + At\cos t + Bt\sin t = \cos t$.

Then $A = 0$ and $B = \frac{1}{2}$ and $\boldsymbol{y_p = \frac{1}{2}t\sin t}$.

13 (a) $y'' - 4y' + 3y = e^t$ (b) $y'' - 4y' + 3y = e^{3t}$

Solution (a) Look for $y_p = cte^t$ with $y_p' = c(t+1)e^t$ and $y_p'' = c(t+2)e^t$.

$$y_p'' - 4y_p' + 3y_p = (2c - 4c)e^t = e^t \text{ with } c = -\frac{1}{2} \text{ and } \boldsymbol{y_p = -\frac{1}{2}te^t}.$$

(b) Look for $y_p = cte^{3t}$ with $y_p' = c(3t+1)e^{3t}$ and $y_p'' = c(9t+6)e^{3t}$.

$$y_p'' - 4y_p' + 3y_p = (6c - 4c)e^{3t} = e^{3t} \text{ with } c = \frac{1}{2} \text{ and } \boldsymbol{y_p = \frac{1}{2}te^{3t}}.$$

14 (a) $y' - y = e^t$ (b) $y' - y = te^t$ (c) $y' - y = e^t\, \cos\, t$

Solution (a) Look for $y_p = cte^t$ with $y_p' = c(t+1)e^t$.

$$\text{Then } y_p' - y_p = ce^t = e^t \text{ when } c = 1 \text{ and } \boldsymbol{y_p = te^t}.$$

(b) Look for $y_p = ct^2e^t$ with $y_p' = c(t^2 + 2t)e^t$.

$$\text{Then } y_p' - y_p = c(t^2 + 2t - t^2)e^t = te^t \text{ when } c = \frac{1}{2} \text{ and } \boldsymbol{y_p = \frac{1}{2}t^2e^t}.$$

(c) Look for $y_p = Ae^t\cos t + Be^t\sin t$. Then

$y_p' = Ae^t\cos t - Ae^t\sin t + Be^t\sin t + Be^t\cos t$.

$y_p' - y_p = -Ae^t\sin t + Be^t\cos t = e^t\cos t$ when $A = 0, B = 1,$ and $\boldsymbol{y_p = e^t\sin t}$.

15 For $y'' + 4y = e^t \sin t$ (exponential times sinusoidal) we have two choices:

1 (Real) Substitute $y_p = Me^t \cos t + Ne^t \sin t$: determine M and N
2 (Complex) Solve $z'' + 4z = e^{(1+i)t}$. Then y is the imaginary part of z.

Use both methods to find the same $y(t)$—which do you prefer?

Solution Method 1 has $y_p' = Me^t \cos t - Me^t \sin t + Ne^t \sin t + Ne^t \cos t = (M+N)e^t \cos t + (-M+N)e^t \sin t$.

$$\text{Then } y_p'' + 4y_p = (M+N)\,e^t\cos t - (M+N)\,e^t \sin t + (-M+N)\,e^t \sin t + (-M+N)\,e^t \cos t + 4M\,e^t \cos t + 4N\,e^t \sin t.$$

This equals $e^t \sin t$ when $2N + 4M = 0$ and $-2M + 4N = 1$.

Then $N = -2M$ and $-2M - 8M = 1$ and $M = -\frac{1}{10}, N = \frac{2}{10}$, $y_p = \boldsymbol{-\frac{1}{10}e^t \cos t + \frac{2}{10}e^t \sin t}$.

Method 2 Look for $z_p = Ze^{(1+i)t}$. Then $z_p'' + 4z_p = Z\left[(1+i)^2 + 4\right]e^{(1+i)t} = e^{(1+i)t}$ gives $Z = 1/(4+2i)$.

Take the imaginary part of z_p:

$$\operatorname{Im}\frac{e^{(1+i)t}}{4+2i} = \operatorname{Im}\frac{e^t(\cos t + i \sin t)(4-2i)}{16+4} = \boldsymbol{\frac{e^t}{20}(-2\cos t + 4 \sin t)}.$$

This complex method was shorter and easier. It produced the same y_p.

16 (a) Which values of c give resonance for $y'' + 3y' - 4y = te^{ct}$?

Solution $c^2 + 3c - 4 = (c-1)(c+4)$. So $c = 1$ and $c = -4$ will give resonance.

(b) What form would you substitute for $y(t)$ if there is no resonance?

Solution With no resonance look for $y_p = (at+b)e^{ct}$.

(c) What form would you use when c produces resonance?

Solution With resonance look for $y_p = (at^2 + bt)e^{ct}$. If we also look for de^{ct}, this will be a null solution and we cannot determine d.

17 This is the rule for equations $P(D)y = e^{ct}$ with resonance $P(c) = 0$:

If $P(c) = 0$ and $P'(c) \neq 0$, look for a solution $y_p = Cte^{ct}$ $(m = 1)$
If c is a root of multiplicity m, then y_p has the form _____.

Solution If c is a root of P with multiplicity m, then multiply the usual Ye^{ct} by $\boldsymbol{t^m}$.

18 (a) To solve $d^4y/dt^4 - y = t^3e^{5t}$, what form do you expect for $y(t)$?

(b) If the right side becomes $t^3 \cos 5t$, which 8 coefficients are to be determined?

Solution (a) The exponent $c = 5$ is not a root of $P(D) = D^4 - 1$ $(5^4 \neq 1)$. So look for $y_p = (at^3 + bt^2 + ct + d)e^{5t}$.

(b) If the right side is $t^3 \cos 5t$ then

$y_p = (at^3 + bt^2 + ct + d)\cos 5t + (et^3 + ft^2 + gt + h)\sin 5t$.

19 For $y' - ay = f(t)$, the method of undetermined coefficients is looking for all right hand sides $f(t)$ so that the usual formula $y_p = e^{at}\int e^{-as}f(s)ds$ is easy to integrate. Find these integrals for the "nice functions" $f = e^{ct}$, $f = e^{i\omega t}$, and $f = t$:

$$\int e^{-as}e^{cs}ds \qquad \int e^{-as}e^{i\omega s}ds \qquad \int e^{-as}sds$$

Solution The equation has $y' - ay$ so the growth factor (the impulse response) is $g(t) = e^{at}$. This problem connects the method of undetermined coefficients to the ordinary formula $y_p = \int g(t-s)f(s)\,ds$. The integral $\int e^{a(t-s)}f(s)\,ds$ is easy for:

$$\int e^{-as}e^{cs}\,ds = \frac{e^{(c-a)s}}{(c-a)} \qquad \int e^{-as}e^{i\omega s}\,ds = \frac{e^{(i\omega-a)s}}{i\omega - a}$$

$$\int se^{-as}\,ds = -\left(\frac{s}{a} + \frac{1}{a^2}\right)e^{-as}.$$

Problems 20–27 develop the method of variation of parameters.

20 Find two solutions y_1, y_2 to $y'' + 3y' + 2y = 0$. Use those in formula (13) to solve

(a) $y'' + 3y' + 2y = e^t$ (b) $y'' + 3y' + 2y = e^{-t}$

Solution (a) $y'' + 3y' + 2y$ leads to $s^2 + 3s + 2 = (s+1)(s+2)$. The null solutions are $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. The Variation of Parameters formula is

$$y_p = -y_1 \int \frac{y_2 f}{W} + y_2 \int \frac{y_1 f}{W} \text{ with } W = y_1y_2' - y_2y_1' = (-2-1)e^{-t}e^{-2t} = \boldsymbol{-3e^{-3t}}.$$

$$f = e^t \text{ gives } y_p = +\frac{e^{-t}}{3}\int \frac{e^{-2t}e^t}{e^{-3t}} - \frac{e^{-2t}}{3}\int \frac{e^{-t}e^t}{e^{-3t}} = \frac{e^{-t}}{3}\frac{e^{2t}}{2} - \frac{e^{-2t}}{3}\frac{e^{3t}}{3} = \left(\frac{1}{6} - \frac{1}{9}\right)e^t = \boldsymbol{\frac{1}{18}e^t}.$$

(b) Again $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. Now $f = e^{-t}$ gives resonance and t appears:

$$y_p = +\frac{e^{-t}}{3}\int \frac{e^{-2t}e^{-t}}{e^{-3t}} - \frac{e^{-2t}}{3}\int \frac{e^{-t}e^{-t}}{e^{-3t}} = \frac{e^{-t}}{3}\boldsymbol{t} - \frac{e^{-2t}}{3}e^t = \boldsymbol{\frac{1}{3}(t-1)e^{-t}}.$$

21 Find two solutions to $y'' + 4y' = 0$ and use variation of parameters for

(a) $y'' + 4y' = e^{2t}$ (b) $y'' + 4y' = e^{-4t}$

Solution (a) $y'' + 4y' = 0$ has null solutions $y_1 = 1 = e^{0t}$ and $y_2 = e^{-4t}$. Then $W = y_1y_2' - y_2y_1' = \boldsymbol{-4e^{-4t}}$. The equation has $f = e^{2t}$.

$$\text{From (13): } y_p = -1\int \frac{e^{-4t}e^{2t}}{-4e^{-4t}} + e^{-4t}\int \frac{(1)e^{2t}}{-4e^{-4t}} = \frac{e^{2t}}{8} + e^{-4t}\left(\frac{e^{6t}}{-24}\right) = \boldsymbol{\frac{e^{2t}}{12}}.$$

(b) $f = e^{-4t}$ is also a null solution: expect resonance and a factor t.

$$y_p = -1\int \frac{e^{-4t}e^{-4t}}{-4e^{-4t}} + e^{-4t}\int \frac{(1)e^{-4t}}{-4e^{-4t}} = \boldsymbol{-\frac{e^{-4t}}{16} - e^{-4t}\left(\frac{t}{4}\right)}.$$

22 Find an equation $y'' + By' + Cy = 0$ that is solved by $y_1 = e^t$ and $y_2 = te^t$. If the right side is $f(t) = 1$, what solution comes from the VP formula (13)?

Solution With $y_1 = e^t$ and $y_2 = te^t$, the exponent $s = 1$ must be a double root:

$$As^2 + Bs + C = A(s-1)^2 \text{ and the equation can be } y'' - 2y' + y = f(t).$$

With $f(t) = 1$ and $W = y_1 y_2' - y_2 y_1' = e^t(e^t + te^t) - te^t(e^t) = \boldsymbol{e^{2t}}$, eq. (13) gives

$$y_p = -e^t \int \frac{te^t(1)}{e^{2t}} + te^t \int \frac{e^t(1)}{e^{2t}} = -e^t(-te^{-t} - e^{-t}) + te^t(-e^{-t}) = \mathbf{1}$$

$y_p = 1$ is a good solution to $y'' - 2y' + y = 1$.

23 $y'' - 5y' + 6y = 0$ is solved by $y_1 = e^{2t}$ and $y_2 = e^{3t}$, because $s = 2$ and $s = 3$ come from $s^2 - 5s + 6 = 0$. Now solve $y'' - 5y' + 6y = 12$ in two ways:

1. Undetermined coefficients (or inspection) **2.** Variation of parameters using (13)

The answers are different. Are the initial conditions different?

Solution Solving $y'' - 5y' + 6y = 12$ gives $\boldsymbol{y_p = 2}$ by inspection or undetermined coefficients.

Using $s^2 - 5s + 6 = (s-2)(s-3)$ we have $y_1 = e^{2t}$ and $y_2 = e^{3t}$ and $W = e^{5t}$. Then set $f = 12$:

$$y_p = -e^{2t} \int \frac{e^{3t}(12)}{e^{5t}} + e^{3t} \int \frac{e^{2t}(12)}{e^{5t}} = -e^{2t}\left(\frac{12e^{-2t}}{-2}\right) + e^{3t}\left(\frac{12e^{-3t}}{-3}\right) = 6 - 4 = \mathbf{2}$$

But if those two integrals are computed from 0 to t, the lower limit gives a different y_p:

$$-e^{2t}\int_0^t e^{-2t}(12) + e^{3t}\int_0^t e^{-3t}(12) = e^{2t}\left[\frac{12e^{-2t}}{-2}\right]_0^t + e^{3t}\left[\frac{12e^{-3t}}{-3}\right]_0^t$$

$$= \mathbf{2 - 6e^{2t} + 4e^{3t} = 2 + \text{ null solution}}.$$

24 What are the initial conditions $y(0)$ and $y'(0)$ for the solution (13) coming from variation of parameters, starting from any y_1 and y_2?

Solution Every integral $I(t) = \displaystyle\int_0^t h(s)\,ds$ starts from $I(0) = 0$ and $I'(0) = h(0)$ by the Fundamental Theorem of Calculus. For equation (13), this gives $y_p(0) = 0$ and $y_p'(0) = 0$ (which can be checked for $y_p = 2 - 6e^{2t} + 4e^{3t}$ in Problem 23).

25 The equation $y'' = 0$ is solved by $y_1 = 1$ and $y_2 = t$. Use variation of parameters to solve $y'' = t$ and also $y'' = t^2$.

Solution Those null solutions $y_1 = 1$ and $y_2 = t$ give $W = y_1 y_2' = \mathbf{1}$. Then

$$\textbf{for } \boldsymbol{f = t} \quad y_p = -1\int t^2 + t\int t \quad = -\tfrac{t^3}{3} + \tfrac{t^3}{2} = \boldsymbol{t^3/6}$$

$$\textbf{for } \boldsymbol{f = t^2} \quad y_p = -1\int t\,t^2 + t\int t^2 = -\tfrac{t^4}{4} + \tfrac{t^4}{3} = \boldsymbol{t^4/12}$$

Those are correct solutions to $y'' = t$ and $y'' = t^2$.

26 Solve ${y_s}'' + y_s = 1$ for the step response using variation of parameters, starting from the null solutions $y_1 = \cos\, t$ and $y_2 = \sin\, t$.

Solution The Wronskian of $y_1 = \cos t$ and $y_2 = \sin t$ is $W = (\cos t)(\sin t)' - (\sin t)(\cos t)' = \mathbf{1}$. Set $f = 1$ and $W = 1$ in equation (13):

$$y_p = -\cos t \int_0^t \frac{(\sin t)(1)}{1} + \sin t \int_0^t \frac{(\cos t)(1)}{1} = -\cos t(-\cos t + 1) + \sin t(\sin t)$$
$$= \mathbf{1 - \cos t} \ : \ \text{Step response}$$

27 Solve ${y_s}'' + 3{y_s}' + 2y_s = 1$ for the step response starting from the null solutions $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.

Solution The Wronskian of $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ is
$W = e^{-t}(-2e^{-2t}) - e^{-2t}(-e^{-t}) = \boldsymbol{-e^{-3t}}$. Set $f = 1$ in (13):

$$y_p = -e^{-t} \int_0^t \frac{e^{-2t}(1)}{-e^{-3t}}\, dt + e^{-2t} \int_0^t \frac{e^{-t}(1)}{-e^{-3t}}\, dt = +e^{-t}[e^t - 1] + e^{-2t}\left[\frac{1}{2}e^{2t} + \frac{1}{2}\right]$$
$$= \boldsymbol{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}.$$

The steady state is $y_p(\infty) = \frac{1}{2}$. This agrees with $y'' + 3y' + 2y = 1$ when $y =$ constant.

28 Solve $Ay'' + Cy = \cos\omega t$ when $A\omega^2 = C$ (the case of resonance). Example 4 suggests to substitute $y = Mt\cos\omega t + Nt\sin\omega t$. Find M and N.

Solution $y = Mt\cos\omega t + Nt\sin\omega t$ has

$y' = M(\cos\omega t - \omega t\sin\omega t) + N(\sin\omega t + \omega t\cos\omega t)$.

Now compute $Ay'' + Cy$ when $C = A\omega^2$. The result is

$AM(-2\omega\sin\omega t - \omega^2 t\cos\omega t) + A\omega^2 Mt\cos\omega t + AN(2\omega\cos\omega t - \omega^2 t\sin\omega t) + A\omega^2 N\sin\omega t = \cos\omega t$.

Simplify to $AM(-2\omega\sin\omega t) + AN(2\omega\cos\omega t) = \cos\omega t$. Then $\boldsymbol{M = 0}$ and $\boldsymbol{N = 1/2A\omega}$.

29 Put $g(t)$ into the great formulas (17)-(18) to see the equations above them.

Solution The equation above (17) came from the V of P equation (13):

Particular solution
Constant coefficients
$$y_p(t) = \frac{-e^{s_1 t}}{s_2 - s_1} \int_0^t e^{-s_1 T} f(T)dT + \frac{e^{s_2 t}}{s_2 - s_1} \int_0^t e^{-s_2 T} f(T)dT$$

This is the integral of $\dfrac{-e^{s_1(t-T)}}{s_2 - s_1} f(T) + \dfrac{e^{s_2(t-T)}}{s_2 - s_1} f(T)$ which is exactly $g(t-T)f(T)$.

For equal roots $s_1 = s_2$, the equation after (17) is the V of P equation:

Particular solution $\boldsymbol{y_p}$
Null solutions $\boldsymbol{e^{st}, te^{st}}$
$$y_p(t) = -e^{st} \int_0^t Te^{-sT} f(T)dT + te^{st} \int_0^t e^{-sT} f(T)dT$$

This is the integral of $-Te^{s(t-T)}f(T) + te^{s(t-T)}f(T)\,dt = (t-T)e^{s(t-T)}f(T)$.
This is exactly $g(t-T)f(T)$ when $g(t) = te^{st}$ in the equal roots case.
Neat conclusion : **Variation of Parameters gives exactly** $\boldsymbol{\int g(t-T)f(T)dT}$.

Problem Set 2.7, page 148

1 Take the Laplace transform of each term in these equations and solve for $Y(s)$, with $y(0) = 0$ and $y'(0) = 1$. Find the roots s_1 and s_2 — the poles of $Y(s)$:

Undamped	$y'' + \ 0y' + 16y = 0$
Underdamped	$y'' + \ 2y' + 16y = 0$
Critically damped	$y'' + \ 8y' + 16y = 0$
Overdamped	$y'' + 10y' + 16y = 0$

For the overdamped case use PF2 to write $Y(s) = A/(s-s_1) \ + \ B/(s-s_2)$.

Solution (a) Taking the Laplace Transform of $y'' + 0y' + 16y = 0$ gives :

$$s^2Y(s) - sy(0) - y'(0) + 0\cdot sY(s) - 0\cdot y(0) + 16Y(s) = 0$$
$$s^2Y(s) - 1 + 16Y(s) = 0$$
$$Y(s)(s^2+16) = 1$$
$$Y(s) = \boldsymbol{\frac{1}{s^2+16}}$$

The poles of Y = roots of s^2+16 are $s = 4i$ and $-4i$.

(b) Taking the Laplace Transform of $y'' + 2y' + 16y = 0$ gives :

$$s^2Y(s) - sy(0) - y'(0) + 2\cdot sY(s) - 2\cdot y(0) + 16Y(s) = 0$$
$$s^2Y(s) - 1 + 2sY(s) + 16Y(s) = 0$$
$$Y(s)(s^2+2s+16) = 1$$
$$Y(s) = \boldsymbol{\frac{1}{s^2+2s+16}}$$

The roots of $s^2+2s+16$ are $-1-i\sqrt{15}$ and $-1+i\sqrt{15}$. Underdamping.

(c) Taking the Laplace Transform of $y'' + 8y' + 16y = 0$ gives :

$$s^2Y(s) - sy(0) - y'(0) + 8\cdot sY(s) - 2\cdot y(0) + 16Y(s) = 0$$
$$s^2Y(s) - 1 + 8sY(s) + 16Y(s) = 0$$
$$Y(s)(s^2+8s+16) = 1$$
$$Y(s) = \frac{1}{s^2+8s+16} = \boldsymbol{\frac{1}{(s+4)^2}}$$

There is a double pole at $s = -4$. Critical damping.

(d) Taking the Laplace Transform of $y'' + 10y' + 16y = 0$ gives :

$$s^2Y(s) - sy(0) - y'(0) + 10 \cdot sY(s) - 10 \cdot y(0) + 16Y(s) = 0$$
$$s^2Y(s) - 1 + 10sY(s) + 16Y(s) = 0$$
$$Y(s)(s^2 + 10s + 16) = 1$$
$$Y(s) = \frac{1}{s^2 + 10s + 16} = \frac{1}{(s+2)(s+8)} = \frac{1}{6(s+2)} - \frac{1}{6(s+8)}$$

The poles of $Y(s)$ are -2 and -8: Overdamping.

2 Invert the four transforms $Y(s)$ in Problem 1 to find $y(t)$.

Solution (a) $Y(s) = \dfrac{1}{s^2+16} = \dfrac{1}{4} \cdot \dfrac{4}{s^2+16}$ inverts to $y(t) = \boldsymbol{\frac{1}{4}\sin(4t)}$.

(b) $Y(s) = \dfrac{1}{s^2+2s+16} = \dfrac{1}{(s+1)^2+15}$ inverts by equation (28) to $y(t) = \boldsymbol{e^{-t}\cos(\sqrt{15}t)/\sqrt{15}}$.

(c) $Y(s) = \dfrac{1}{(s+4)^2}$ inverts to $y(t) = \boldsymbol{te^{-4t}}$.

(d) $Y(s) = \dfrac{1}{6(s+2)} - \dfrac{1}{6(s+8)}$ inverts to $y(t) = \boldsymbol{\frac{1}{6}e^{-2t} - \frac{1}{6}e^{-8t}}$.

3 (a) Find the Laplace Transform $Y(s)$ from the equation $y' = e^{at}$ with $y(0) = A$.

(b) Use PF2 to break $Y(s)$ into two fractions $C_1/(s-a) + C_2/s$.

(c) Invert $Y(s)$ to find $y(t)$ and check that $y' = e^{at}$ and $y(0) = A$.

Solution (a) Taking the Laplace Transform of $y' = e^{at}$ gives:

$$sY(s) - y(0) = \frac{1}{s-a}$$
$$sY(s) - A = \frac{1}{s-a}$$
$$Y(s) = \frac{A}{s} + \frac{1}{s(s-a)}$$

(b) By using partial fractions $Y(s) = \dfrac{A}{s} + \dfrac{\frac{1}{a}}{(s-a)} + \dfrac{\frac{-1}{a}}{s}$

(c) The inverse Laplace Transform of each term gives:

$$\boldsymbol{y(t) = A + \frac{1}{a}e^{at} - \frac{1}{a}}$$

Differentiating gives: $y'(t) = a\dfrac{1}{a}e^{at} = e^{at}$ with $y(0) = A + \dfrac{1}{a} - \dfrac{1}{a} = A$.

4 (a) Find the transform $Y(s)$ when $y'' = e^{at}$ with $y(0) = A$ and $y'(0) = B$.

(b) Split $Y(s)$ into $C_1/(s-a) + C_2/(s-a)^2 + C_3/s$.

(c) Invert $Y(s)$ to find $y(t)$. Check $y'' = e^{at}$ and $y(0) = A$ and $y'(0) = B$.

Solution (a) The Laplace Transform of $y'' = e^{at}$ gives:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) &= \frac{1}{s-a} \\ s^2Y(s) &= sA + B + \frac{1}{s-a} \\ Y(s) &= \frac{A}{s} + \frac{B}{s^2} + \frac{1}{s^2(s-a)} \end{aligned}$$

(b) $\dfrac{1}{s^2(s-a)} = \dfrac{Cs+D}{s^2} + \dfrac{E}{s-a} = \dfrac{(s-a)(Cs+D)+Es^2}{s^2(s-a)}.$

That numerator matches 1 when $D = -\dfrac{1}{a}, C = -\dfrac{1}{a^2}, E = \dfrac{1}{a^2}.$

(c) $y(t) = A + Bt + C + Dt + Ee^{at} = \boldsymbol{A + Bt - \dfrac{1}{a^2} - \dfrac{t}{a} + \dfrac{1}{a^2}e^{at}}.$

5 Transform these differential equations to find $Y(s)$:

(a) $y'' - y' = 1$ with $y(0) = 4$ and $y'(0) = 0$

(b) $y'' + y = \cos \omega t$ with $y(0) = y'(0) = 0$ and $\omega \neq 1$

(c) $y'' + y = \cos t$ with $y(0) = y'(0) = 0$. What changed for $\omega = 1$?

Solution (a) The Laplace Transform of $y'' - y' = 1$ is

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) &= \frac{1}{s} \\ s^2Y(s) - 4s - sY(s) + 4 &= \frac{1}{s} \\ Y(s)(s^2 - s) &= \frac{1}{s} + 4s - 4 \\ Y(s) &= \frac{\frac{1}{s} + 4s - 4}{s^2 - s} \\ Y(s) &= \frac{4s^2 - 4s + 1}{s^3 - s^2} \\ Y(s) &= \frac{(2s-1)^2}{s^2(s-1)} \\ Y(s) &= \boldsymbol{-\frac{1}{s^2} + \frac{3}{s} + \frac{1}{s-1}} \end{aligned}$$

(b) The Laplace Transform of $y'' + y = \cos \omega t$ with $y(0) = 0$ and $y'(0) = 0$:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{s}{s^2 + \omega^2} \\ s^2Y(s) + Y(s) &= \frac{s}{s^2 + \omega^2} \\ Y(s) &= \frac{s}{(s^2 + \omega^2)(s^2 + 1)} \end{aligned}$$

(c) The Laplace Transform of $y'' + y = \cos t$ with $y(0) = 0$ and $y'(0) = 0$:

$$s^2Y(s) + Y(s) = \frac{s}{s^2+1}$$
$$Y(s) = \frac{\boldsymbol{s}}{\mathbf{(s^2+1)^2}} \text{ : Double poles from resonance}$$

6 Find the Laplace transforms F_1, F_2, F_3 of these functions f_1, f_2, f_3 :

(a) $f_1(t) = e^{at} - e^{bt}$ (b) $f_2(t) = e^{at} + e^{-at}$ (c) $f_3(t) = t\cos t$

Solution (a) The Laplace Transform of $e^{at} - e^{bt}$ is $\dfrac{1}{s-a} - \dfrac{1}{s-b} = \dfrac{a-b}{(s-a)(s-b)}$.

(b) The Laplace Transform of $e^{at} + e^{-at}$ is $\dfrac{1}{s-a} + \dfrac{1}{s+a} = \dfrac{2s}{s^2-a^2}$.

(c) The Laplace Transform of te^{at} is $\frac{1}{(s-a)^2}$ by equation (19). With $a = i$, write $t\cos t = \frac{1}{2}te^{it} + \frac{1}{2}te^{-it}$. Then the transform of $t\cos t$ is

$$\frac{1}{2}\frac{1}{(s-i)^2} + \frac{1}{2}\frac{1}{(s+i)^2} = \frac{1}{2}\frac{(s+i)^2+(s-i)^2}{(s-i)^2(s+i)^2} = \frac{\mathbf{s^2-1}}{\mathbf{(s^2+1)^2}}.$$

7 For any real or complex a, the transform of $f = te^{at}$ is _____. By writing $\cos\omega t$ as $(e^{i\omega t} + e^{-i\omega t})/2$, transform $g(t) = t\cos\omega t$ and $h(t) = te^t\cos\omega t$. (*Notice that the transform of h is new.*)

Solution The transform of te^{at} is $\frac{1}{(s-a)^2}$ by equation (19). Here $a = i\omega$. Then $t\cos\omega t = \frac{1}{2}te^{i\omega t} + \frac{1}{2}te^{-i\omega t}$ transforms to

$$\frac{1}{2}\frac{1}{(s-i\omega)^2} + \frac{1}{2}\frac{1}{(s+i\omega)^2} = \frac{1}{2}\frac{(s+i\omega)^2+(s-i\omega)^2}{(s-i\omega)^2(s+i\omega)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

Similarly $te^t\cos\omega t = \frac{1}{2}te^{(1+i\omega)t} + \frac{1}{2}te^{(1-i\omega)t}$ transforms to

$$\frac{1}{2}\frac{1}{(s-1-i\omega)^2} + \frac{1}{2}\frac{1}{(s-1+i\omega)^2} = \frac{1}{2}\frac{(s-1+i\omega)^2+(s-1-i\omega)^2}{(s-1-i\omega)^2(s-1+i\omega)^2} = \frac{(s-1)^2-\omega^2}{((s-1)^2+\omega^2)^2}.$$

8 Invert the transforms F_1, F_2, F_3 using PF2 and PF3 to discover f_1, f_2, f_3 :

(a) $F_1(s) = \dfrac{1}{(s-a)(s-b)}$ (b) $F_2(s) = \dfrac{s}{(s-a)(s-b)}$ (c) $F_3(s) = \dfrac{1}{s^3-s}$

Solution (a) $F_1(s) = \dfrac{1}{(s-a)(s-b)} = \dfrac{1}{(a-b)(s-a)} + \dfrac{1}{(b-a)(s-b)}$.

The inverse transform is $f_1 = \dfrac{\mathbf{1}}{\boldsymbol{(a-b)}}\boldsymbol{e^{at}} + \dfrac{\mathbf{1}}{\boldsymbol{(b-a)}}\boldsymbol{e^{bt}}$.

(b) $F_2(s) = \dfrac{s}{(s-a)(s-b)} = \dfrac{a}{(a-b)(s-a)} + \dfrac{b}{(b-a)(s-b)}$.

The inverse transform is $f_2 = \dfrac{\boldsymbol{a}}{\boldsymbol{(a-b)}}\boldsymbol{e^{at}} + \dfrac{\boldsymbol{b}}{\boldsymbol{(b-a)}}\boldsymbol{e^{bt}}$.

(c) $F_3(s) = \dfrac{1}{s^3-s} = \dfrac{1}{(s-1)(s+1)s} = -\dfrac{1}{s} + \dfrac{\frac{1}{2}}{s+1} + \dfrac{\frac{1}{2}}{s-1}$ using **PF3**.

The inverse transform is $f_3 = \boldsymbol{-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}}$.

9 Step 1 transforms these equations and initial conditions. Step 2 solves for $Y(s)$. Step 3 inverts to find $y(t)$:

(a) $y' - ay = t$ with $y(0) = 0$

(b) $y'' + a^2y = 1$ with $y(0) = 1$ and $y'(0) = 2$

(c) $y'' + 3y' + 2y = 1$ with $y(0) = 4$ and $y'(0) = 5$.

What particular solution y_p to (c) comes from using "undetermined coefficients"? $\boldsymbol{y_p = \frac{1}{2}}$.

Solution (a) $y' - ay = t$ transforms to $sY(s) - y(0) - aY(s) = \dfrac{1}{s^2}$ with $y(0) = 0$.

$$Y(s) = \frac{1}{s^2(s-a)} = \frac{-\frac{1}{a^2}}{s} + \frac{-\frac{1}{a}}{s^2} + \frac{\frac{1}{a^2}}{s-a}$$

The inverse transform is $y(t) = \boldsymbol{-\dfrac{1}{a^2} - \dfrac{1}{a}t + \dfrac{1}{a^2}e^{at}}$.

(b) $y'' + a^2y = 1$ transforms to $s^2Y(s) - sy(0) - y'(0) + a^2Y(s) = \dfrac{1}{s}$ with $y(0) = 1$ and $y'(0) = 2$. This is $(s^2+a^2)Y(s) = y'(0) + sy(0) + \dfrac{1}{s}$:

$$Y(s) = \frac{2}{s^2+a^2} + \frac{s}{s^2+a^2} + \frac{1}{s(s^2+a^2)} = \frac{2}{a}\frac{a}{s^2+a^2} + \frac{s}{s^2+a^2} + \frac{1}{a^2s} - \frac{1}{a^2}\frac{s}{s^2+a^2}.$$

The inverse transform is $y(t) = \boldsymbol{\dfrac{2}{a}\sin(at) + \cos(at) + \dfrac{1}{a^2} - \dfrac{1}{a^2}\cos(at)}$.

(c) $y'' + 3y' + 2y = 1$ becomes $s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = \dfrac{1}{s}$.

Then $y(0) = 4$ and $y'(0) = 5$ give

$$Y(s) = \frac{1}{s(s^2+3s+2)} + \frac{4s+5}{(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} + \frac{4(s+1)+1}{(s+1)(s+2)}.$$

The inverse transform can come from **PF3** on page 143. It comes much more quickly and directly (without Laplace transforms!) from knowing that $y = y_p + y_n = \frac{1}{2} + c_1e^{-t} + c_2e^{-2t}$:

$y(0) = \frac{1}{2} + c_1 + c_2 = 4$ and $y'(0) = -c_1 - 2c_2 = 5$ add to $\frac{1}{2} - c_2 = \frac{18}{2}$ and $y(t) = \boldsymbol{\frac{1}{2} + 12e^{-t} - \frac{17}{2}e^{-2t}}$.

Questions 10-16 are about partial fractions.

10 Show that PF2 in equation (9) is correct. Multiply both sides by $(s-a)(s-b)$:

$$(*) \qquad 1 = \_\_\_\_\_ + \_\_\_\_\_.$$

(a) What do those two fractions in $(*)$ equal at the points $s = a$ and $s = b$?

(b) The equation $(*)$ is correct at those two points a and b. It is the equation of a straight _____. So why is it correct for every s ?

Solution (using b instead of c in PF2) :

$\boldsymbol{1 = \dfrac{s-b}{a-b} + \dfrac{s-a}{b-a}}$ after multiplying equation (9) by $(s-a)(s-b)$.

(a) At $s = a$ we get $1 = \frac{a-b}{a-b}$. At $s = b$ we get $1 = \frac{b-a}{b-a}$.

(b) When the equation of a *straight line* is correct for two values $s = a$ and $s = b$, it is correct for all values of s.

11 Here is the PF2 formula with numerators. Formula $(*)$ had $K = 1$ and $H = 0$:

$$\textbf{PF2}' \qquad \frac{Hs+K}{(s-a)(s-b)} = \frac{Ha+K}{(s-a)(a-b)} + \frac{Hb+K}{(b-a)(s-b)}$$

To show that PF2′ is correct, multiply both sides by $(s-a)(s-b)$. You are left with the equation of a straight _____. Check your equation at $s = a$ and at $s = b$. Now it must be correct for all s, and PF2′ is proved.

Solution Multiplying by $(s-a)(s-b)$ produces

$$\textbf{(*)}\ Hs + K = \frac{(Ha+K)(s-b)}{a-b} + \frac{(Hb+K)(s-a)}{b-a}.$$

At $s = a$ this is $Ha + K = Ha + K + 0$: correct. Similarly correct at $s = b$. Since **(*)** is linear in s, it is the equation of a straight line. When correct at 2 points $s = a$ and $s = b$, it is correct for every s.

12 Break these functions into two partial fractions using PF2 and PF2′ :

(a) $\dfrac{1}{s^2-4}$ (b) $\dfrac{s}{s^2-4}$ (c) $\dfrac{Hs+K}{s^2-5s+6}$

Solution (a) $\dfrac{1}{s^2-4} = \dfrac{1}{(s-2)(s+2)} = \dfrac{1}{(s-2)(2+2)} + \dfrac{1}{(s+2)(-4)}$

$$= \boldsymbol{\frac{1}{4(s-2)} - \frac{1}{4(s+2)}}$$

(b) $\dfrac{s}{s^2-4} = \dfrac{s}{(s-2)(s+2)} = \dfrac{2}{(s-2)(2+2)} + \dfrac{-2}{(-4)(s+2)}$

$$= \boldsymbol{\frac{1}{2(s-2)} + \frac{1}{2(s+2)}}$$

(c) $\dfrac{Hs+K}{s^2-5s+6} = \dfrac{Hs+K}{(s-2)(s-3)}$

$$= \frac{2H+K}{(s-2)(2-3)} + \frac{3H+K}{(3-2)(s-3)}$$

$$= \boldsymbol{-\frac{2H+K}{s-2} + \frac{3H+K}{s-3}}$$

13 Find the integrals of $(a)(b)(c)$ in Problem 12 by integrating each partial fraction. The integrals of $C/(s-a)$ and $D/(s-b)$ are logarithms.

Solution (a) $$\int \frac{1}{s^2-4}\,ds = \int \frac{1}{4(s-2)} - \frac{1}{4(s+2)}\,ds$$
$$= \frac{1}{4}\ln(s-2) - \frac{1}{4}\ln(s+2) = \boldsymbol{\frac{1}{4}\ln\frac{s-2}{s+2}}$$

(b) $$\int \frac{s}{s^2-4}\,ds = \int \frac{1}{2(s-2)} + \frac{1}{2(s+2)}\,ds$$
$$= \frac{1}{2}\ln(s-2) + \frac{1}{2}\ln(s+2) = \boldsymbol{\frac{1}{2}\ln(s^2-4)}$$

(c) $$\int \frac{Hs+K}{s^2-5s+6}\,ds = \int -\frac{2H+K}{s-2} + \frac{3H+K}{s-3}\,ds$$
$$= \boldsymbol{-(2H+K)\ln(s-2) + (3H+K)\ln(s-3)}$$

14 Extend PF3 to PF3$'$ in the same way that PF2 extended to PF2$'$:

$$\textbf{PF3}' \qquad \frac{Gs^2+Hs+K}{(s-a)(s-b)(s-c)} = \frac{Ga^2+Ha+K}{(s-a)(a-b)(a-c)} + \frac{?}{?} + \frac{?}{?}.$$

Solution We want $\dfrac{Gs^2+Hs+K}{(s-a)(s-b)(s-c)} = \dfrac{A}{s-a} + \dfrac{B}{s-b} + \dfrac{C}{s-c}$.

We can multiply both sides by $(s-a)(s-b)(s-c)$ and solve for A, B, C. Or we can use A as given in the problem statement—and permute letters a, b, c to get B and C from A. That way is easier, and our three fractions are

$$\boldsymbol{\frac{a^2G+aH+K}{(a-b)(a-c)}\frac{1}{s-a} + \frac{b^2G+bH+K}{(b-a)(b-c)}\frac{1}{s-b} + \frac{c^2G+cH+K}{(c-a)(c-b)}\frac{1}{s-c}}$$

15 The linear polynomial $(s-b)/(a-b)$ equals 1 at $s=a$ and 0 at $s=b$. Write down a quadratic polynomial that equals 1 at $s=a$ and 0 at $s=b$ and $s=c$.

Solution $\dfrac{(s-b)(s-c)}{(a-b)(a-c)}$ equals 0 for $s=b$ and $s=c$. It equals 1 for $s=a$.

16 What is the number C so that $C(s-b)(s-c)(s-d)$ equals 1 at $s=a$?

Note A complete theory of partial fractions must allow double roots (when $b=a$). The formula can be discovered from l'Hôpital's Rule (in PF3 for example) when b approaches a. Multiple roots lose the beauty of PF3 and PF3$'$—we are happy to stay with simple roots a, b, c.

Solution Choose $C = \boldsymbol{\dfrac{1}{(a-b)(a-c)(a-d)}}$.

Questions 17-21 involve the transform $\boldsymbol{F(s)=1}$ of the delta function $\boldsymbol{f(t)=\delta(t)}$.

17 Find $F(s)$ from its definition $\int\limits_0^\infty f(t)e^{-st}dt$ when $f(t) = \delta(t-T)$, $T \geq 0$.

Solution The transform of $\delta(t-T)$ is $F(s) = \int\limits_0^\infty \delta(t-T)\, e^{-st}dt = \boldsymbol{e^{-sT}}$.

18 Transform $y'' - 2y' + y = \delta(t)$. The **impulse response** $y(t)$ transforms into $Y(s) =$ **transfer function**. The double root $s_1 = s_2 = 1$ gives a double pole and a new $y(t)$.

Solution With $y(0) = y'(0) = 0$, the transform is $(s^2 - 2s + 1)Y(s) = 1$. Then $Y(s) = \frac{1}{(s-1)^2}$ and the inverse transform is the impulse response $y(t) = g(t) = \boldsymbol{te^t}$.

19 Find the inverse transforms $y(t)$ of these transfer functions $Y(s)$:

(a) $\dfrac{s}{s-a}$ (b) $\dfrac{s}{s^2-a^2}$ (c) $\dfrac{s^2}{s^2-a^2}$

Solution (a) $Y(s) = \dfrac{s}{s-a} = \dfrac{s-a+a}{s-a} = 1 + \dfrac{a}{s-a}$

$$y(t) = \boldsymbol{\delta(t) + ae^{at}}$$

(b) Using **PF2** we have $Y(s) = \dfrac{s}{s^2-a^2} = \dfrac{s}{(s-a)(s+a)} = \dfrac{1}{2(s-a)} + \dfrac{1}{2(s+a)}$

The inverse transform is $y(t) = \boldsymbol{\dfrac{1}{2}e^{at} + \dfrac{1}{2}e^{-at}} = \cosh at$

(c) $Y(s) = \dfrac{s^2}{s^2-a^2} = \dfrac{s^2-a^2+a^2}{s^2-a^2} = 1 + \dfrac{a^2}{s^2-a^2} = 1 + \dfrac{a}{2(s-a)} - \dfrac{a}{2(s+a)}$

$$y(t) = \boldsymbol{\delta(t) + \frac{a}{2}e^{at} - \frac{a}{2}e^{-at} = \delta(t) + a\sinh(at)}$$

20 Solve $y'' + y = \delta(t)$ by Laplace transform, with $y(0) = y'(0) = 0$. If you found $y(t) = \sin t$ as I did, this involves a serious mystery: *That sine solves* $y'' + y = 0$, and it doesn't have $y'(0) = 0$. *Where does* $\delta(t)$ *come from*? In other words, what is the derivative of $y' = \cos t$ if all functions are zero for $t < 0$?

If $y = \sin t$, explain why $y'' = -\sin t + \delta(t)$. Remember that $y = 0$ **for** $t < 0$.

Problem (20) connects to a remarkable fact. The same impulse response $y = g(t)$ solves both of these equations: **An impulse at $t = 0$ makes the velocity $y'(0)$ jump by 1**. Both equations start from $y(0) = 0$.

$$y'' + By' + Cy = \boldsymbol{\delta(t)} \text{ with } y'(0) = \mathbf{0} \quad y'' + By' + Cy = \mathbf{0} \text{ with } y'(0) = \mathbf{1}.$$

Solution $y'' + y = \delta(t)$ transforms into $s^2Y(s) + Y(s) = 1$.

Then $Y(s) = \frac{1}{s^2+1}$ has the inverse transform $y(t) = \boldsymbol{\sin t}$.

At time $t = 0$ the derivative of $y' = \cos(t)$ is not $y'' = \sin(0) = 0$, but rather $\boldsymbol{y'' = \sin(0) + \delta(t)}$, since the function $y' = \cos(t)$ jumps from 0 to 1 at $t = 0$.

21 (Similar mystery) These two problems give the same $Y(s) = s/(s^2+1)$ and the same impulse response $y(t) = g(t) = \cos t$. How can this be ?

(a) $y' = -\sin t$ with $y(0) = \mathbf{1}$ (b) $y' = -\sin t + \boldsymbol{\delta(t)}$ with "$y(0) = 0$"

Solution (a) The Laplace transform of $y'(t) = -\sin(t)$ with $y(0) = 1$ is

$$sY(s) - 1 = -\frac{1}{s^2+1}$$
$$sY(s) = 1 - \frac{1}{s^2+1} = \frac{s^2+1-1}{s^2+1} = \frac{s^2}{s^2+1}$$
$$Y(s) = \frac{s}{s^2+1}$$

(b) The Laplace transform of $y'(t) = -\sin(t) + \delta(t)$ with $y(0) = 0$ is

$$sY(s) - y(0) = -\frac{1}{s^2+1^2} + 1$$
$$sY(s) - 0 = \frac{s^2+1-1}{s^2+1} = \frac{s^2}{s^2+1}$$
$$Y(s) = \frac{s}{s^2+1}$$

These two problems (a) and (b) give the same $Y(s)$ and therefore the same $y(t)$. The reason is that $\delta(t)$ in the derivative y' gives the same result as an initial condition $y(0) = 1$. Both cause a jump from $y = 0$ before $t = 0$ to $y = 1$ right after $t = 0$. And both transform to 1.

Problems 22-24 involve the Laplace transform of the integral of $\boldsymbol{y(t)}$.

22 If $f(t)$ transforms to $F(s)$, what is the transform of the integral $h(t) = \int_0^t f(T)dT$? Answer by transforming the equation $dh/dt = f(t)$ with $h(0) = 0$.

Solution If $h(t) = \int_0^t f(T)\,dT$ then $dh/dt = f(t)$ with $h(0) = 0$. Taking the Laplace Transform gives :

$$sH(s) = F(s) \text{ and } H(s) = \frac{\boldsymbol{F(s)}}{\boldsymbol{s}}.$$

23 Transform and solve the integro-differential equation $y' + \int_0^t y\,dt = 1$, $y(0) = 0$.

A mystery like Problem 20 : $y = \cos t$ seems to solve $y' + \int_0^t y dt = 0, y(0) = 1$.

Solution The Laplace transform of $y' + \int_0^t y\,dt = 1$ with $y(0) = 0$ is

$$sY(s) - y(0) + \frac{Y(s)}{s} = \frac{1}{s}$$
$$Y(s) = \frac{1}{\left(s + \frac{1}{s}\right)s} = \frac{1}{s^2+1}$$

The inverse transform of $Y(s)$ is $\boldsymbol{y(t) = \sin(t)}$

About the mystery: The derivative of $\cos t$ is $-\sin t + \delta(t)$ because $\cos t$ jumps at $t = 0$ from zero for $t < 0$ (by convention) to 1. But I am not seeing a new mystery.

24 Transform and solve the amazing equation $dy/dt + \int_0^t y\,dt = \delta(t)$.

Solution The transform of $\dfrac{dy}{dt} + \displaystyle\int_0^t y\,dt = \delta(t)$ is $sY(s) + \dfrac{Y(s)}{s} = 1.$

Then $Y(s) = \dfrac{1}{\left(s + \frac{1}{s}\right)s} = \dfrac{s}{s^2+1}$ and $\boldsymbol{y(t) = \cos t}$.

Note that this follows from Problem 20, where we found that $\cos(t)$ has integral $\sin(t)$ and derivative $-\sin(t) + \delta(t)$.

25 The derivative of the delta function is not easy to imagine—it is called a "doublet" because it jumps up to $+\infty$ and back down to $-\infty$. Find the Laplace transform of the doublet $d\delta/dt$ from the rule for the transform of a derivative.

A doublet $\delta'(t)$ is known by its integral: $\int \delta'(t)F(t)dt = -\int \delta(t)F'(t)dt = \boldsymbol{-F'(0)}$.

Solution The Laplace transform of $\delta(t)$ is 1. The Laplace transform of the derivative is $sY(s) - y(0)$. The Laplace transform of the doublet $\delta'(t) = d\delta/dt$ is therefore $\boldsymbol{s}$.

26 (Challenge) What function $y(t)$ has the transform $Y(s) = 1/(s^2+\omega^2)(s^2+a^2)$? First use partial fractions to find H and K:

$$Y(s) = \frac{H}{s^2+\omega^2} + \frac{K}{s^2+a^2}$$

Solution $Y(s) = \dfrac{1}{(s^2+\omega^2)(s^2+a^2)} = \dfrac{1}{(s^2+\omega^2)(a^2-\omega^2)} - \dfrac{1}{(s^2+a^2)}\dfrac{1}{(a^2-\omega^2)}.$

Then $y(t) = \dfrac{\boldsymbol{\sin \omega t}}{\boldsymbol{\omega(a^2-\omega^2)}} - \dfrac{\boldsymbol{\sin at}}{\boldsymbol{a(a^2-\omega^2)}}.$

27 Why is the Laplace transform of a unit step function $H(t)$ the same as the Laplace transform of a constant function $f(t) = 1$?

Solution The step function and the constant function are the same for $t \geq 0$.

Problem Set 3.1, page 160

1 (a) Why do two isoclines $f(t, y) = s_1$ and $f(t, y) = s_2$ never meet ?

(b) Along the isocline $f(t, y) = s$, what is the slope of all the arrows ?

(c) Then all solution curves go only one way across an _____ .

Solution (a) Isoclines can't meet because $f(t, y)$ has one fixed value along an isocline.

(b) The slope of the arrows is fixed at s along the isocline $f(t, y) = s$.

(c) All solution curves go one way (with slope s) across the isocline $f(t, y) = s$.

2 (a) Are isoclines $f(t, y) = s_1$ and $f(t, y) = s_2$ always parallel ? Always straight ?

(b) An isocline $f(t, y) = s$ is a solution curve when its slope equals _____ .

(c) The zerocline $f(t, y) = 0$ is a solution curve only when y is _____ : slope 0.

Solution (a) In case $f(t, y)$ does not depend on t (autonomous equation) the isoclines are horizontal lines. In general isoclines need to be parallel or straight.

(b) If the slope of the isoclines $f(t, y) = s$ happens to be s (slope of arrows equals slope of curve, so the arrows go along the isocline) then the isocline is actually a solution curve. Example: A steady state where $f(y) = 0$ has arrows of slope zero. That horizontal isocline is also the graph of the constant solution $y(t) = Y$.

(c) The zerocline is a solution curve when the slope is zero and y is **constant**.

3 If $y_1(0) < y_2(0)$, what continuity of $f(t, y)$ assures that $y_1(t) < y_2(t)$ for all t ?

Solution Two solution curves $y_1(t)$ and $y_2(t)$ can't meet or cross if they are continuous curves : this will be true if f and $\partial f/\partial y$ are continuous.

4 The equation $dy/dt = t/y$ is completely safe if $y(0) \neq 0$. Write the equation as $y\,dy = t\,dt$ and find its unique solution starting from $y(0) = -1$. The solution curves are hyperbolas—can you draw two on the same graph ?

Solution $dy/dt = t/y$ leads to $\int y\,dy = \int t\,dt$ and $y^2 = t^2 + C$. If $y(0) = -1$ then $y(t) = -\sqrt{t^2 + 1}$. The hyperbolas $y^2 = t^2 + C$ are asymptotic to the 45° and -45° lines $y = t$ and $y = -t$.

5 The equation $dy/dt = y/t$ has many solutions $y = Ct$ in case $y(0) = 0$. It has no solution if $y(0) \neq 0$. When you look at all solution curves $y = Ct$, which points in the t, y plane have no curve passing through ?

Solution The solution curves $y = Ct$ (allowing all numbers C) go through all points (t, y) with suitable $C = y/t$—**except** the points on the vertical line $t = 0$ (other than the origin $(0, 0)$ that all the lines $y = Ct$ will pass through). You cannot solve $dy/dt = y/t$ with an initial value like $y(0) = 1$, because the right side y/t would be $1/0$.

6 For $y' = ty$ draw the isoclines $ty = 1$ and $ty = 2$ (those will be hyperbolas). On each isocline draw four arrows (they have slopes 1 and 2). Sketch pieces of solution curves that fit your picture between the isoclines.

Solution The solution curves $dy/dt = ty$ have $dy/y = t\,dt$ and $\ln y = \frac{1}{2}t^2 + c$ and $y = \exp\left(\frac{1}{2}t^2 + c\right) = C\exp\left(\frac{1}{2}t^2\right)$. Solution curves cross isoclines $f(t, y) = s$ with that slope s! **The arrows with that slope are tangent to the curves as they cross the isocline**.

7 The solutions to $y' = y$ are $y = Ce^t$. Changing C gives a higher or lower curve. But $y' = y$ is autonomous, its solution curves should be shifting right and left!

Draw $y = 2e^t$ and $y = -2e^t$ to show that they really are *right-left shifts* of $y = e^t$ and $y = -e^t$. The shifted solutions to $y' = y$ are e^{t+C} and $-e^{t+C}$.

Solution For all autonomous equations $dy/dt = f(y)$, the solution curves are horizontal shifts of each other. In particular for $f(y) = y$, the curves $y = Ce^t$ shift right-left as C increases-decreases.

8 For $y' = 1 - y^2$ the flat lines $y =$ constant are isoclines $1 - y^2 = s$. Draw the lines $y = 0$ and $y = 1$ and $y = -1$. On each line draw arrows with slope $1 - y^2$. The picture says that $y =$ _____ and $y =$ _____ are steady state solutions. From the arrows on $y = 0$, guess a shape for the solution curve $y = (e^t - e^{-t})/(e^t + e^{-t})$.

Solution The picture will show the horizontal lines $y = 1$ and $y = -1$ as "zeroclines" where $f(t, y) = s = 1 - y^2 = 0$. So those are steady state solution curves $y(t) = Y = 1$ or -1.

The isocline $y = 0$ is the x-axis, along with $f(t, y) = 1 - y^2 = 1 = s$. (The arrows cross the x-axis at $45\,°$, with slope $s = 1$.) So the solution curves are S-curves going up from the line $y = -1$ to the line $y = 1$, rising at $45\,°$ along the x-axis halfway between those two lines.

9 The parabola $y = t^2/4$ and the line $y = 0$ are both solution curves for $\boldsymbol{y' = \sqrt{|y|}}$. Those curves meet at the point $t = 0$, $y = 0$. What continuity requirement is failed by $f(y) = \sqrt{|y|}$, to allow more than one solution through that point ?

Solution The function $f(y) = \sqrt{|y|}$ is continuous at $y = 0$ but its derivative $df/dy = 1/2\sqrt{|y|}$ blows up (because of $1/0$.) So two solutions can start from the same initial value $y(0) = 0$, and they do.

10 Suppose $y = 0$ up to time T is followed by the curve $y = (t - T)^2/4$. Does this solve $\boldsymbol{y' = \sqrt{|y|}}$? Draw this $y(t)$ going through flat isoclines $\sqrt{|y|} = 1$ and 2.

Solution Yes, $y' = \sqrt{|y|}$ is solved by the constant $y(t) = 0$. It is also solved by the curve $y(t) = (t - T)^2/4$ because $dy/dt = (t - T)/2$ equals the square root of $|y(t)|$. So solution curves can lift off the x-axis $y = 0$ anywhere they want, and start upwards on a parabola.

11 The equation $\boldsymbol{y' = y^2 - t}$ is often a favorite in MIT's course 18.03: not too easy. Why do solutions $y(t)$ rise to their maximum on $y^2 = t$ and then descend ?

Solution Below the parabola $y^2 = t$ (which opens to the right instead of opening upwards) the right side of $dy/dt = y^2 - t$ will be negative. The solution curves have negative slope and they can't cross the rising parabola.

12 Construct $f(t, y)$ with two isoclines so solution curves go *up* through the higher isocline and other solution curves go *down* through the lower isocline. *True or false*: Some solution curve will stay between those isoclines: **A continental divide**.

Solution We want the isocline $f(t, y) = s = 1$ to be *above* the isocline $f(t, y) = s = -1$. A simple example would be $\boldsymbol{f(t, y) = y}$. Then the equation $dy/dt = y$ has solution curves $y = Ce^t, C > 0$ going *up* through the isocline $f(t, y) = 1$ (which is the flat line $y = 1$). The curves $y = Ce^t$ with $C < 0$ go down through $y = -1$. The **continental divide** is the solution curve $y(t) = 0$ with $C = 0$. Certainly $y(t) = 0$ does solve $dy/dt = y$.

There is always a "continental divide" where solution curves (like water in the Rockies) can't choose between the Atlantic and the Pacific.

Problem Set 3.2, page 168

1 Draw Figure 3.6 for a sink (the missing middle figure) with $y = c_1e^{-2t} + c_2e^{-t}$. Which term dominates as $t \to \infty$? The paths approach the dominating line as they go in toward zero. **The slopes of the lines are -2 and -1** (the numbers s_1 and s_2).

Solution The c_2e^{-t} term dominates at $t \to \infty$ since it decays at a slower rate.

$$\text{Then } y(t) = \frac{\sin \omega t}{\omega(a^2 - \omega^2)} - \frac{\sin at}{a(a^2 - \omega^2)}.$$

2 Draw Figure 3.7 for a spiral sink (the missing middle figure) with roots $s = -1 \pm i$. The solutions are $y = C_1e^{-t}\cos t + C_2e^{-t}\sin t$. They approach zero because of the factor e^{-t}. They spiral around the origin because of $\cos t$ and $\sin t$.

Solution The spiral goes clockwise in toward $(0, 0)$. Not easy to draw to scale, by hand!

3 Which path does the solution take in Figure 3.6 if $y = e^t + e^{t/2}$? Draw the curve $(y(t), y'(t))$ more carefully starting at $t = 0$ where $(y, y') = (2, 1.5)$.

Solution As $t \to \infty$, the path of the point $(y(t), y'(t))$ comes closer and closer to the path for $y = e^t$—**because e^t dominates the other term $e^{t/2}$**. The path for $y = e^t$ has points $(y, y') = (e^t, e^t)$ so it is a straight $45°$ line in the (y, y') plane.

4 Which path does the solution take around the saddle in Figure3.6 if $y = e^{t/2} + e^{-t}$? Draw the curve more carefully starting at $t = 0$ where $(y, y') = (2, -\frac{1}{2})$.

Solution The function $y = e^{t/2} + e^{-t}$ comes from exponents $\frac{1}{2}$ and -1 (positive and negative will give a **saddle point**. The graph shows the spiral is unwinding clockwise as it leaves the tight spiral and goes outward. For large t the dominant part of (y, y') will be $(e^{t/2}, \frac{1}{2}e^{t/2})$ from the growing term $e^{t/2}$ in y.

5 Redraw the first part of Figure 3.6 when the roots are equal: $s_1 = s_2 = 1$ and $y = c_1e^t + c_2te^t$. *There is no s_2-line*. Sketch the path for $y = e^t + te^t$.

Solution $y = e^t + te^t$ has $y' = 2e^t + te^t$. The larger term te^t gives $(y, y') \approx (te^t, te^t)$ on the $45°$ line in the y, y' plane. At $t = 0$ it starts from $(y(0), y'(0)) = (1, 2)$.

6 The solution $y = e^{2t} - 4e^t$ gives a source (Figure 3.6), with $y' = 2e^{2t} - 4e^t$. Starting at $t = 0$ with $(y, y') = (-3, -2)$, where is (y, y') when $e^t = 1.1$ and $e^t = .25$ and $e^t = 2$?

Solution Substituting the values $t = \ln 1.1$ and $\ln 0.25$ and $\ln 2$, we get:

1. For $e^t = 1.1$ we have $(y, y') = (-3.19, -1.98)$
2. For $e^t = .25$ we have $(y, y') = (-0.9375, -0.875)$
3. For $e^t = 2$ we have $(y, y') = (-4, 0)$

Those early times don't show the situation for large t, when the dominant term e^{2t} gives $(y, y') = (e^{2t}, 2e^{2t})$ and the path approaches a straight line **with slope 2**.

7 The solution $y = e^t(\cos t + \sin t)$ has $y' = 2e^t \cos t$. This spirals out because of e^t. Plot the points (y, y') at $t = 0$ and $t = \pi/2$ and $t = \pi$, and try to connect them with a spiral. Note that $e^{\pi/2} \approx 4.8$ and $e^{\pi} \approx 23$.

Solution

1. For $t = 0, (y, y') = (1, 2)$
2. For $t = \frac{\pi}{2}, (y, y') = (e^{\pi/2}, 0) \approx (4.8, 0)$
3. For $t = \pi, (y, y') = (-e^{\pi}, -2e^{\pi}) \approx (-23.1, -46.2)$

Maybe we can see the path better by writing $(y, y') = e^t(\cos t, \cos t) + e^t(\sin t, \cos t)$. The first term goes forward and back on the $45\,^\circ$ line. the second term circles around and spirals out because of e^t. So we have a big circle around a moving slider.

8 The roots s_1 and s_2 are $\pm 2i$ when the differential equation is _____. Starting from $y(0) = 1$ and $y'(0) = 0$, draw the path of $(y(t), y'(t))$ around the center. Mark the points when $t = \pi/2, \pi, 3\pi/2, 2\pi$. Does the path go clockwise ?

Solution The differential equation is $\boldsymbol{y'' + 4y = 0}$. The solution starting at $(y, y') = (1, 0)$ is $(y(t), y'(t)) = \mathbf{(\cos 2t, -2 \sin 2t)}$. This is an ellipse in the equation

$$y^2 + \frac{1}{4}(y')^2 = \cos^2 2t + sin^2 2t = 1.$$

The path is clockwise around that elliptical center.

9 The equation $y'' + By' + y = 0$ leads to $s^2 + Bs + 1 = 0$. For $B = -3, -2, -1, 0$, 1, 2, 3 decide which of the six figures is involved. For $B = -2$ and 2, why do we not have a perfect match with the source and sink figures ?

Solution To determine which figure is involved, we solve the quadratic equation:

$$s_1 \text{ and } s_2 \text{ are } \frac{-B \pm \sqrt{B^2 - 4}}{2}$$

$\boldsymbol{B = -3}$ has $s_1 = \frac{3-\sqrt{5}}{2} \approx 0.38$ and $s_2 = \frac{3+\sqrt{5}}{2} \approx 2.6$. **Source** with $\mathbf{0 < s_1 < s_2}$

$\boldsymbol{B = -2}$ has $s_1 = 1$ and $s_2 = 1$. Since $0 < s_1 = s_2$ we have a **source**

$\boldsymbol{B = -1}$ has $s_1 = \frac{1+\sqrt{3}i}{2}$ and $s_2 = \frac{1+\sqrt{3}i}{2}$. **Spiral Source (outward)** $\text{Re}(s_1) = \text{Re}(s_2) > 0$

$\boldsymbol{B =}$ **0** has $s_1 = i$ and $s_2 = -i$. Since $0 = \text{Re}(s_1) = \text{Re}(s_2)$ we have a **center**

$\boldsymbol{B =}$ **1** has $s_1 = \frac{-1+\sqrt{3}i}{2}$ and $s_2 = \frac{-1+\sqrt{3}i}{2}$. **Spiral Sink (inward)** $\text{Re}(s_1) = \text{Re}(s_2) < 0$

$\boldsymbol{B =}$ **2** has $s_1 = -1$ and $s_2 = -1$. Since $s_1 = s_2 < 0$ we have a **sink**

$\boldsymbol{B =}$ **3** has $s_1 = \frac{-3-\sqrt{5}}{2} \approx -2.6$ and $s_2 = \frac{-3+\sqrt{5}}{2} \approx -0.38$. $s_1 < s_2 < 0$. This is a **sink**

The special case $B = 2$ and $B = -2$ gave **equal roots** $\boldsymbol{s_1 = s_2}$. Then there will be a factor "t" in the null solution. The path won't close on itself like a circle or ellipse. As it turns, it will go slowly outward from that factor t.

10 For $y'' + y' + Cy = 0$ with damping $B = 1$, the characteristic equation will be $s^2 + s + C = 0$. Which C gives the changeover from a *sink* (overdamping) to a spiral *sink* (underdamping)? Which figure has $C < 0$?

Solution The solutions to the quadratic equation $s^2 + s + C = 0$ are

$$s_1 \text{ and } s_2 \text{ are } \frac{-1 \pm \sqrt{1-4C}}{2}$$

The change from a sink to a spiral sink occurs at $C = \frac{1}{4}$. Those are sinks because the real part of s is negative. When C is less than zero, we change to one positive root and one negative root. Then the path becomes a **saddle**.

Problems 11–18 are about $dy/dt = Ay$ with companion matrices $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\boldsymbol{C} & -\boldsymbol{B} \end{bmatrix}$.

11 The eigenvalue equation is $\boldsymbol{\lambda^2 + B\lambda + C = 0}$. Which values of B and C give complex eigenvalues? Which values of B and C give $\lambda_1 = \lambda_2$?

Solution Look at the solution to the quadratic equation $\lambda^2 + B\lambda + C = 0$:

$$\lambda_1 \text{ and } \lambda_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

Therefore when $B^2 < 4C$ we get complex eigenvalues.

On the other hand, when $B^2 = 4C$ we get $\lambda_1 = \lambda_2 = -B/2$ (the square root is 0).

12 Find λ_1 and λ_2 if $B = 8$ and $C = 7$. Which eigenvalue is more important as $t \to \infty$? Is this a sink or a saddle?

Solution We solve the quadratic eigenvalue equation for λ_1 and λ_2:

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-8 \pm \sqrt{64 - 28}}{2} \text{ gives } \boldsymbol{\lambda_1 = -7} \text{ and } \boldsymbol{\lambda_2 = -1}.$$

Since $s_1 < s_2 < 0$ we have a **sink**. The more negative λ_2 gives slower decay as $t \to \infty$.

13 Why do the eigenvalues have $\lambda_1 + \lambda_2 = -B$? Why is $\lambda_1\lambda_2 = C$?

Solution This refers to the eigenvalues of the companion matrix:

$A = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$ comes from $\begin{array}{l} y_1' = y_2 \\ y_2' = -Cy - By_2 \end{array}$. Then $y_1'' = y_2'$ is $y_1'' + By_1' + Cy_1 = 0$.

The eigenvalues λ_1 and λ_2 are the roots of $\lambda^2 + B\lambda + C = 0$ just as the roots s_1 and s_2 are the roots of $s^2 + Bs + C = 0$. We know from factoring into $(s - s_1)(s - s_2)$ or $(\lambda - \lambda_1)(\lambda - \lambda_2)$ that the coefficient of λ^2 is 1, the coefficient of λ is $B = -\lambda_1 - \lambda_2$, and the constant form is $C = \lambda_1$ times λ_2.

14 Which second order equations did these matrices come from?

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (saddle)} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ (center)}$$

Solution Write the matrix equation $y' = Ay$ as two coupled first order equations. For A we get

$$y_1' = y_2$$
$$y_2' = y_1$$

Then $y_1'' = y_2' = y_1$ and the second order equation is $\boldsymbol{y'' = y}$.

The second matrix A_2 gives $y_1' = y_2$ and $y_2' = -y_1$.

Then $y_1'' = y_2' = -y_1$ and the second order equation is $\boldsymbol{y'' + y = 0}$. (Notice that we also find $y_2'' = -y_2$.)

15 The equation $y'' = 4y$ produces a saddle point at $(0,0)$. Find $s_1 > 0$ and $s_2 < 0$ in the solution $y = c_1e^{s_1t} + c_2e^{s_2t}$. If $c_1c_2 \neq 0$, this solution will be (large) (small) as $t \to \infty$ and also as $t \to -\infty$.

The only way to go toward the saddle $(y, y') = (0,0)$ as $t \to \infty$ is $c_1 = 0$.

Solution Assuming a solution of the form $y(t) = e^{st}$ gives:

$$\begin{aligned} y'' - 4y &= 0 \\ s^2e^{st} - 4e^{st} &= 0 \\ s^2 - 4 &= 0 \\ s &= \pm 2 \end{aligned}$$

Therefore $s_1 = 2$ and $s_2 = -2$. The solution becomes $y = \boldsymbol{c_1e^{2t} + c_2e^{-2t}}$. As $t \to \infty$, the e^{2t} term will grow unless $c_1 = 0$. In that case $(y, y') = (c_2e^{-2t}, -2c_2e^{-2t})$ goes to the saddle point $(0,0)$.

16 If $B = 5$ and $C = 6$ the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. The vectors $\boldsymbol{v} = (1,3)$ and $\boldsymbol{v} = (1,2)$ are *eigenvectors* of the matrix A: Multiply $A\boldsymbol{v}$ to get $3\boldsymbol{v}$ and $2\boldsymbol{v}$.

Solution $v = (1,3)$ is an eigenvector with eigenvalue $\lambda_1 = 3$:

$$Av = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3v.$$

Similarly $v = (1,2)$ is an eigenvector with eigenvalue $\lambda_2 = 2$:

$$\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Notice that these eigenvectors of the companion matrix A have the form $\boldsymbol{v = (1, \lambda)}$.

17 In Problem 16, write the two solutions $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$ to the equations $\boldsymbol{y}' = A\boldsymbol{y}$. Write the complete solution as a combination of those two solutions.

Solution The eigenvectors $\boldsymbol{v_1} = (1,3)$ and $\boldsymbol{v_2} = (1,2)$ give two pure exponential solutions $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$:

$$\boldsymbol{y_1} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \quad \text{and} \quad \boldsymbol{y_2} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}.$$

The complete solution is $\boldsymbol{y}(t) = c_1\boldsymbol{y_1} + c_2\boldsymbol{y_2}$. Two constants to match two components of the initial vector $\boldsymbol{y}(0)$ at $t = 0$. Then $\boldsymbol{y}(0) = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}$.

18 The eigenvectors of a companion matrix have the form $v = (1, \lambda)$. Multiply by A to show that $Av = \lambda v$ gives one trivial equation and the characteristic equation $\lambda^2 + B\lambda + C = 0$.

$$\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \qquad \text{is} \qquad \begin{aligned} \lambda &= \lambda \\ -C - B\lambda &= \lambda^2 \end{aligned}$$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution The eigenvectors of a companion matrix have the special form $\boldsymbol{v} = (1, \lambda)$, as the problem statement shows—because $-C - B\lambda = \lambda^2$ from the eigenvalue equation $\boldsymbol{\lambda^2 + B\lambda + C = 0}$.

The example A is *not* a companion matrix!

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ has eigenvectors } \boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ with } \lambda_1 = \mathbf{4} \text{ and } \lambda_2 = \mathbf{2}.$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The equation for λ is $\lambda^2 - 6\lambda + 8 = 0$ with 6 coming from the trace $3 + 3$ and 8 coming from the determinant $9 - 1$.

19 An equation is stable and all its solutions $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ go to $y(\infty) = 0$ exactly when

$(s_1 < 0$ **or** $s_2 < 0)$ $\qquad$ $(s_1 < 0$ **and** $s_2 < 0)$ $\qquad$ $(\text{Re } s_1 < 0$ **and** $\text{Re } s_2 < 0)$?

Solution The correct answer is $\quad (\text{Re} s_1 < 0$ **and** $\text{Re} s_2 < 0)$.

20 If $Ay'' + By' + Cy = D$ is stable, what is $y(\infty)$?

Solution The steady state solution to this equation is the constant $\boldsymbol{y(\infty) = D/C}$. Because the equation is stable, the null solution $y_n(t)$ will go to zero as $t \to \infty$. The roots s_1 and s_2 have negative real parts.

Problem Set 3.3, page 182

1 If $y' = 2y + 3z + 4y^2 + 5z^2$ and $z' = 6z + 7yz$, how do you know that $Y = 0$, $Z = 0$ is a critical point ? What is the 2 by 2 matrix A for linearization around $(0, 0)$? This steady state is certainly unstable because _____ .

Solution Here $y' = f(y, z)$ and $z' = g(y, z)$ have $f = g = 0$ at the point $(y, z) = (0, 0)$. Then this point is a critical point (stationary point). The Jacobian matrix of derivatives at that point $(0, 0)$ is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 2 + 8y & 3 + 10z \\ 7z & 6 + 7y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \quad \text{at } (y, z) = (0, 0).$$

The eigenvalues of this triangular matrix are 2 and 6 (on the diagonal). Any positive eigenvalue means growth and instability.

2 In Problem 1, change $2y$ and $6z$ to $-2y$ and $-6z$. What is now the matrix A for linearization around $(0,0)$? How do you know this steady state is stable ?

Solution
$A=\begin{bmatrix} -2+8y & 3+10z \\ 7z & -6+7y \end{bmatrix}=\begin{bmatrix} -2 & 3 \\ 0 & -6 \end{bmatrix}$ now has eigenvalues $\lambda=-2,-6$: **stable**.

3 The system $y' = f(y,z) = 1-y^2-z$, $z' = g(y,z) = -5z$ has a critical point at $Y=1$, $Z=0$. Find the matrix A of partial derivatives of f and g at that point: stable or unstable ?

Solution Here $f=g=0$ when $(Y,Z)=(1,0)$.

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} -2y & -1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}.$$ **Stable**.

4 This linearization is wrong but the zero derivatives are correct. *What is missing* ? $Y=0, Z=0$ is not a critical point of $y' = \cos(ay+bz)$, $z' = \cos(cy+dz)$.

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} -a\sin 0 & -b\sin 0 \\ -c\sin 0 & -d\sin 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

Solution At the point $(Y,Z)=(0,0)$, the functions $f=\cos(0+0)$ and $g=\cos(0+0)$ are equal to **1**. This is not a critical point.

5 Find the linearized matrix A at every critical point. Is that point stable ?

(a) $\begin{matrix} y' = 1-yz \\ z' = y-z^3 \end{matrix}$ (b) $\begin{matrix} y' = -y^3-z \\ z' = y+z^3 \end{matrix}$

Solution (a) $f(y,z)=1-yz$ and $g(y,z)=y-z^3$ are both zero when $y=z^3$ and then $1-yz=1-z^4=0$. Then $Z=1$ goes with $Y=1$ and $Z=-1$ goes with $Y=-1$: **two critical points**.

$$A = \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} -z & -y \\ 1 & -3z^2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \textbf{ OR } \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}.$$

The eigenvalues solve $\det(A-\lambda I)=0$.

At $(1,1)$ $\det\begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2+4\lambda+4=0,\ \boldsymbol{\lambda=-2,-2}$

At $(-1,-1)$ $\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2+2\lambda-4=0,\ \boldsymbol{\lambda=-1\pm\sqrt{5}}$

Then $(Y,Z)=(1,1)$ is **stable** but $(-1,-1)$ is **unstable** (because $-1+\sqrt{5}>0$).

(b) $f=-y^3-z$ and $g=y+z^3$ are both zero at $(Y,Z)=\mathbf{(0,0)}$ and $\mathbf{(1,-1)}$ and $\mathbf{(-1,1)}$: three critical points because $f=0$ gives $z=-y^3$ and then $g=0$ gives $y=y^a$, leading to $y=0,1,$ or -1. The stability test applies to the matrix of derivatives:

$$A = \begin{bmatrix} -3y^2 & -1 \\ 1 & 3z^2 \end{bmatrix} \text{ has } \det(A-\lambda I) = \lambda^2+\lambda(3y^2-3z^2)+1-9y^2z^2.$$

At $(0,0)$ $\lambda^2+1=0$ and $\lambda=\pm i$ **Unstable** (neutrally stable)

At $(1,-1)$ and $(-1,1)$ $\lambda^2-8=0$ **Unstable** with $\lambda=\sqrt{8}$.

6 Can you create two equations $y' = f(y, z)$ and $z' = g(y, z)$ with four critical points: $(1, 1)$ and $(1, -1)$ and $(-1, 1)$ and $(-1, -1)$?

I don't think all four points could be stable ? This would be like a surface with four minimum points and no maximum.

Solution An example would be $y' = \boldsymbol{y^2 - z^2}$ and $z' = \boldsymbol{1 - z^2}$. Then $z^2 - 1 = 0$ and $y^2 - z^2 = 0$ have the four points $(Y, Z) = (\pm 1, \pm 1)$ as critical points. In this case the linearized matrix (Jacobian matrix) is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 2y & -2z \\ 0 & -2z \end{bmatrix} \text{ and only } (Y, Z) = (-1, 1) \text{ is stable.}$$

7 The second order nonlinear equation for a damped pendulum is $y'' + y' + \sin y = 0$. Write z for the damping term y', so the equation is $z' + z + \sin y = 0$.

Show that $Y = 0$, $Z = 0$ is a stable critical point at the bottom of the pendulum.
Show that $Y = \pi$, $Z = 0$ is an unstable critical point at the top of the pendulum.

8 Those pendulum equations $y' = z$ and $z' = -\sin y - z$ have infinitely many critical points ! What are two more and are they stable ?

Solutions to 7 and 8 The system $y' = z$ and $z' = -z - \sin y$ has critical points when $z = 0$ and $\sin y = 0$ (**this allows all values $\boldsymbol{y = n\pi}$**).

The Jacobian matrix of derivatives of z and $-z - \sin y$ is a companion matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -\cos y & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \textbf{ or } \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

We have $-\cos y = -1$ at $y = 0, \pm 2\pi, \pm 4\pi, \ldots$ and $-\cos y = +1$ at $y = \pm\pi, \pm 3\pi, \ldots$

The eigenvalues satisfy $\lambda^2 + \lambda + 1 = 0$ or $\lambda^2 + \lambda - 1 = 0$:

$$\lambda = \tfrac{1}{2}(-1 \pm \sqrt{-3}) = \tfrac{1}{2}(-1 \pm i\sqrt{-3}) \text{ is } \textbf{stable at } \boldsymbol{y = 2n\pi}.$$

$$\lambda = \tfrac{1}{2}(-1 \pm \sqrt{5}) \text{ is } \textbf{unstable at } \boldsymbol{y = (2n+1)\pi}.$$

The pendulum is stable hanging straight down (at 6:00) and unstable when balanced directly upward (at 12:00).

9 The Liénard equation $y'' + p(y)y' + q(y) = 0$ gives the first order system $y' = z$ and $z' =$ _____. What are the equations for a critical point ? When is it stable ?

Solution The coupled equations are $y' = z$ and $z' = -p(y)z - q(y)$. These right sides are zero (critical point) when $z = 0$ and $q(y) = 0$.

The first derivative matrix is

$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -p'z - q' & -p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}.$$

That companion matrix is stable (according to Section 3.1) when $\boldsymbol{B > 0}$ **and** $\boldsymbol{C > 0}$.

10 Are these matrices stable or neutrally stable or unstable (source or saddle) ?

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & -1 \end{bmatrix}$$

Solution The stability tests are **trace < 0** and **determinant > 0**. This is because determinant $= (\lambda_1)(\lambda_2)$ and trace $=$ sum down the main diagonal $= \lambda_1 + \lambda_2$. Apply these tests to find

stable, **unstable** (saddle with $\det < 0$), **stable,** **unstable,** **stable**.

The second matrix has $\lambda = \pm 3i$ which gives undamped oscillation and neutral stability.

11 Suppose a predator x eats a prey y that eats a smaller prey z :

$dx/dt = -x + xy$	Find all critical points X, Y, Z
$dy/dt = -xy + y + yz$	Find A at each critical point
$dz/dt = -yz + 2z$	(9 partial derivatives)

Solution The right hand sides are $x(1-y)$ and $y(-x+1+z)$ and $z(-y+z)$. These are all zero at **three critical points** (X, Y, Z) : $\mathbf{(0,0,0)}$ $\mathbf{(0,2,-1)}$, $\mathbf{(1,1,0)}$

(Follow the two possibilities $X = 0$ or $Y = 1$ needed for $X(1-Y) = 0$.) The matrix of first derivatives of those right hand sides is

$$\begin{bmatrix} 1-y & -x & 0 \\ -y & -x+1+z & y \\ 0 & -z & 2-y \end{bmatrix}. \text{ Substitute the three critical vectors } (X, Y, Z) :$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

12 The damping in $y'' + (y')^3 + y = 0$ depends on the velocity $y' = z$. Then $z' + z^3 + y = 0$ completes the system. Damping makes this nonlinear system stable—is the linearized system stable ?

Solution $y' = z$ and $z' = -y - z^3$ has only $(Y, Z) = (0, 0)$ as critical point :

$A =$ first derivative matrix $= \begin{bmatrix} 0 & 1 \\ -1 & -3z^2 \end{bmatrix}$ has determinant $= 1$, trace $= -3z^2$: **unstable**.

13 Determine the stability of the critical points $(0, 0)$ and $(2, 1)$:

(a) $\begin{aligned} y' &= -y + 4z + yz \\ z' &= -y - 2z + 2yz \end{aligned}$ (b) $\begin{aligned} y' &= -y^2 + 4z \\ z' &= y - 2x^4 \end{aligned}$

Solution (a) The first derivative matrix at $(y, z) = (0, 0)$ or $(2, 1)$ is

$$A = \begin{bmatrix} z-1 & 4+y \\ z-1 & 2y-2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & -2 \end{bmatrix} \textbf{(stable) or} \begin{bmatrix} 0 & 6 \\ 1 & 2 \end{bmatrix} \begin{matrix} \textbf{(unstable)} \\ \text{(trace 2)} \end{matrix}$$

(b) The first derivative matrix at $(y, z) = (0, 0)$ or $(2, 1)$ is (**replace $\boldsymbol{x}$ by $\boldsymbol{z}$**)

$$A = \begin{bmatrix} -2y & 4 \\ 1 & -8z^3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{matrix} \textbf{(unstable)} \\ \text{(trace 0)} \end{matrix} \quad \text{or} \quad \begin{bmatrix} -4 & 4 \\ 1 & -8 \end{bmatrix} \textbf{(stable)}.$$

Problems 14–17 are about Euler's equations for a tumbling box.

14 The correct coefficients involve the moments of inertia I_1, I_2, I_3 around the axes. The unknowns x, y, z give the angular momentum around the three principal axes:

$$\begin{aligned} dx/dt &= ayz \qquad \text{with} \quad a = (1/I_3 - 1/I_2) \\ dy/dt &= bxz \qquad \text{with} \quad b = (1/I_1 - 1/I_3) \\ dz/dt &= cxy \qquad \text{with} \quad c = (1/I_2 - 1/I_1). \end{aligned}$$

Multiply those equations by x, y, z and add. This proves that $x^2 + y^2 + z^2$ is _____.

Solution Multiply by x, y, and z to get

$$\begin{aligned} xx' &= axyz \\ yy' &= bxyz \\ zz' &= cxyz \end{aligned}$$

$$\frac{1}{2}(x^2 + y^2 + z^2)' = (a + b + c)xyz = \mathbf{0} \text{ for the given } a, b, c.$$

Then $x^2 + y^2 + z^2 =$ **constant** because its derivative is zero.

15 Find the 3 by 3 first derivative matrix from those three right hand sides f, g, h. What is the matrix A in the 6 linearizations at the same 6 critical points?

Solution The first derivative matrix in Problem 14 is

$$\begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} 0 & az & ay \\ bz & 0 & bx \\ cy & cx & 0 \end{bmatrix}.$$

The 3 right sides are zero at the 6 critical points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \pm b \\ 0 & \pm c & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \pm a \\ 0 & 0 & 0 \\ \pm c & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \pm a & 0 \\ \pm b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

All six points are neutrally stable ($\text{Re}\,\lambda = 0$).

16 You almost always catch an unstable tumbling book at a moment when it is flat. That tells us: The point $x(t), y(t), z(t)$ spends most of its time (near) (far from) the critical point $(0, 1, 0)$. This brings the travel time t into the picture.

Solution This neat observation was explained to me by Alar Toomre. The velocity $(f, g, h) = (ayz, bxz, cxy)$ is low near a critical point where x, y, z are small. Then the book spends most time **near** the point where the book is flat and easy to catch.

17 In reality what happens when you

(a) throw a baseball with no spin (a knuckleball) ?

(b) hit a tennis ball with overspin ?

(c) hit a golf ball left of center ?

(d) shoot a basketball with underspin (a free throw) ?

Solution (a) The knuckleball is unstable–hard for the batter to judge.

(b) The topspin brings the tennis ball down faster with a higher bounce.

(c) The golf ball slices to the right off the fairway.

(d) The basketball with underspin is more stable with less bounce around the rim. It is more likely to end up in the basket.

Problem Set 3.4, page 189

1 Apply Euler's method $y_{n+1} = y_n + \Delta t f_n$ to find y_1 and y_2 with $\Delta t = \frac{1}{2}$:

(a) $y' = y$ (b) $y' = y^2$ (c) $y' = 2ty$ (all with $y(0) = y_0 = 1$)

Solution (a) $y_1 = y_0 + \Delta t \quad y_0 = 1 + \Delta t = 1.5 \quad y_2 = (1 + \Delta t)^2 = y_n = (1 + \Delta t)^R = 2.25$

(b) $y_1 = y_0 + \Delta t \quad y_0^2 = 1 + \Delta t = 1.5 \quad y_2 = y_1 + \Delta t y_1^2 = 1 + \Delta t + \Delta t(1 + 2\Delta t + \Delta t^2) = (1 + \Delta t)(1 + \Delta t + \Delta t^2) = (1.5)(1.75)$

(c) $y_1 = (1 + 2t + \Delta t)y_0 = 1$ because $t = 0 \quad y_2 = (1 + 2t + \Delta t)y_1 = 1.5$ because $t = \Delta t$.

2 For the equations in Problem 3, find y_1 and y_2 with the step size reduced to $\Delta t = \frac{1}{4}$. Now the value y_2 is an approximation to the exact $y(t)$ at what time t? Then y_2 in this question corresponds to which y_n in Problem 3?

Solution With $\Delta t = \frac{1}{4}, y_2$ will now be an approximation to the true solution $y(\frac{1}{2})$ because $2\Delta t = \frac{1}{2}$.

(a) $y_1 = 1 + \Delta t = 5/4 = 1.35$ $\qquad y_2 = (1 + \Delta t)^2 = 25/16$

(b) $y_1 = 1 + \Delta t = 1.25$ $\qquad y_2 = \left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{4} + \frac{1}{16}\right) = \left(\frac{5}{4}\right)\left(\frac{21}{16}\right)$

(c) $y_1 = 1$ $\qquad y_2 = (1 + 2t + \Delta t)y_1 = \left(1 + \frac{2}{19}\right) = \left(\frac{9}{8}\right)$

3 (a) For $dy/dt = y$ starting from $y_0 = 1$, what is Euler's y_n when $\Delta t = 1$?

(b) Is it larger or smaller than the true solution $y = e^t$ at time $t = n$?

(c) What is Euler's y_{2n} when $\Delta t = \frac{1}{2}$? This is closer to the true $y(n) = e^n$.

Solution (a) $y_{n+1} = (1 + \Delta t)y_n = 2y_n$ so $y_n = 2^n$

(b) 2^n is smaller than e^n

(c) $y_{n+1} = (1 + \Delta t)y_n = \frac{3}{2}y_n$. Then $y_{2n} = \left(1 + \frac{1}{2}\right)^{2n}$ is above 2^n because $\left(1 + \frac{1}{2}\right)^2 > 2$.

4 For $dy/dt = -y$ starting from $y_0 = 1$, what is Euler's approximation y_n after n steps of size Δt? Find all the y_n's when $\Delta t = 1$. Find all the y_n's when $\Delta t = 2$. Those time steps are *too large* for this equation.

Solution $y_{n+1} = Y_n - \Delta t y_n$ so $y_n = (1 - \Delta t)^n y_0$.

If $\Delta t = 1$ then all of $Y_1, Y_2, Y_3, \ldots$ are zero.

If $\Delta t = 2$ then $Y_{n+1} = -y_n$ and $y_n = (-1)^n$.

The approximation will blow up for $\Delta t > 2$.

In reality it seems useless for $\Delta t > 0.1$.

5 The true solution to $y' = y^2$ starting from $y(0) = 1$ is $y(t) = 1/(1 - t)$. This explodes at $t = 1$. Take 3 steps of Euler's method with $\Delta t = \frac{1}{3}$ and take 4 steps with $\Delta t = \frac{1}{4}$. Are you seeing any sign of explosion?

Solution With $\Delta t = \frac{1}{3}$, Euler's method for $y' = y^2$ becomes $y_{n+1} = y_n + \Delta t y_n^2$. Three steps with $\Delta t = \frac{1}{3}$ and four steps with $\Delta t = \frac{1}{4}$ give

$y_1 = \frac{4}{3}$, $y_2 = \frac{52}{27}$, $y_3 = \_\_$ $\qquad$ $y_1 = \frac{5}{4}$, $y_2 = \frac{105}{64}$, $y_3 = \_\_$ $y_4 = \_\_$

We are not reaching infinity at time $t = n\Delta t = 1$ but as $\Delta t \to 0$ and $n = 1/\Delta t$ the numbers y_n will keep growing past any bound.

6 The true solution to $dy/dt = -2ty$ with $y(0) = 1$ is the bell-shaped curve $y = e^{-t^2}$. It decays quickly to zero. Show that step $n + 1$ of Euler's method gives $y_{n+1} = (1 - 2n\Delta t^2)y_n$. Do the y_n's decay toward zero ? Do they stay there ?

Solution A step of Euler's method starting at time $t = n\Delta t$ gives $y_{n+1} = y_n - 2(n\Delta t)y_n$. In the early steps we are multiplying y_n by $1 - 2n\Delta t$ which is normally less than 1. So the y_n are decreasing at first. But when n is larger than $1/\Delta t$, we are multiplying by a number below -1. At that point the y_n start growing and changing sign at every step : serious *instability*.

7 The equations $y' = -y$ and $z' = -10z$ are uncoupled. If we use Euler's method for both equations with the same Δt between $\frac{2}{10}$ and 2, show that $y_n \to 0$ but $|z_n| \to \infty$. The method is failing on the solution $z = e^{-10t}$ that should decay fastest.

Solution The Euler formulas are $y_{n+1} = (1 - \Delta t)y_n$ and $z_{n+1} = (1 - 10\Delta t)z_n$. For time steps Δt between $\frac{2}{10}$ and 2, the y factor has $|1 - \Delta t| < 1$. But the z factor has $|1 - 10\Delta t| > 1$. The true solutions are $y = Ce^{-t}$ and $z = Ce^{-10t}$.

But that quickly decreasing z has a quickly increasing z_n when $|1 - 10\Delta t| > 1$: instability.

8 What values y_1 and y_2 come from *backward Euler* for $dy/dt = -y$ starting from $y_0 = 1$? Show that $y_1^B < 1$ and $y_2^B < 1$ even if Δt is very large. We have *absolute stability* : no limit on the size of Δt.

Solution Backward Euler for $y' = -y$ is $y_{n+1} - y_n = -\Delta t y_{n+1}$ (**not** $\boldsymbol{-\Delta t} y_n$). Then $y_{n+1} = y_n/(1 + \Delta t)$. For any At that factor $1/(1 + \Delta t)$ is **less than** 1 : absolute stability.

9 The logistic equation $y' = y - y^2$ has an S-curve solution in Section 1.7 that approaches $y(\infty) = 1$. This is a steady state because $y' = 0$ when $y = 1$.

Write Euler's approximation $y_{n+1} = \_\_\_\_\_$ to this logistic equation, with stepsize Δt. Show that this has the same steady state : y_{n+1} equals y_n if $y_n = 1$.

Solution $y' = y - y^2$ is approximated by $y_{n+1} = y_n + \Delta t(y_n - y_n^2)$. This equation has a steady state when $y_{n+1} = y_n$—and this requires the Δt factor to be zero : $\boldsymbol{y_n - y_n^2} = 0$. So the two steady states are ($y_n = 1$ forever) and ($y_n = 0$ forever).

10 The important question in Problem 3 is whether the steady state $y_n = 1$ is stable or unstable. Subtract 1 from both sides of Euler's $y_{n+1} = y_n + \Delta t(y_n - y_n^2)$:

$$y_{n+1} - 1 = y_n + \Delta t(y_n - y_n^2) - 1 = (y_n - 1)(1 - \Delta t y_n).$$

Each step multiplies the distance from 1 by $(1 - \Delta t y_n)$. Near the steady $y_\infty = 1$, $1 - \Delta t\, y_n$ has size $|1 - \Delta t|$. For which Δt is this smaller than 1 to give stability ?

Solution $y_n - 1$ is the distance from steady state. The equation in the problem shows that this distance is multiplied at each step by a factor $\boldsymbol{1 - \Delta t y_n}$. This factor has $|1 - \Delta t y_n| < 1$ when $\boldsymbol{0 < \Delta t y_n < 2}$. When y_n is near 1, this means Δt can be almost **2** for stability.

11 Apply backward Euler $y^B_{n+1} = y_n + \Delta t f^B_{n+1} = y_n + \Delta t \left[y^B_{n+1} - \left(y^B_{n+1} \right)^2 \right]$ to the logistic equation $y' = f(y) = y - y^2$. What is y^B_1 if $y_0 = \frac{1}{2}$ and $\Delta t = \frac{1}{4}$? You have to solve a quadratic equation to find y^B_1. I am finding two answers for y^B_1. A computer code might choose the answer closer to y_0.

Solution At each new time step, Backward Euler becomes a quadratic equation for y_{n+1} in the logistic equation. If $y_0 = \frac{1}{2}$ and $\Delta t = \frac{1}{4}$ the equation for $y_1 (= y^B_1)$ is

$$\Delta t (y_1)^2 + (1 - \Delta t) y_1 - y_0 = 0 \quad \text{OR} \quad \frac{1}{4} y_1^2 + \frac{3}{4} y_1 - \frac{1}{2} = 0.$$

Multiply by 4. The solutions of $y_1^2 + 3y_1 - 2 = 0$ are

$$y_1 = \frac{-3 \pm \sqrt{17}}{2}. \text{ The better choice } \left(\text{near } \frac{1}{2} \right) \text{ is } y^B_1 = \frac{-3 + \sqrt{17}}{2}.$$

12 For the bell-shaped curve equation $y' = -2ty$, show that backward Euler divides y_n by $1 + 2n(\Delta t)^2$ to find y^B_{n+1}. As $n \to \infty$, what is the main difference from forward Euler in Problem 3 ?

Solution Backward Euler for $y' = -2ty$ is $y_{n+1} - y_n = -2t \Delta t y_{n+1}$ or $y_{n+1} = y_n / (1 + 2t + \Delta t)$.

That fraction is smaller than 1 for all t and Δt. Then the numbers y_n are steadily decreasing as $n \to \infty$, like the true solution $y(t) = e^{-t^2}$. (Forward Euler was hopeless in Problem 6, with Y_n increasing and changing sign at every step beyond $n = 1/\Delta t$.)

13 The equation $y' = \sqrt{|y|}$ has *many solutions* starting from $y(0) = 0$. One solution stays at $y(t) = 0$, another solution is $y = t^2/4$. (Then $y' = t/2$ agrees with $\sqrt{y}$.) Other solutions can stay at $y = 0$ up to $t = T$, and then switch to the parabola $y = (t - T)^2/4$. As soon as y leaves the bad point $y = 0$, where $f(y) = y^{1/2}$ has infinite slope, the equation has only one solution.

Backward Euler $y_1 - \Delta t \sqrt{|y_1|} = y_0 = 0$ gives two correct values $y^B_1 = 0$ and $y^B_1 = (\Delta t)^2$. What are the three possible values of y^B_2 ?

Solution Backward Euler for y^B_2 will be $\boldsymbol{y_2 - \Delta t \sqrt{|y_2|} = Y_1}$. If $y^B_1 = 0$ then y^B_2 can be 0 or $(\Delta t)^2$. If $y^B_1 = (\Delta t)^2$ then $x = \sqrt{|y^B_2|}$ solves $x^2 - \Delta t x - (\Delta t)^2 = 0$. Again two possibilities:

$$x = \frac{1}{2} \left(1 \pm \sqrt{5} \right) \Delta t.$$

Because $\sqrt{|y|}$ is continuous but its derivative blows up at $y = 0$, multiple solutions are possible.

14 Every finite difference person will think of averaging forward and backward Euler:

$$\textbf{Centered Euler / Trapezoidal} \qquad \boldsymbol{y^C_{n+1} - y_n = \Delta t \left(\frac{1}{2} f_n + \frac{1}{2} f^C_{n+1} \right)}.$$

For $y' = -y$ the key questions are **accuracy** and **stability**. Start with $y(0) = 1$.

$$y^C_1 - y_0 = \Delta t \left(-\frac{1}{2} y_0 - \frac{1}{2} y^C_1 \right) \quad \text{gives} \quad \boldsymbol{y^C_1 = \frac{1 - \Delta t/2}{1 + \Delta t/2} \, y_0}.$$

Stability Show that $|1 - \Delta t/2| < |1 + \Delta t/2|$ for all Δt. *No stability limit on* Δt.

Accuracy For $y_0 = 1$ compare the exact $y_1 = e^{-\Delta t} = 1 - \Delta t + \frac{1}{2}\Delta t^2 - \cdots$ with $y_1^C = (1 - \frac{1}{2}\Delta t)/(1 - \frac{1}{2}\Delta t) = (1 - \frac{1}{2}\Delta t)(1 - \frac{1}{2}\Delta t + \frac{1}{4}\Delta t^2 - \cdots)$.
An extra power of Δt is correct: *Second order accuracy*. A good method.

Solution Stability is $|\boldsymbol{y_{n+1}}| \leq |\boldsymbol{y_n}|$ for an equation like $y' = -y$ where the true solution $y = e^{-t}$ is decreasing. In this problem

$$y_1^C = \frac{1-\Delta t/2}{1+\Delta t/2} y_0 \text{ has growth factor } \left|\frac{1-\Delta t/2}{1+\Delta t/2}\right| < 1 \text{ because } \left|1 + \frac{\Delta t}{2}\right| > \left|1 - \frac{\Delta t}{2}\right|$$

Accuracy is decided by comparing y_1^C to the exact y_1. The two series agree in the terms 1 and $-\Delta t$ and $\frac{1}{2}(\Delta t)^2$: **Second order accuracy** because the $(\Delta t)^3$ error appears in $1/\Delta t$ time steps to reach the typical time $t = 1$. Sign correction in text to:

$$y_1^C = \left(1 - \frac{1}{2}\Delta t\right) / \left(1 + \frac{1}{2}\Delta t\right) = \cdots$$

The rest is correct and produces $1 - \Delta t + \frac{1}{2}(\Delta t)^2 \ldots$ as required.

The website has codes for Euler and Backward Euler and Centered Euler. Those methods are slow and steady with first order and second order accuracy. The test problems give comparisons with faster methods like Runge-Kutta.

Problem Set 3.5, page 194

Runge-Kutta can only be appreciated by using it. A simple code is on math.mit.edu/dela. Professional codes are ode 45 (in MATLAB) and ODEPACK and many more.

1 For $y' = y$ with $y(0) = 1$, show that simplified Runge-Kutta and full Runge-Kutta give these approximations y_1 to the exact $y(\Delta t) = e^{\Delta t}$:

$$y_1^S = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 \qquad y_1^{RK} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \frac{1}{24}(\Delta t)^4$$

Solution Simplified Runge-Kutta (equation (1) in this section) when $y' = f(t,y) = \boldsymbol{y}$:

$$\begin{aligned} y_{n+1} &= y_n + \Delta t\left[\frac{1}{2}f(t_n, y_n) + \frac{1}{2}f\left(t_{n+1}, y_{n+1}^{\text{Euler}}\right)\right] \\ &= y_n + \Delta t\left[\frac{1}{2}y_n + \frac{1}{2}\left(y_n + \Delta t y_n\right)\right] \\ &= y_n + \Delta t y_n + \frac{1}{2}(\Delta t^2) y_n \ (\textbf{3 good terms of} \boldsymbol{e^{\Delta t} y_n}) \end{aligned}$$

Full Runge-Kutta is in equation (5)—now applied when $f(t,y) = \boldsymbol{y}$:

$$\boldsymbol{k_1} = \frac{1}{2}y_n \qquad \boldsymbol{k_3} = \frac{1}{2}\left(y_n + \frac{\Delta t}{2}\left(y_n + \frac{\Delta t}{2}y_n\right)\right)$$

$$\boldsymbol{k_2} = \frac{1}{2}\left(y_n + \frac{\Delta t}{2}y_n\right) \quad \boldsymbol{k_4} = \frac{1}{2}\left(y_n + \Delta t\left(y_n + \frac{\Delta t}{2}\left(y_n + \frac{\Delta t}{2}y_n\right)\right)\right)$$

Then the Runge-Kutta choice for y_{n+1} is correct through $(\Delta t)^4$!

$$y_n + \frac{\Delta t}{3}(k_1 + 2k_2 + 2k_3 + k_4) = y_n\left[1 + \frac{\Delta t}{6} + \frac{\Delta t}{3}\left(1 + \frac{\Delta t}{2}\right) + \right.$$
$$\left.\frac{\Delta t}{3}\left(1 + \frac{\Delta t}{2}\left(1 + \frac{\Delta t}{2}\right)\right) + \frac{\Delta t}{6}\left(1 + \Delta t + \frac{(\Delta t)^2}{2}\left(1 + \frac{\Delta t}{2}\right)\right)\right]$$
$$= y_n\left[1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \frac{1}{24}(\Delta t)^4\right].$$

2 With $\Delta t = 0.1$ compute those numbers y_1^S and y_1^{RK} and subtract from the exact $y = e^{\Delta t}$. The errors should be close to $(\Delta t)^3/6$ and $(\Delta t)^5/120$.

Solution When $y_0 = 1$ and $\Delta t = \frac{1}{10}$, the first step in the solution above gives

Simplified Runge-Kutta $1 + \frac{1}{10} + \frac{1}{2}\left(\frac{1}{10}\right)^2 = \mathbf{1.105}$.

Runge-Kutta $1 + \frac{1}{10} + \frac{1}{2}\left(\frac{1}{10}\right)^2 + \frac{1}{6}\left(\frac{1}{10}\right)^3 + \frac{1}{24}\left(\frac{1}{10}\right)^4 = \frac{11}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{1}{240000} = \mathbf{1.1051708}$.

The exact growth factor is $\exp\left(\frac{1}{10}\right) = \mathbf{1.1051709}$. Error 10^{-7} is near $10^{-5}/120$.

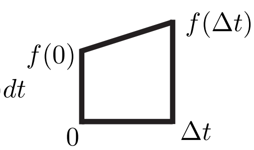
3 Those values y_1^S and y_1^{RK} have errors of order $(\Delta t)^3$ and $(\Delta t)^5$. Errors of this size at every time step will produce total errors of size _____ and _____ at time T, from N steps of size $\Delta t = T/n$.

Those estimates of total error are correct provided errors don't grow (*stability*).

Solution Local errors of size $(\Delta t)^3$ or $(\Delta t)^5$ produce global errors of size $\boldsymbol{(\Delta t)^2}$ **or** $\boldsymbol{(\Delta t)^4}$ after $1/\Delta t$—**provided** the system is stable and local errors don't grow.

4 $dy/dt = f(t)$ with $y(0) = 0$ is solved by integration when f does not involve y. From time $t = 0$ to Δt, simplified Runge-Kutta approximates the integral of $f(t)$:

$$y_1^S = \Delta t\left(\frac{1}{2}f(0) + \frac{1}{2}f(\Delta t)\right) \text{ is close to } y(\Delta t) = \int_0^{\Delta t} f(t)dt$$



Suppose the graph of $f(t)$ is a straight line as shown. Then the region is a *trapezoid*. Check that its area is exactly y_1^S. Second order means exact for linear f.

Solution The area of a trapezoid is $(\text{base})(\text{average height}) = (\Delta t)(f(0)+f(\Delta t))/2$. This is exactly the answer chosen by simplified Runge-Kutta.

5 Suppose again that f does not involve y, so $dy/dt = f(t)$ with $y(0) = 0$. Then full Runge-Kutta from $t = 0$ to Δt approximates the integral of $f(t)$ by y_1^{RK} :

$$y_1^{RK} = \Delta t\,(c_1 f(0) + c_2 f(\Delta t/2) + c_3 f(\Delta t)). \qquad \textbf{Find } \boldsymbol{c_1, c_2, c_3}.$$

This approximation to $\int_0^{\Delta t} f(t)\,dt$ is called Simpson's Rule. It has 4th order accuracy.

Solution Full Runge-Kutta allows the top edge of the trapezoid to be *curved* : it is the graph of a nonlinear $f(t)$. The area under this curve is well approximated by Simpson's Rule :

$$\text{area} \;\approx \Delta t\left[\frac{\mathbf{1}}{\mathbf{6}}f(0)+\frac{\mathbf{4}}{\mathbf{6}}f\left(\frac{\Delta t}{2}\right)+\frac{\mathbf{1}}{\mathbf{6}}f(\Delta t)\right].$$

If you apply Runge-Kutta to $y' = f(t)$ from 0 to Δt, with the right hand side independent of y, the result is

$$\boldsymbol{k_1} = \frac{1}{2}f(0) \qquad \boldsymbol{k_2} = \frac{1}{2}f\left(\frac{\Delta t}{2}\right) \qquad \boldsymbol{k_3} = \frac{1}{2}f\left(\frac{\Delta t}{2}\right) \qquad \boldsymbol{k_4} = \frac{1}{2}f\left(\Delta t\right)$$

$$\frac{\Delta t}{3}(k_1+2k_2+2k_3+k_4) = \frac{\Delta t}{\mathbf{6}}f(0)+\frac{4\Delta t}{\mathbf{6}}f\left(\frac{\Delta t}{2}\right)+\frac{\Delta t}{\mathbf{6}}f\left(\Delta t\right) \;:\; \textbf{Simpson's Rule}$$

6 Reduce these second order equations to first order systems $\boldsymbol{y'} = \boldsymbol{f}(t,y)$ for the vector $\boldsymbol{y} = (y, y')$. Write the two components of $\boldsymbol{y}_1^E$ (Euler) and $\boldsymbol{y}_1^S$.

(a) $y'' + yy' + y^4 = 1$ (b) $my'' + by' + ky = \cos t$

Solutions to Problems 6 and 7 **Write $\boldsymbol{z}$ for $\boldsymbol{y'}$**. The first order systems are

$$\text{(a) } y' = z \qquad\qquad \text{(b) } y' = z$$
$$z' = 1 - yz - y^4 \qquad\qquad mz' = -ky - bz + \cos t$$

Then Euler's method gives (y_1^E, z_1^E) from (y_0, z_0):

$$\begin{bmatrix} y_1^E \\ z_1^E \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} + \Delta t \begin{bmatrix} z_0 \\ 1 - y_0 z_0 - (z_0)^4 \end{bmatrix}$$

$$\begin{bmatrix} y_1^E \\ mz_1^E \end{bmatrix} = \begin{bmatrix} y_0 \\ mz_0 \end{bmatrix} + \Delta t \begin{bmatrix} z_0 \\ -ky_0 - bz_0 + \cos 0 \end{bmatrix}$$

Simplified Runge-Kutta finds (y_1^S, z_1^S) from (y_0, z_0) by adding *half* of those Euler corrections *plus half* of the updated correction:

$$\text{(a)} \begin{bmatrix} y_1^S \\ z_1^S \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} + \frac{\Delta t}{2} \begin{bmatrix} z_0 \\ 1 - y_0 z_0 - (z_0)^4 \end{bmatrix} + \frac{\Delta t}{2} \begin{bmatrix} z_1^E \\ 1 - y_1^E z_1^E - (z_1^E)^4 \end{bmatrix}$$

$$\text{(b)} \begin{bmatrix} y_1^S \\ mz_1^S \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} + \frac{\Delta t}{2} \begin{bmatrix} z_0 \\ -ky_0 \ldots \end{bmatrix} + \frac{\Delta t}{2} \begin{bmatrix} z_1^E \\ -ky_1^E - bz_1^E + \mathbf{cos}\,\boldsymbol{\Delta t} \end{bmatrix}$$

8 For $y' = -y$ and $y_0 = 1$ the exact solution $y = e^{-t}$ is approximated at time Δt by 2 or 3 or 5 terms:

$$y_1^E = 1-\Delta t \quad y_1^S = 1-\Delta t+\frac{1}{2}(\Delta t)^2 \quad y_1^{RK} = 1-\Delta t+\frac{1}{2}(\Delta t)^2-\frac{1}{6}(\Delta t)^3+\frac{1}{24}(\Delta t)^4$$

(a) With $\Delta t = 1$ compare those three numbers to the exact e^{-1}. What error E?

(b) With $\Delta t = 1/2$ compare those three numbers to $e^{-1/2}$. Is the error near $E/16$?

Solution (a) $\Delta t = 1$ gives $y_1^E = \mathbf{0}$ $y_1^S = \frac{\mathbf{1}}{\mathbf{2}}$ $y_1^{\text{RK}} = \frac{\mathbf{9}}{\mathbf{24}} = \mathbf{.375}$ compared to the exact $e^{-1} = \mathbf{.368}$ $E^{RK} = \mathbf{.007}$.

(b) $\Delta t = \frac{1}{2}$ gives $y_1^E = \mathbf{\frac{1}{2}}$ $\quad y_1^S = \mathbf{\frac{5}{8}}$ $\quad y_1^{\text{RK}} = \mathbf{\frac{233}{(24)(16)}} = \mathbf{.60677}$ $\quad e^{-1/2} = \mathbf{.60653}$ $E^{RK} = \mathbf{.00024}$.

Two steps with $\Delta t = \frac{1}{2}$ would leave an error about $2(.00024) = -.00048$ which is close to $.007/16$.

9 For $y' = ay$, simplified Runge-Kutta gives $y_{n+1}^S = (1 + a\Delta t + \frac{1}{2}(a\Delta t)^2)y_n$. This multiplier of y_n reaches $1 - 2 + 2 = \mathbf{1}$ when $a\Delta t = -2$: *the stability limit*.

(**Computer experiment**) For $N = 1, 2, \ldots, 10$ discover the stability limit $L = L_N$ when the series for e^{-L} is cut off after $N + 1$ terms :

$$\left| 1 - L + \frac{1}{2}L^2 - \frac{1}{6}L^3 + \cdots \pm \frac{1}{N!}L^N \right| = 1.$$

We know $L = 2$ for $N = 1$ and $N = 2$. Runge-Kutta has $L = 2.78$ for $N = 4$.

Solution The stability limits L_N for $N = 1, . . ., 10$ come from MATLAB:
2.0 2.0 2.513 2.785 3.217 3.55 3.954 4.314 4.701 5.070.

Problem Set 4.1, page 206

1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $\boldsymbol{v} = (x, y, z) = (2, 3, 4)$:

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Draw the four vectors in the column picture. Two times column 1 plus three times column 2 plus four times column 3 equals the right side $\boldsymbol{b}$.

The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 If the equations in Problem 1 are multiplied by $2, 3, 4$ they become $D\boldsymbol{V} = \boldsymbol{B}$:

$$\begin{aligned} 2x + 0y + 0z &= 4 \\ 0x + 3y + 0z &= 9 \\ 0x + 0y + 4z &= 16 \end{aligned} \qquad \text{or} \qquad D\boldsymbol{V} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = \boldsymbol{B}$$

Why is the row picture the same? Is the solution $\boldsymbol{V}$ the same as $\boldsymbol{v}$? What is changed in the column picture—the columns or the right combination to give $\boldsymbol{B}$?

The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

3 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

5 The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line $\mathbf{L}$). Find three solutions on $\mathbf{L}$.

If x, y, z satisfy the first two equations they also satisfy the third equation. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

6 Move the third plane in Problem 5 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.

Equation $1 +$ equation $2 -$ equation 3 is now $0 = -4$. Line misses plane; *no solution*.

7 In Problem 5 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b} = (2, 3, 5)$. This is only possible for $\boldsymbol{b} = (4, 6, c)$ if $c =$ _____.

Column $3 =$ Column 1 makes the matrix singular. Solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns).

8 Normally 4 "planes" in 4-dimensional space meet at a _____. Normally 4 vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\boldsymbol{b} = (3, 3, 3, 2)$?

Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$.

Problems 9–14 are about multiplying matrices and vectors.

9 Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

(a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

(a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Compute each $A\boldsymbol{x}$ in Problem 9 as a combination of the columns:

$$\text{9(a) becomes} \quad A\boldsymbol{x} = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

How many separate multiplications for $A\boldsymbol{x}$, when the matrix is "3 by 3"?

Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x}$. By rows or by columns: **9** separate multiplications for 3 by 3.

11 Find the two components of $A\boldsymbol{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

$A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 Multiply A times $\boldsymbol{x}$ to find three components of $A\boldsymbol{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.

(b) The planes from the m equations $A\boldsymbol{x} = \boldsymbol{b}$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.

(a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space, but the columns are in m-dimensional space.

14 Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector $\boldsymbol{x} = (x, y, z, t)$ to produce $\boldsymbol{b}$. The solutions $\boldsymbol{x}$ fill a plane or "hyperplane" in 4-dimensional space. *The plane is* 3*-dimensional with no* 4*D volume*.

$2x+3y+z+5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [\,2 \;\; 3 \;\; 1 \;\; 5\,]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

Problems 15–22 ask for matrices that act in special ways on vectors.

15 (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} \mathtt{x} \\ \mathtt{y} \end{bmatrix}$ equals $\begin{bmatrix} \mathtt{x} \\ \mathtt{y} \end{bmatrix}$.

(b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} \mathtt{x} \\ \mathtt{y} \end{bmatrix}$ equals $\begin{bmatrix} \mathtt{y} \\ \mathtt{x} \end{bmatrix}$.

(a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 (a) What 2 by 2 matrix R rotates every vector by 90° ? R times $\begin{bmatrix} \mathtt{x} \\ \mathtt{y} \end{bmatrix}$ is $\begin{bmatrix} \mathtt{y} \\ -\mathtt{x} \end{bmatrix}$.

(b) What 2 by 2 matrix R^2 rotates every vector by 180° ?

90° rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$

17 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z). Q is the inverse of P.

18 What 2 by 2 matrix E subtracts the first component from the second component ? What 3 by 3 matrix does the same ?

$$E \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad E \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z + x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z - x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1}, the two results are (_____) and (_____).

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$ and $E^{-1}E\boldsymbol{v}$ recovers $(3, 4, 5)$.

20 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (_____) and (_____).

$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ has $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 What 2 by 2 matrix R rotates every vector through $45°$? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

$R = \dfrac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by $45°$. The columns of R are the results from rotating $(1, 0)$ and $(0, 1)$!

22 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication $A\boldsymbol{v}$. The matrix A has one row. The solutions to $A\boldsymbol{v} = \mathbf{0}$ lie on a _____ perpendicular to the vector _____. The columns of A are only in _____-dimensional space.

The dot product $A\boldsymbol{x} = [\,1 \;\; 4 \;\; 5\,] \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 In MATLAB notation, write the commands that define this matrix A and the column vectors $\boldsymbol{v}$ and $\boldsymbol{b}$. What command would test whether or not $A\boldsymbol{v} = \boldsymbol{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$A = [\,1 \;\; 2 \;\; ; \;\; 3 \;\; 4\,]$ and $\boldsymbol{x} = [\,5 \;\; -2\,]'$ and $\boldsymbol{b} = [\,1 \;\; 7\,]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

24 If you multiply the 4 by 4 all-ones matrix A = ones(4) and the column v = ones(4,1), what is A∗v ? (Computer not needed.) If you multiply B = eye(4) + ones(4) times w = zeros(4,1) + 2∗ones(4,1), what is B∗w ?

ones$(4, 4) *$ **ones**$(4, 1) = [\,4 \;\; 4 \;\; 4 \;\; 4\,]'$; $B * \boldsymbol{w} = [\,10 \;\; 10 \;\; 10 \;\; 10\,]'$.

Questions 25–27 review the row and column pictures in 2, 3, and 4 dimensions.

25 Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.

The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) = 4$(column 1) $+ 2$(column 2) $=$ right side $(0, 6)$.

26 For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.

The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a ***line***.

27 For four linear equations in two unknowns x and y, the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.

The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

Challenge Problems

28 Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be $8, 3, 4$. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if a 4 by 4 magic matrix has entries $1, \ldots, 16$?

$$M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}; M_3(1,1,1) = (15,15,15);$$

$M_4(1,1,1,1) = (34,34,34,34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

29 Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are the first two columns of a 3 by 3 matrix A. Which third columns $\boldsymbol{w}$ would make this matrix singular ? Describe a typical column picture of $A\boldsymbol{v} = \boldsymbol{b}$ in that singular case, and a typical row picture (for a random $\boldsymbol{b}$).

A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution*.

30 **Multiplying by $\boldsymbol{A}$ is a "linear transformation".** Those important words mean:

If $\boldsymbol{w}$ is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

It is this "***linearity***" $A\boldsymbol{w} = cA\boldsymbol{u} + dA\boldsymbol{v}$ that gives us the name *linear algebra*.

If $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\boldsymbol{u}$ and $A\boldsymbol{v}$ are the columns of A.

Combine $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. **If** $\boldsymbol{w} = \begin{bmatrix} \mathbf{5} \\ \mathbf{7} \end{bmatrix}$ **how is** $A\boldsymbol{w}$ **connected to** $A\boldsymbol{u}$ **and** $A\boldsymbol{v}$?

$\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

31 A 9 by 9 ***Sudoku matrix*** S has the numbers $1, \ldots, 9$ in every row and column, and in every 3 by 3 block. For the all-ones vector $\boldsymbol{v} = (1, \ldots, 1)$, what is $S\boldsymbol{v}$?

A better question is: **Which row exchanges will produce another Sudoku matrix** ? Also, which exchanges of block rows give another Sudoku matrix ?

Section 4.5 will look at all possible permutations (reorderings) of the rows. I see 6 orders for the first 3 rows, all giving Sudoku matrices. Also 6 permutations of the next 3 rows, and of the last 3 rows. And 6 block permutations of the block rows ?

$\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

32 Suppose the second row of A is some number c times the first row :

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}.$$

Then if $a \neq 0$, the second column of A is what number d times the first column ? **A square matrix with dependent rows will also have dependent columns**. This is a crucial fact coming soon.

The second column is $d = b/a$ times the first column. So the columns are "dependent" when the rows are "dependent".

Problem Set 4.2, page 215

Problems 1–10 are about elimination on 2 by 2 systems.

1 What multiple ℓ_{21} of equation 1 should be subtracted from equation 2 ?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this step, solve the triangular system by back substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution ?

Multiply by $\ell_{21} = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$. The pivots to circle are 2 and -6. If the right hand side is multiplied by 4, the solution is multiplied by 4.

2 If you find solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ to $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{c}$, what is the solution $\boldsymbol{u}$ to $A\boldsymbol{u} = \boldsymbol{b} + \boldsymbol{c}$? What is the solution $\boldsymbol{U}$ to $A\boldsymbol{U} = 3\boldsymbol{b} + 4\boldsymbol{c}$? (We saw superposition for linear differential equations, it works in the same way for all linear equations.)

If $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{c}$ then $A(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{b} + \boldsymbol{c}$. The solution to $A\boldsymbol{U} = 3\boldsymbol{b} + 4\boldsymbol{c}$ is $U = 3\boldsymbol{v} + 4\boldsymbol{w}$.

3 What multiple of equation 1 should be *subtracted* from equation 2 ?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution ?

Subtract $-\frac{1}{2}$ times equation 1 from equation 2. This leaves $0x + 3y = 3$. Then $y = \mathbf{1}$ and the first equation becomes $2x - 4 = 6$ to give $x = 5$.
If the right side changes from $(6, 0)$ to $(-6, 0)$ the solution changes from $(5, 1)$ to $(-5, -1)$.

4 What multiple ℓ of equation 1 should be subtracted from equation 2 to remove cx ?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot ? The second pivot is missing when $ad = bc$: that is the *singular case*.

Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions ?

Singular system

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

$6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. (The two lines in the row picture are the same line, containing all solutions).

6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 For which a does elimination break down (1) permanently or (2) temporarily ?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= \quad 6. \end{aligned}$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 For which three numbers k does elimination break down ? Which is fixed by a row exchange ? In these three cases, is the number of solutions 0 or 1 or ∞ ?

$$\begin{aligned} kx + 3y &= \quad 6 \\ 3x + ky &= -6. \end{aligned}$$

If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 What test on b_1 and b_2 decides whether these two equations allow a solution ? How many solutions will they have ? Draw the column picture for $\boldsymbol{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).

10 In the xy plane, draw the lines $x+y=5$ and $x+2y=6$ and the equation $y=$ _____ that comes from elimination. The line $5x-4y=c$ will go through the solution of these equations if $c=$ _____.

The equation $y=1$ comes from elimination (subtract $x+y=5$ from $x+2y=6$). Then $x=4$ and $5x-4y=c=16$.

11 (Recommended) A system of linear equations can't have exactly two solutions. If (x,y) and (X,Y) are two solutions to $A\boldsymbol{v}=\boldsymbol{b}$, what is another solution?

If $\boldsymbol{v}=(x,y)$ and also $\boldsymbol{V}=(X,Y)$ solve the system $A\boldsymbol{v}=\boldsymbol{b}$, then another solution is $\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{V}$. (All combinations $\boldsymbol{u}=c\boldsymbol{v}+(1-c)\boldsymbol{V}$ will be solutions since $A\boldsymbol{u}=cA\boldsymbol{v}+(1-c)A\boldsymbol{V}=c\boldsymbol{b}+(1-c)\boldsymbol{b}=\boldsymbol{b}$.)

Problems 12–20 study elimination on 3 by 3 systems (and possible failure).

12 Reduce this system to upper triangular form by two row operations:

$$\begin{array}{lrl} & & 2x+3y+z\ \ =8 \\ \text{Eliminate } x & \rightarrow & 4x+7y+5z=20 \\ \text{Eliminate } y & \rightarrow & \quad -2y+2z=0. \end{array}$$

Circle the pivots. Solve by back substitution for z,y,x.

Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{array}{rcrl} 2x+3y+\ \ z=8 & & x=2 & \\ y+3z=4 & \text{gives} & y=1 & \text{If a zero is at the start of row 2 or 3,} \\ 8z=8 & & z=1 & \text{that avoids a row operation.} \end{array}$$

13 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x-3y\qquad &=3 \\ 4x-5y+\ \ z&=7 \\ 2x-\ \ y-3z&=5. \end{aligned}$$

List the three row operations: Subtract _____ times row _____ from row _____.

$$\begin{array}{rcrcrcr} 2x-3y\qquad =3 & & 2x-3y=3 & & 2x-3y=3 & & x=3 \\ 4x-5y+\ \ z=7 & \text{gives} & y+\ \ z=1 & \text{and} & y+\ \ z=1 & \text{and} & y=1 \\ 2x-\ \ y-3z=5 & & 2y+3z=2 & & -5z=0 & & z=0 \end{array}$$

Subtract $2\times$ row 1 from row 2, subtract $1\times$ row 1 from row 3, subtract $2\times$ row 2 from row 3

14 Which number d forces a row exchange? What is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x+5y+z&=0 \\ 4x+dy+z&=2 \\ y-z&=3. \end{aligned}$$

Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

15 Which number b leads later to a row exchange? Which b leads to a singular problem that row exchanges cannot fix? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \quad &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

16 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form.

(b) Construct a 3 by 3 system that needs a row exchange for pivot 2, but breaks down for pivot 3.

(a) **Example of 2 exchanges**
$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$
(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down**
$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$
(rows 1 and 3 are not consistent)

17 If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

Equal rows
$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned}$$

Equal columns
$$\begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

If row 1 $=$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 $=$ column 1, then column 2 has no pivot.

18 Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0, 0, 0)$?

Example $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions.

19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q+4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Three planes can fail to have an intersection point, *even if no planes are parallel*. The system is singular if row 3 is a combination of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1{+}2{=}$row 3 on the left side but not the right side: $x{+}y{+}z{=}0$, $x{-}2y{-}z{=}1$, $2x{-}y{=}1$. No parallel planes but still no solution.

21 Find the pivots and the solution for both systems ($A\boldsymbol{v} = \boldsymbol{b}$ and $S\boldsymbol{w} = \boldsymbol{b}$):

$$\begin{aligned} 2x + y \quad\quad &= 0 \\ x + 2y + z \quad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned} \qquad\qquad \begin{aligned} 2x - y \quad\quad &= 0 \\ -x + 2y - z \quad &= 0 \\ -y + 2z - t &= 0 \\ -z + 2t &= 5. \end{aligned}$$

(a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x+y=0$, $\frac{3}{2}y+z=0$, $\frac{4}{3}z+t=0$, $\frac{5}{4}t=5$ after elimination. Back substitution gives $t=4, z=-3, y=2, x=-1$.

(b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1,2,3,4)$ instead of $(-1,2,-3,4)$.

22 If you extend Problem 21 following the $1,2,1$ pattern or the $-1,2,-1$ pattern, what is the fifth pivot? What is the nth pivot? S is my favorite matrix.

The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If elimination leads to $x+y=1$ and $2y=3$, find three possible original problems.

If ordinary elimination leads to $x+y=1$ and $2y=3$, the original second equation could be $2y+\ell(x+y)=3+\ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y=3$.

24 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a=2$ or $a=0$.

25 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

$a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

26 Look for a matrix that has row sums 4 and 8, and column sums 2 and s:

$$\text{Matrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ _____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{v} = (4,8,2,s)$ with $\boldsymbol{v} = (a,b,c,d)$ and make A triangular by elimination.

Solvable for $s=10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2+s$ on the right sides). The four equations for a,b,c,d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

27 Elimination in the usual order gives what matrix U and what solution (x, y, z) to this "lower triangular" system ? We are really solving by *forward substitution* :

$$\begin{aligned} 3x \quad\quad\quad &= 3 \\ 6x + 2y \quad\; &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

Elimination leaves the diagonal matrix $\text{diag}(3, 2, 1)$ in $3x = 3, 2y = 2, z = 4$. Then $x = 1, y = 1, z = 4$.

28 Create a MATLAB command `A(2, : ) = ...` for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.

$A(2,:) = A(2,:) - 3 * A(1,:)$ subtracts 3 times row 1 from row 2.

29 If the last corner entry of A is $A(5, 5) = 11$ and the last pivot of A is $U(5, 5) = 4$, what different entry $A(5, 5)$ would have made A singular ?

A change up or down in $A(5, 5)$ produces the same change in $U(5, 5)$. If $A(5, 5) = 11$ gave $U(5, 5) = 4$, then subtract 4: $A(5, 5) = \mathbf{7}$ will give $U(5, 5) = 0$ and a singular matrix—zero in the last pivot position $U(5, 5)$.

Challenge Problems

30 Suppose elimination takes A to U without row exchanges. Then row i of U is a combination of which rows of A ? If $A\boldsymbol{v} = \mathbf{0}$, is $U\boldsymbol{v} = \mathbf{0}$? If $A\boldsymbol{v} = \boldsymbol{b}$, is $U\boldsymbol{v} = \boldsymbol{b}$?

Row j of U is a combination of rows $1, \ldots, j$ of A. If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

31 Start with 100 equations $A\boldsymbol{v} = \mathbf{0}$ for 100 unknowns $\boldsymbol{v} = (v_1, \ldots, v_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property : Some linear combination of the 100 **rows** is _____ .

(b) Singular systems $A\boldsymbol{v} = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is _____ .

(c) Invent a 100 by 100 singular matrix with no zero entries.

(d) For your matrix, describe in words the row picture and the column picture of $A\boldsymbol{v} = \mathbf{0}$. Not necessary to draw 100-dimensional space.

The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A = \mathbf{eye}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 4.3, page 223

Problems 1–16 are about the laws of matrix multiplication .

1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results ?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B+C).$$

If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**$(3,1)$ is 3 by 1. DBA and $A(B+C)$ are not defined.

2 What rows or columns or matrices do you multiply to find

(a) the third column of AB ?

(b) the first row of AB ?

(c) the entry in row 3, column 4 of AB ?

(d) the entry in row 1, column 1 of CDE ?

(a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B)
(d) (Row 1 of C)D(column 1 of E).

3 Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

$AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 In Problem 3, multiply A times BC. Then multiply AB times C.

$A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ from column 1 of AB and row 2 of C (multiply columns times rows).

5 Compute A^2 and A^3. Make a prediction for A^5 and A^n :

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

(a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\qquad} + B^2$.

$(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 True or false. Give a specific example when false:

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

(d) $(AB)^2 = A^2B^2$.

(a) True (b) False (c) True (d) False: usually $(AB)^2 \neq A^2B^2$.

8 How is each row of DA and EA related to the rows of A, when

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}?$$

How is each column of AD and AE related to the columns of A?

The rows of DA are 3 (row 1 of A) and 5 (row 2 of A). Both rows of EA are row 2 of A. The columns of AD are 3 (column 1 of A) and 5 (column 2 of A). The first column of AE is zero, the second is column 1 of A + column 2 of A.

9 Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$. Notice E and F in boldface.

$$EA = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$(EA)F = (EA) \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+c & a+c+b+d \end{bmatrix}.$$

Do those steps in the opposite order, first multiply AF and then $E(AF)$. Compare with $(EA)F$. What law is obeyed by matrix multiplication?

$AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 Row 1 of A is added to row 2 to produce EA. Then F adds row 2 of EA to row 1. Now F is on the left, for row operations. The result is $F(EA)$:

$$F(EA) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix}.$$

Do those steps in the opposite order: first add row 2 to row 1 by FA, then add row 1 of FA to row 2. What law is or is not obeyed by matrix multiplication?

$FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative.

11 (3 by 3 matrices) Choose the only B so that for every matrix A

(a) $BA = 4A$

(b) $BA = 4B$ (tricky)

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A.

(a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

12 Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are $A =$ multiple of I.

$AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \boldsymbol{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with B and C (and all other matrices) are multiples of I: $A = aI$.

13 Which of the following matrices are guaranteed to equal $(A - B)^2$: $A^2 - B^2$, $(B - A)^2$, $A^2 - 2AB + B^2$, $A(A - B) - B(A - B)$, $A^2 - AB - BA + B^2$?

$(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A - B)^2$.

14 True or false:

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

(a) True (A^2 is only defined when A is square) (b) False (if A is m by n and B is n by m, then AB is m by m and BA is n by n). (c) True (d) False (take $B = 0$).

15 If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector $\boldsymbol{x}$ with n components?

(b) A multiplies an n by p matrix B?

(c) A multiplies itself to produce A^2? Here $m = n$ and A is square.

(a) mn (use every entry of A) (b) $mnp = p\times$part (a) (c) n^3 (n^2 dot products).

16 For $A = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{4} \\ \mathbf{1} & \mathbf{0} & \mathbf{6} \end{bmatrix}$, compute these answers *and nothing more*:

(a) column 2 of AB (b) row 2 of AB (c) row 2 of A^2

(d) row 2 of A^3.

(a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

Problems 17–19 use a_{ij} for the entry in row i, column j of A.

17 Write down the 3 by 3 matrix A whose entries are

(a) $a_{ij} =$ minimum of i and j (b) $a_{ij} = (-1)^{i+j}$ (c) $a_{ij} = i/j$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i,j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$ (this will be an example of a *rank one matrix*).

18 What words would you use to describe each of these classes of matrices? Give a 3 by 3 example in each class. Which matrix belongs to all four classes?

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i < j$ (c) $a_{ij} = a_{ji}$
(d) $a_{ij} = a_{1j}$.

Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot?
(b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3?
(c) the new entry that replaces a_{32} after that subtraction?
(d) the second pivot?

(a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

Problems 20–24 involve powers of A.

20 Compute A^2, A^3, A^4 and also $A\boldsymbol{v}$, $A^2\boldsymbol{v}$, $A^3\boldsymbol{v}$, $A^4\boldsymbol{v}$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

$$A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^4 = \text{zero matrix for } \textit{strictly triangular } A.$$

$$\text{Then } A\boldsymbol{v} = A\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix},\ A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix},\ A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix},\ A^4\boldsymbol{v} = 0.$$

21 Find all the powers $A^2, A^3, \ldots$ and $AB, (AB)^2, \ldots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$A = A^2 = A^3 = \cdots = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ but $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$ and $(AB)^2 =$ zero matrix!

22 By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \qquad BC = 0 \qquad DE = -ED \text{ (not allowing } DE = 0).$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

23 (a) Find a nonzero matrix A for which $A^2 = 0$.
(b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. Note: Any matrix A = column times row = $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ will have $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = 0$ if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but $A^3 = 0$; strictly triangular as in Problem 20.

24 By experiment with $n = 2$ and $n = 3$ predict A^n for these matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

$(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

Problems 25–31 use column-row multiplication and block multiplication.

25 Multiply A times I using columns of A (3 by 3) times rows of I.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad} = \underline{\qquad}.$$

Columns of A **times rows of** B $\begin{bmatrix}1\\2\\2\end{bmatrix}[3 \quad 3 \quad 0]+\begin{bmatrix}0\\4\\1\end{bmatrix}[1 \quad 2 \quad 1]=\begin{bmatrix}3&3&0\\6&6&0\\6&6&0\end{bmatrix}+\begin{bmatrix}0&0&0\\4&8&4\\1&2&1\end{bmatrix}=$
$\begin{bmatrix}3&3&0\\10&14&4\\7&8&1\end{bmatrix}=AB.$

27 Show that the product of two upper triangular matrices is always upper triangular:

$$AB=\begin{bmatrix}x&x&x\\0&x&x\\0&0&x\end{bmatrix}\begin{bmatrix}x&x&x\\0&x&x\\0&0&x\end{bmatrix}=\begin{bmatrix}x&&\\0&&\\0&0&x\end{bmatrix}.$$

Proof using dot products (*Row-times-column*) (Row 2 of A) $\cdot$ (column 1 of B)$=0$. Which other dot products give zeros?

Proof using full matrices (*Column-times-row*) Draw x's and 0's in (column 2 of A) times (row 2 of B). Also show (column 3 of A) times (row 3 of B).

(a) (row 3 of A) $\cdot$ (column 1 of B) and (row 3 of A) $\cdot$ (column 2 of B) are both zero.

(b) $\begin{bmatrix}x\\x\\0\end{bmatrix}\begin{bmatrix}0&x&x\end{bmatrix}=\begin{bmatrix}0&x&x\\0&x&x\\0&0&0\end{bmatrix}$ and $\begin{bmatrix}x\\x\\x\end{bmatrix}\begin{bmatrix}0&0&x\end{bmatrix}=\begin{bmatrix}0&0&x\\0&0&x\\0&0&x\end{bmatrix}$: **both upper**.

28 If A is 2 by 3 with rows 1, 1, 1 and 2, 2, 2, and B is 3 by 4 with columns 1, 1, 1 and 2, 2, 2 and 3, 3, 3 and 4, 4, 4, use each of the four multiplication rules to find AB:

(1) Rows of A times columns of B. **Inner products** (each entry in AB)

(2) Matrix A times columns of B. **Columns of AB**

(3) Rows of A times the matrix B. **Rows of AB**

(4) Columns of A times rows of B. **Outer products** (3 matrices add to AB)

$$AB=\begin{bmatrix}1&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}1&2&3&4\\1&2&3&4\\1&2&3&4\end{bmatrix}=\begin{bmatrix}3&6&9&12\\6&12&18&24\end{bmatrix}.$$

(1) Two rows of A times four columns of $B=$ **eight** numbers

(2) A times the first column of B gives $\begin{bmatrix}3\\6\end{bmatrix}$. The later columns are multiplied by 2, 3, and 4.

(3) The first row of A is multiplied by B to give 3, 6, 9, 12. The second row of A is doubled so the second row of AB is doubled.

(4) Column times row multiplication gives three matrices (in this case they are all the same!)

$\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1&2&3&4\end{bmatrix}=\begin{bmatrix}1&2&3&4\\2&4&6&8\end{bmatrix}$ times 3 gives AB.

29 Which matrices E_{21} and E_{31} produce zeros in the $(2,1)$ and $(3,1)$ positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ produce zeros in the $2,1$ and $3,1$ entries.

Multiply E's to get $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. Then $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is the result of both E's since $(E_{31}E_{21})A = E_{31}(E_{21}A)$.

30 **Block multiplication** produces zeros below the pivot in one big step :

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & D - \boldsymbol{c}\boldsymbol{b}/a \end{bmatrix} \text{ with vectors } \mathbf{0}, \boldsymbol{b}, \boldsymbol{c}.$$

In Problem 29, what are $\boldsymbol{c}$ and D and what is the block $D - \boldsymbol{c}\boldsymbol{b}/a$?

In **29**, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{c}\boldsymbol{b}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

31 With $i^2 = -1$, the product of $(A + iB)$ and $(\boldsymbol{x} + i\boldsymbol{y})$ is $A\boldsymbol{x} + iB\boldsymbol{x} + iA\boldsymbol{y} - B\boldsymbol{y}$. Use blocks to separate the real part without i from the imaginary part that multiplies i :

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ B\boldsymbol{x} + A\boldsymbol{y} \end{bmatrix}$ real part / imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.

32 (*Very important*) Suppose you solve $A\boldsymbol{v} = \boldsymbol{b}$ for three special right sides $\boldsymbol{b}$:

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are the columns of a matrix X, what is A times X ?

A times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

33 If the three solutions in Question 32 are $\boldsymbol{v}_1 = (1,1,1)$ and $\boldsymbol{v}_2 = (0,1,1)$ and $\boldsymbol{v}_3 = (0,0,1)$, solve $A\boldsymbol{v} = \boldsymbol{b}$ when $\boldsymbol{b} = (3,5,8)$. Challenge problem : What is A ?

$\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

34 **Practical question** Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The same answer comes from A times BC, now with $mnq + npq$ separate multiplications. Notice npq for BC.

(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?

(b) With N-component vectors, would you choose $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ or $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$?

(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

Multiplying $AB = (m \text{ by } n)(n \text{ by } p)$ needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC = (n \text{ by } p)(p \text{ by } q)$ needs npq and then $A(BC)$ needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$ with the larger number $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$. So AB first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are N by 1, then $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ needs $2N$ multiplications but $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$ needs N^2 to find $\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}$ and N^2 more to multiply by the row vector $\boldsymbol{u}^{\mathrm{T}}$. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1} + n^{-1}$ with $p^{-1} + m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying $\boldsymbol{v}$ for $AB\boldsymbol{v}$, **don't multiply the matrices first**.

35 **Unexpected fact** A friend in England looked at powers of a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \quad A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \quad A^4 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

He noticed that the ratios $2/3$ and $10/15$ and $54/81$ are all the same. This is true for all powers. It doesn't work for an $n \times n$ matrix, unless A is tridiagonal. One neat proof is to look at the equal $(1,1)$ entries of $A^n A$ and AA^n. Can you use that idea to show that $B/C = 2/3$ in this example?

The off-diagonal ratio $\frac{2}{3}$ in $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ stays the same for all powers of A^n. Peter Larcombe gave a proof by induction. Ira Gessel compared the $(1,1)$ entries on the left and right sides of the true equation $A^n A = AA^n$:

$$A^n A = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The $(1,1)$ entries give $A + 3B = A + 2C$ and therefore $B/C = 2/3$. This ratio stays the same for A^{-1}.

The same idea applies when the matrix A is N by N, provided it is tridiagonal (three nonzero diagonals):

$$\text{The } (1,1) \text{ entry of } A^n A = \begin{bmatrix} A & B & E \\ C & D & F \\ G & H & I \end{bmatrix} \begin{bmatrix} 1 & 2 & \\ 3 & 4 & 5 \\ & 6 & 7 \end{bmatrix} \text{ is still } A + 3B.$$

Problem Set 4.4, page 234

1 Find the inverses of A, B, C (directly or from the 2 by 2 formula):

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

A simple row exchange has $P^2 = I$ so $P^{-1} = P$. Here $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} = "transpose" of P, coming in Section 2.7.

3 Solve for the first column (x, y) and second column (t, z) of A^{-1}:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination.

4 Show that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1}:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(*For a different* A, could column 1 of A^{-1} be possible to find but not column 2?)

The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.

5 Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

An upper triangular U with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any a. And also $-U$.

6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, find two different matrices such that $AB = AC$.

(a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{v} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{v} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

(a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Why is no third pivot ?

(a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \boldsymbol{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

9 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how would you find B^{-1} from A^{-1}?

If you exchange rows 1 and 2 of A to reach B, you exchange **columns** 1 and 2 of A^{-1} to reach B^{-1}. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this P.

10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B).

11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

(a) If $B = -A$ then certainly $A+B$ = zero matrix is not invertible. (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

Multiply $C = AB$ on the right by C^{-1} and on the left by A^{-1} to get $A^{-1} = BC^{-1}$.

13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

$M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by C and the right by A : $B^{-1} = CM^{-1}A$.

14 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

$$\text{Notice the order.}\quad \text{The inverse of}\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A \quad \text{is} \quad \underline{\qquad}.$$

$B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

15 Prove that a matrix with a column of zeros cannot have an inverse.

If A has a column of zeros, so does BA. Then $BA = I$ is impossible. There is no A^{-1}.

16 Multiply $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} \mathbf{d} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{a} \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

17 (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E$. Reverse the order and change -1 to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} =$ $L = E^{-1}$. Notice the 1's unchanged by multiplying in this order.

18 If B is the inverse of A^2, show that AB is the inverse of A.

$A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

19 (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

$$A^{-1} = \begin{bmatrix} 1 & -a & 0 & 0 \\ & 1 & -b & 0 \\ & & 1 & -c \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ab & abc \\ & 1 & b & bc \\ & & 1 & c \\ & & & 1 \end{bmatrix}.$$

20 Find the numbers a and b that give the inverse of 5 ∗ eye(4) − ones(4,4) :

$$[\,5I - \mathsf{ones}\,]^{-1} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 ∗ eye(5) − ones(5,5) ? In MATLAB, $I =$ eye.

The $(1,1)$ entry requires $4a - 3b = 1$; the $(1,2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b = a$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

21 Sixteen 2 by 2 matrices contain only 1's and 0's. How many of them are invertible?

Six of the sixteen $0-1$ matrices are invertible, including all four with three 1's.

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22 Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix};$$
$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$$

23 Follow the 3 by 3 text example of Gauss-Jordan but with all plus signs in A. Eliminate above and below the pivots to reduce $[\,A \;\; I\,]$ to $[\,I \;\; A^{-1}\,]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

$$[\boldsymbol{A}\;\; \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] =$$
$[\boldsymbol{I}\; \boldsymbol{A}^{-1}]$.

24 Use Gauss-Jordan elimination on $[\,U \;\; I\,]$ to find the upper triangular U^{-1}:

$$\boldsymbol{U}\boldsymbol{U}^{-1} = \boldsymbol{I} \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

25 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \;\; I\,]$ and $[\,B \;\; I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{\mathbf{-1}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so B^{-1} does not exist.

26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{6} \end{bmatrix}$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

$E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. $E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach $DE_{12}E_{21}A = I$. Then $A^{-1} = DE_{12}E_{21} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$.

27 Invert these matrices A by the Gauss-Jordan method starting with $[\,A \quad I\,]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$.
This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$: row exchanges are certainly allowed in Gauss-Jordan.

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's down the main diagonal is invertible.
(c) If A is invertible then A^{-1} and A^2 are invertible.

(a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible) (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows) (c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

This A is not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \quad I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. When the triangular A alternates 1 and -1 on its diagonal, A^{-1} is *bidiagonal* with 1's on the diagonal and first superdiagonal.

33 **(Puzzle)** Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

A can be invertible with diagonal zeros. B is singular because each row adds to zero.

34 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

$\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

Problem Set 4.5, Page 245

Questions 1–9 are about transposes A^{T} and symmetric matrices $S = S^{\mathrm{T}}$.

1 Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$;
$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \dfrac{1}{c^2}\begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

2 (a) Find 2 by 2 symmetric matrices A and B so that AB is not symmetric.

(b) With $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$, show that $AB = BA$ ensures that AB will now be symmetric. The product is symmetric only when A commutes with B.

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ give $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad$ and $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) If $AB = BA$ and $A^{\mathrm{T}} = A, B^{\mathrm{T}} = B$ then $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = BA = AB$. Thus AB is symmetric when A and B commute.

3 (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order*?

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is _____ triangular.

(a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.
(b) If U is upper triangular, so is U^{-1}: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

4 Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

5 Every square matrix A has a symmetric part and an antisymmetric part :

$$A = \text{symmetric} + \text{antisymmetric} = \left(\frac{A + A^{\mathrm{T}}}{2}\right) + \left(\frac{A - A^{\mathrm{T}}}{2}\right).$$

Transpose the antisymmetric part to get *minus* that part. Split these in two parts :

$$A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Transposing $\frac{1}{2}(A - A^{\mathrm{T}})$ gives $\frac{1}{2}(A^{\mathrm{T}} - A)$: this part is antisymmetric.

$$\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix}.$$

6 The transpose of a block matrix $M = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is $M^{\mathrm{T}} =$ _____. Test an example to be sure. Under what conditions on A, B, C, D is the block matrix symmetric?

$M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

7 True or false:

(a) The block matrix $\left[\begin{smallmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{smallmatrix}\right]$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

(a) False: $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$. (b) False: The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$ when A and B are symmetric $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ transposes to $\begin{bmatrix} 0 & A^{\mathrm{T}} \\ A^{\mathrm{T}} & 0 \end{bmatrix}$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$. (c) True: Invertible symmetric matrices have symmetric in verses! Easiest proof is to transpose $AA^{-1} = I$. (d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices A, B, and C).

8 (a) How many entries of S can be chosen independently, if $S = S^{\mathrm{T}}$ is 5 by 5 ?

(b) How many entries can be chosen if A is *skew-symmetric* ? $(A^{\mathrm{T}} = -A)$.

Answers: **15** and **10**. If $S = S^{\mathrm{T}}$ is 5 by 5, its 5 diagonal entries and 10 entries above the diagonal are free to choose. If $A^{\mathrm{T}} = -A$, the 5 diagonal entries of A must be zero.

9 Transpose the equation $A^{-1}A = I$. The result shows that the inverse of A^{T} is _____. If S is symmetric, **how does this show that S^{-1} is also symmetric** ?

$A^{-1}A = I$ transposes to $A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = I$. This shows that the inverse of A^{T} is $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$. If S is symmetric ($S^{\mathrm{T}} = S$) then this statement becomes $S^{-1} = (S^{-1})^{\mathrm{T}}$. Therefore **$S^{-1}$ is symmetric**.

Questions 10–14 are about permutation matrices.

10 Why are there $n!$ permutation matrices of size n ? They give $n\,!$ orders of $1, \ldots, n$.

The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

11 If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

$$P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ but } P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If P_3 and P_4 exchange *different* pairs of rows, $P_3P_4 = P_4P_3$ does both exchanges.

12 There are 12 "*even*" permutations of $(1, 2, 3, 4)$, with an *even number of exchanges*. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, just order the numbers.

$(3, 1, 2, 4)$ and $(2, 3, 1, 4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2, 1, 4, 3)$ and $(3, 4, 1, 2)$ exchange 2 pairs. $(1, 2, 3, 4)$, $(4, 3, 2, 1)$ make 12 even P's.

13 If P has 1's on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP. Is P even?

The "reverse identity" P takes $(1, \ldots, n)$ into $(n, \ldots, 1)$. When rows and also columns are reversed, $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$. In particular $(PAP)_{11}$ is A_{nn}.

14 (a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

(b) Find a 4 by 4 permutation with $P^4 \neq I$.

A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1, 2, 3) \to (2, 3, 1) \to (3, 1, 2) \to (1, 2, 3)$. $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same P has $\widehat{P}^4 = \widehat{P} \neq I$.

Questions 15–18 are about first differences A and second differences $A^{\mathrm{T}}A$ and AA^{T}.

15 Write down the 5 by 4 backward difference matrix A.

(a) Compute the symmetric second difference matrices $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$.

(b) Show that S is invertible by finding S^{-1}. Show that L is singular.

$$A = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad S = A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$L = AA^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}.$$

L (5 by 5) is singular: $L\boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} = (1,1,1,1,1)$.

S (4 by 4) is invertible: $S^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

16 In Problem 15, find the pivots of S and L (4 by 4 and 5 by 5). The pivots of S in equation (8) are $2, 3/2, 4/3$. The pivots of L in equation (10) are $1, 1, 1, 0$ (fail).

The pivots of S are $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$. Multiply those pivots to find determinant $= 5$. This explains 1/5 in S^{-1}.

The pivots of L are $1, 1, 1, 1, 0$ (no pivot).

17 (Computer problem) Create the 9 by 10 backward difference matrix A. Multiply to find $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$. If you have linear algebra software, ask for the determinants $\det(S)$ and $\det(L)$.

Challenge : By experiment find $\det(S)$ when $S = A^{\mathrm{T}}A$ is n by n.

Correction The backward difference matrix A will be **10 by 9**. Then $S = A^{\mathrm{T}}A$ is 9 by 9 (the $-1, 2, -1$ matrix) with $\det S = 10$. In general $\det S = n$ when A is n by $n-1$.
$L = AA^{\mathrm{T}}$ is 10 by 10 (the $-1, 2 - -1$ matrix except that $L_{11} = 1$ and $L_{nn} = 1$). Then L is singular and $\det L = 0$.

18 (Infinite computer problem) Imagine that the second difference matrix S is infinitely large. The diagonals of 2's and -1's go from minus infinity to plus infinity:

$$\textbf{Infinite tridiagonal matrix} \qquad S = \begin{bmatrix} \cdot & \cdot & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \cdot & \cdot & \end{bmatrix}$$

(a) Multiply S times the infinite *all-ones* vector $\boldsymbol{v} = (\ldots, 1, 1, 1, 1, \ldots)$

(b) Multiply S times the infinite *linear* vector $\boldsymbol{w} = (\ldots, 0, 1, 2, 3, \ldots)$

(c) Multiply S times the infinite *squares* vector $\boldsymbol{u} = (\ldots, 0, 1, 4, 9, \ldots)$.

(d) Multiply S times the infinite *cubes* vector $\boldsymbol{c} = (\ldots, 0, 1, 8, 27, \ldots)$.

The answers correspond to second derivatives (with minus sign) of 1 and x^2 and x^3.

S times **all-ones**	gives the zero vector
S times **linear $\boldsymbol{w}$**	gives the zero vector
S times **squares $\boldsymbol{u}$**	gives -2 times **all-ones**
S times **cubes $\boldsymbol{c}$**	gives -6 times **linear $\boldsymbol{w}$**

Those correspond to $0, 0, -2, -6x =$ **minus** the second derivatives of $1, x, x^2, x^3$.

Questions 19–28 are about matrices with $\boldsymbol{Q^{\mathrm{T}}Q = I}$. If Q is square, then it is an orthogonal matrix and $\boldsymbol{Q^{\mathrm{T}} = Q^{-1}}$ and $\boldsymbol{QQ^{\mathrm{T}} = I}$.

19 Complete these matrices to be orthogonal matrices :

$$\text{(a)}\quad Q = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix} \quad \text{(b)}\quad Q = \tfrac{1}{3}\begin{bmatrix} -1 & & \\ 2 & & \\ 2 & & \end{bmatrix} \quad \text{(c)}\quad Q = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & -1 & & \\ 1 & -1 & & \end{bmatrix}.$$

$$Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \quad Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Note: You could complete to Q with different columns than these.

20 (a) Suppose Q is an orthogonal matrix. Why is $Q^{-1} = Q^{\mathrm{T}}$ also an orthogonal matrix ?

(b) From $Q^{\mathrm{T}}Q = I$, the columns of Q are orthogonal unit vectors (orthonormal vectors). Why are the rows of Q (square matrix) also orthonormal vectors ?

(a) Q^{-1} is also orthogonal because $(Q^{-1})^{\mathrm{T}}(Q^{-1}) = (Q^{\mathrm{T}})^{\mathrm{T}}Q^{\mathrm{T}} = QQ^{\mathrm{T}} = I$.

(b) The rows of Q are orthonormal vectors because $QQ^{\mathrm{T}} = I$. For square matrices, Q^{T} is a right-inverse of Q whenever it is a left-inverse of Q. So rows are orthonormal when columns are orthonormal.

21 (a) Which vectors can be the first column of an orthogonal matrix ?

(b) If $Q_1^{\mathrm{T}}Q_1 = I$ and $Q_2^{\mathrm{T}}Q_2 = I$, is it true that $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = I$? Assume that the matrix shapes allow the multiplication Q_1Q_2.

(a) Any unit vector (length 1) can be the first column of Q.

(b) YES, $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = Q_2^{\mathrm{T}}(Q_1^{\mathrm{T}}Q_1)Q_2 = Q_2^{\mathrm{T}}Q_2 = I$.

22 If $\boldsymbol{u}$ is a unit column vector (length 1, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$), show why $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is

(a) a symmetric matrix : $H = H^{\mathrm{T}}$ (b) an orthogonal matrix : $H^{\mathrm{T}}H = I$.

The Householder matrix $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric (because $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric) and also orthogonal (because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$) :

$$H^{\mathrm{T}}H = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = I.$$

23 If $\boldsymbol{u} = (\cos\theta, \sin\theta)$, what are the four entries in $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$? Show that $H\boldsymbol{u} = -\boldsymbol{u}$ and $H\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} = (-\sin\theta, \cos\theta)$. This H is a **reflection matrix**: the $\boldsymbol{v}$-line is a mirror and the $\boldsymbol{u}$-line is reflected across that mirror.

$$H = I - 2\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}\begin{bmatrix}\cos\theta & sin\theta\end{bmatrix} = \begin{bmatrix}1 - 2\cos^2\theta & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 1 - 2\sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos^2\theta\end{bmatrix}.$$

$H\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathbf{T}}\boldsymbol{u} = -\boldsymbol{u}$ $\quad\quad$ $H\boldsymbol{v} = \boldsymbol{v} - 2\boldsymbol{u}\boldsymbol{u}^{\mathbf{T}}\boldsymbol{v} = \boldsymbol{v}$ since $\boldsymbol{u}^{\mathbf{T}}\boldsymbol{v} = 0$.

24 Suppose the matrix Q is orthogonal and also upper triangular. What can Q look like? Must it be diagonal?

If Q is orthogonal and upper triangular, its first column must be $\boldsymbol{q}_1 = (\pm 1, 0, \ldots, 0)$. Then its second column $\boldsymbol{q}_2$ must start with 0 to have the orthogonality $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$. Then $\boldsymbol{q}_2 = (0, \pm 1, 0, \ldots, 0)$. Then $\boldsymbol{q}_3$ must start with 0, 0 to have $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_3 = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_3 = 0$ (and so onward). Thus Q is diagonal: $Q = \text{diag}\ (\pm 1, \ldots, \pm 1)$.

25 (a) To construct a 3 by 3 orthogonal matrix Q whose first column is in the direction $\boldsymbol{w}$, what first column $\boldsymbol{q}_1 = c\boldsymbol{w}$ would you choose?

(b) The next column $\boldsymbol{q}_2$ can be any unit vector perpendicular to $\boldsymbol{q}_1$. To find $\boldsymbol{q}_3$, choose a solution $\boldsymbol{v} = (v_1, v_2, v_3)$ to the two equations $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} = 0$. *Why is there always a nonzero solution* $\boldsymbol{v}$?

(a) The first column of Q will be $\boldsymbol{q}_1 = \boldsymbol{w}/||w||$ to have length 1.

(b) The next column $\boldsymbol{q}_2$ has $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ and $||\boldsymbol{q}_2|| = 1$. Then there will be a vector $\boldsymbol{v}$ orthogonal to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ because $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} = 0$ give 2 linear equations in 3 unknowns v_1, v_2, v_3.

26 Why is every solution $\boldsymbol{v}$ to $A\boldsymbol{v} = \mathbf{0}$ orthogonal to every row of A?

Writing out $A\boldsymbol{v} = \mathbf{0}$ shows that every row is orthogonal to $\boldsymbol{v}$:

$$\begin{bmatrix} row\ 1 \\ \cdots \\ row\ n\end{bmatrix}\begin{bmatrix} \\ \boldsymbol{v} \\ \\ \end{bmatrix} = \begin{bmatrix}0 \\ \cdots \\ 0\end{bmatrix}.$$

27 Suppose $Q^{\mathrm{T}}Q = I$ but Q is not square. The matrix $P = QQ^{\mathrm{T}}$ is not I. But show that P is symmetric and $P^2 = P$. This is a **projection matrix**.

If Q has n orthogonal columns and $n < m$, then the m by m matrix $P = QQ^{\mathrm{T}}$ is not I. (Some vector $\boldsymbol{v}$ in R^m will solve the n equations $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$. Then $QQ^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ and $QQ^{\mathrm{T}} \neq I$.) But P is symmetric and $P^2 = QQ^{\mathrm{T}}QQ^{\mathrm{T}} = QIQ^{\mathrm{T}} = P$. Thus P is a **projection matrix**.

28 A 5 by 4 matrix Q can have $Q^{\mathrm{T}}Q = I$ but *it cannot possibly have* $QQ^{\mathrm{T}} = I$. Explain in words why the four equations $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ must have a nonzero solution $\boldsymbol{v}$. Then $\boldsymbol{v}$ is not the same as $QQ^{\mathrm{T}}\boldsymbol{v}$ and I is not the same as QQ^{T}.

The four equations $Q^{\mathrm{T}}\boldsymbol{v} = 0$ have 5 unknowns v_1, v_2, v_3, v_4, v_5. With only 4 rows, Q^{T} cannot have more than 4 pivots. There must be a free column in Q^{T} and a *nonzero* special solution to $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$.

Challenge Problems

29 Can you find a rotation matrix Q so that QDQ^{T} is a permutation ?

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \quad \text{equals} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

With $\theta = 45°$, $\dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix} \qquad \dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$

30 Split an orthogonal matrix ($Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I$) into two rectangular submatrices :

$$Q = [\,Q_1 \mid Q_2\,] \quad \text{and} \quad Q^{\mathrm{T}}Q = \begin{bmatrix}Q_1^{\mathrm{T}}Q_1 & Q_1^{\mathrm{T}}Q_2\\ Q_2^{\mathrm{T}}Q_1 & Q_2^{\mathrm{T}}Q_2\end{bmatrix}$$

(a) What are those four blocks in $Q^{\mathrm{T}}Q = I$?

(b) $QQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} + Q_2Q_2^{\mathrm{T}} = I$ is column times row multiplication. Insert the diagonal matrix $D = \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$ and do the same multiplication for QDQ^{T}.

Note : The description of all symmetric orthogonal matrices S in (**??**) becomes $S = QDQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$. This is exactly the reflection matrix $I - 2Q_2Q_2^{\mathrm{T}}$.

(a) The four blocks in $Q^{\mathrm{T}}Q$ are $I, 0, 0, I$ because all the columns of Q_1 are orthogonal to all the columns of Q_2. (All together they are the columns of the orthogonal matrix Q.)

(b) Column times row multiplication gives

$$\begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}Q_1^{\mathrm{T}}\\ Q_2^{\mathrm{T}}\end{bmatrix} = Q_1Q_1^{\mathrm{T}} + Q_2Q_2^{\mathrm{T}} = I.$$

$$QDQ^{\mathrm{T}} = \begin{bmatrix}Q_1 & Q_2\end{bmatrix} D \begin{bmatrix}Q_1^{\mathrm{T}}\\ Q_2^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}Q_1^{\mathrm{T}}\\ -Q_2^{\mathrm{T}}\end{bmatrix} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$$

$= I - 2Q_2Q_2^{\mathrm{T}}$.

Then QDQ^{T} is both symmetric and orthogonal.

31 The real reason that the transpose "flips A across its main diagonal" is to obey this dot product law : $(A\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (A^{\mathrm{T}}\boldsymbol{w})$. That rule $(A\boldsymbol{v})^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w})$ **becomes integration by parts in calculus**, where $A = d/dx$ and $A^{\mathrm{T}} = -d/dx$.

(a) For 2 by 2 matrices, write out both sides (4 terms) and compare :

$$\left(\begin{bmatrix}a & \boldsymbol{b}\\ \boldsymbol{c} & d\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right) \cdot \begin{bmatrix}w_1\\ w_2\end{bmatrix} \quad \text{is equal to} \quad \begin{bmatrix}v_1\\ v_2\end{bmatrix} \cdot \left(\begin{bmatrix}a & \boldsymbol{c}\\ \boldsymbol{b} & d\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}\right).$$

(b) The rule $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ comes slowly but directly from part (a) :

$$(AB)\,\boldsymbol{v} \cdot \boldsymbol{w} = A(B\boldsymbol{v}) \cdot \boldsymbol{w} = B\boldsymbol{v} \cdot A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v} \cdot B^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w}) = \boldsymbol{v} \cdot (B^{\mathrm{T}}A^{\mathrm{T}})\boldsymbol{w}$$

Steps 1 and 4 are the _____ law. Steps 2 and 3 are the dot product law.

The connection between $(Ax)^{\mathrm{T}}y = x\left(A^{\mathrm{T}}y\right)$ and integration by parts is developed in the Chapter 7 Notes. The idea is that A becomes the derivative d/dx and the dot product becomes an integral:

$$(Af)^{\mathrm{T}}g = \int \frac{df}{dx}\, g(x)\, dx = -\int f(x)\frac{dg}{dx}\, dx = f^{\mathrm{T}}\left(A^{\mathrm{T}}g\right).$$

That last step identifies $A^{\mathrm{T}}g$ as $-dg/dx$. So the first derivative $A = d/dx$ is like an *anti*symmetric matrix. Our functions f and g are zero at the ends of the integration interval, so the "by parts formula" above has zero from the other usual term $[fg]_0^1$.

In $31(b)$, steps 1 and 4 are the **associative law** $(AB)\boldsymbol{v} = A(B\boldsymbol{v})$.

32 How is a matrix $S = S^{\mathrm{T}}$ decided by its entries on and above the diagonal? How is Q with orthonormal columns decided by its entries *below* the diagonal? Together this matches the number of entries in an n by n matrix. So it is reasonable that every matrix can be factored into $A = SQ$ (like $re^{i\theta}$).

If S is symmetric, then the entries on and above the diagonal tell you the entries below the diagonal. If Q is orthogonal, here is how the entries *below the diagonal* decide the matrix. In column 1, the top entry Q_{11} has to complete a unit vector (no choice except a $\pm$ sign). In column 2, the two top entries are decided by (1) orthogonality to column 1 and (2) unit vector. Every column, in order, has no free numbers available on and above the diagonal.

So there are a total of n^2 choices available: on and above the diagonal of S and below the diagonal of Q. This n^2 matches the number of equations in $A = SQ$ (linear equations in $S = AQ^{\mathrm{T}}$). "polar factorization" of a matrix is possible.

Problem Set 5.1, Page 258

Questions 1–10 are about the "subspace requirements": $v + w$ and cv (and then all linear combinations $cv + dw$) stay in the subspace.

1 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{v} + \boldsymbol{w}$ stays in the set but $\frac{1}{2}\boldsymbol{v}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{v}$ stays in the set but $\boldsymbol{v} + \boldsymbol{w}$ may be outside.

(a) The set of vectors with integer components (adding $v + w$ produces integers, multiplying by $\frac{1}{2}$ may not).

(b) One option for the set is to take two lines through $(0, 0)$. Then $c\boldsymbol{v}$ stays on these lines but $\boldsymbol{v} + \boldsymbol{w}$ may not.

2 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

3 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

4 Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

For the plane $v + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

5 Let $\mathbf{P}_0$ be the plane through $(0, 0, 0)$ parallel to the previous plane $\mathbf{P}$. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

The parallel plane $\mathbf{P}_0$ has the equation $v + y - 2z = 0$. Pick two points, for example $(2, 0, 1)$ and $(0, 2, 1)$, and their sum $(2, 2, 2)$ is in $\mathbf{P}_0$.

6 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

(a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0,0)$, and $(0,0)$ by itself (b) The subspaces of $\mathbf{R}^4$ are $\mathbf{R}^4$ itself, three-dimensional planes $\boldsymbol{n}\cdot\boldsymbol{v}=0$, two-dimensional subspaces ($\boldsymbol{n}_1\cdot\boldsymbol{v}=0$ and $\boldsymbol{n}_2\cdot\boldsymbol{v}=0$), one-dimensional lines through $(0,0,0,0)$, and $(0,0,0,0)$ by itself.

7 (a) The intersection of two planes through $(0,0,0)$ is probably a _____ but it could be a _____. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a _____ but it could be a _____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S}\cap\mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S}\cap\mathbf{T}$ consists of the vectors that lie in both subspaces. *Check the requirements on $\boldsymbol{v}+\boldsymbol{w}$ and $c\boldsymbol{v}$.*

(a) Two planes through $(0,0,0)$ probably intersect in a line through $(0,0,0)$
(b) The plane and line probably intersect in the point $(0,0,0)$
(c) If $\boldsymbol{v}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{v}+\boldsymbol{y}$ and $c\boldsymbol{v}$ are in both subspaces.

8 Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space $\mathbf{P}+\mathbf{L}$ containing both $\mathbf{P}$ and $\mathbf{L}$ is either _____ or _____.

The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

9 (a) Show that the set of *invertible* matrices in $\mathbf{M}$ is not a subspace.

(b) Show that the set of *singular* matrices in $\mathbf{M}$ is not a subspace.

(a) The invertible matrices do not include the zero matrix, so they are not a subspace

(b) The sum of singular matrices $\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}$ is not singular: not a subspace.

10 True or false (check addition in each case by an example):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}=A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}=-A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}}\neq A$) form a subspace.

(a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}}=-A$ do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.

Questions 11–19 are about column spaces $C(A)$ and the equation $Av=b$.

11 Describe the column spaces (lines or planes) of these particular matrices:

$$A=\begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix}\qquad B=\begin{bmatrix}1&0\\0&2\\0&0\end{bmatrix}\qquad C=\begin{bmatrix}1&0\\2&0\\0&0\end{bmatrix}.$$

The column space of A is the x-axis $=$ all vectors $(x,0,0)$. The column space of B is the xy plane $=$ all vectors $(x,y,0)$. The column space of C is the line of vectors $(x,2x,0)$.

12 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + 2b_3$ in equation 3: Solution only if $b_3 = -b_1$.

13 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. Which matrices have the same column space? Which have the same *row space*?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space.

14 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

15 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{v} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger? Then it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$.

The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.
$[A \;\; \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{v} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ ($\boldsymbol{b}$ is in column space) ($A\boldsymbol{v} = \boldsymbol{b}$ has a solution)

16 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B = 0$ and $A \neq 0$ is a case when $AB = 0$ has a smaller column space than A.

17 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, then $\boldsymbol{b} + \boldsymbol{b}^*$ is also in $\boldsymbol{C}(A)$.

The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

18 If A is any 5 by 5 invertible matrix, then its column space is ____ . Why ?

The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{v} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

19 True or false (with a counterexample if false) :

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

(a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$. (d) *False*: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

20 Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

$A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $(1,1,1)$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line.

21 If the 9 by 12 system $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A)$ must be ____ .

When $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\mathbf{R}^9$.

Challenge Problems

22 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$. The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Then $\mathbf{S} + \mathbf{T}$ is a vector space.

If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and all vectors from $\mathbf{T}$. Explain this statement : *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

(a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

23 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M ? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\, A \quad B \,]$.

24 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (this has extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$.

The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

25 An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an _____ matrix.

(Key question) The column space of an n by n matrix A is all of R^n exactly when $\boldsymbol{A}$ **is invertible**. In this invertible case, every vector $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$ because we can solve $A\boldsymbol{v} = \boldsymbol{b}$. And if A were not invertible, elimination would lead to a row of zeros—then $A\boldsymbol{v} = \boldsymbol{b}$ could not be solved for some (most !) vectors b.

Problem Set 5.2, Page 269

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

1 Reduce these matrices to their ordinary echelon forms U :

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables ?

(a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables v_2, v_4, v_5 Pivot variables v_1, v_3 (b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free v_3 Pivot v_1, v_2

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

(a) Free variables v_2, v_4, v_5 and solutions $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, $(0,0,-3,0,1)$
(b) Free variable v_3: solution $(1,-1,1)$. Special solution for each free variable.

3 By combining the special solutions in Problem 2, describe every solution to $A\boldsymbol{v} = \boldsymbol{0}$ and $B\boldsymbol{v} = \boldsymbol{0}$. The nullspace contains only $\boldsymbol{v} = \boldsymbol{0}$ when there are no _____.

The complete solution to $A\boldsymbol{v} = \boldsymbol{0}$ is $(-2v_2, v_2, -2v_4 - 3v_5, v_4, v_5)$ with v_2, v_4, v_5 free. The complete solution to $B\boldsymbol{v} = \boldsymbol{0}$ is $(2v_3, -v_3, v_3)$. The nullspace contains only $\boldsymbol{v} = \boldsymbol{0}$ when there are no free variables.

4 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false* : The nullspace of R equals the nullspace of U.

$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

5 By row operations reduce this new A and B to triangular echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}.$$

$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = LU$.

6 For the same A and B, find the special solutions to $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is _____.

(a) Special solutions $(3, 1, 0)$ and $(5, 0, 1)$ (b) $(3, 1, 0)$. Total of pivot and free is n.

7 In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors $\boldsymbol{v}$ that satisfy those equations as combinations of the special solutions.

(a) The nullspace of A in Problem 5 is the plane $-v + 3y + 5z = 0$; it contains all the vectors $(3y + 5z, y, z) = y(3, 1, 0) + z(5, 0, 1) =$ combination of special solutions. (b) The *line* through $(3, 1, 0)$ has equations $-v + 3y + 5z = 0$ and $-2v + 6y + 7z = 0$. The special solution for the free variable v_2 is $(3, 1, 0)$.

8 Reduce the echelon forms U in Problem 5 to R. For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

$R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Questions 9–17 are about free variables and pivot variables.

9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.
(b) An invertible matrix has no free variables.
(c) An m by n matrix has no more than n pivot variables.
(d) An m by n matrix has no more than m pivot variables.

(a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

10 Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.
(b) A has no zero entries but $R = I$.
(c) A has no zero entries but $R = U$.
(d) $A = U = 2R$.

(a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

11 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) 2, 4, 5

(b) 1, 3, 6, 7

(c) 4 and 6.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

12 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) 2, 4, 5, 6

(b) 1, 3, 6, 7, 8.

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

13 Suppose column 4 of a 3 by 5 matrix is all zero. Then v_4 is certainly a _____ variable. The special solution for this variable is the vector $\boldsymbol{s} =$ _____.

If column 4 of a 3 by 5 matrix is all zero then v_4 is a *free* variable. Its special solution is $\boldsymbol{v} = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $A\boldsymbol{v} = \boldsymbol{0}$.

14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.

If column 1 = column 5 then v_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

15 Suppose an m by n matrix has r pivots. The number of special solutions is _____. The nullspace contains only $\boldsymbol{v} = \boldsymbol{0}$ when $r =$ _____. The column space is all of $\mathbf{R}^m$ when $r =$ _____.

If a matrix has n columns and r pivots, there are $\boldsymbol{n} - \boldsymbol{r}$ special solutions. The nullspace contains only $\boldsymbol{v} = \boldsymbol{0}$ when $\boldsymbol{r} = \boldsymbol{n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All important!

16 The nullspace of a 5 by 5 matrix contains only $\boldsymbol{v} = \boldsymbol{0}$ when the matrix has _____ pivots. The column space is $\mathbf{R}^5$ when there are _____ pivots. Explain why.

The nullspace contains only $\boldsymbol{v} = \boldsymbol{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{v} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

17 The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation ? Which are the free variables ? The special solutions are $(3, 1, 0)$ and _____.

$A = [1 \ \ -3 \ \ -1]$ gives the plane $v - 3y - z = 0$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

18 (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Fill in **12** then **4** then **1** to get the complete solution to $v - 3y - z = 12$: $\begin{bmatrix} v \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{4} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{v}_{\text{nullspace}}$.

19 Prove that U and $A = LU$ have the same nullspace when L is invertible:

$$\text{If } U\boldsymbol{v} = \mathbf{0} \text{ then } LU\boldsymbol{v} = \mathbf{0}. \quad \text{If } LU\boldsymbol{v} = \mathbf{0}, \text{ how do you know } U\boldsymbol{v} = \mathbf{0}\,?$$

If $LU\boldsymbol{v} = \mathbf{0}$, multiply by L^{-1} to find $U\boldsymbol{v} = \mathbf{0}$. Then U and LU have the same nullspace.

20 Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free) ? What is the special solution ? What is the nullspace ?

Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

Questions 21–28 ask for matrices (if possible) with specific properties.

21 Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

For special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables v_3, v_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

22 Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through $(4, 3, 2, 1)$.

23 Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $(1, 1, 5)$ and $(0, 3, 1)$ in $\boldsymbol{C}(A)$ and $(1, 1, 2)$ in $\boldsymbol{N}(A)$. *Which other A's?*

24 Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

25 Construct a matrix whose column space contains $(1,1,1)$ and whose nullspace is the line of multiples of $(1,1,1,1)$.

$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1,1,1)$ in $\boldsymbol{C}(A)$ and only the line (c,c,c,c) in $\boldsymbol{N}(A)$.

26 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

$A = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice $\textsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1\,0 \\ 0\,0 \end{bmatrix}$.

27 Why does no 3 by 3 matrix have a nullspace that equals its column space ?

If nullspace = column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

28 (Important) If $AB = 0$ then the column space of B is contained in the _____ of A. Give an example of A and B.

If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ *equals* $\boldsymbol{N}(A)$. (For $B = 0, \boldsymbol{C}(B)$ is smaller.)

29 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What reduced form R is virtually certain if the random A is 4 by 3 ?

For A = random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with fourth row of zeros. What about a random 3 by 4 matrix?

30 Show by example that these three statements are generally *false* :

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form of A then R^{T} is the reduced form of A^{T}.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\textsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31 If the nullspace of A consists of all multiples of $\boldsymbol{v} = (2,1,0,1)$, how many pivots appear in U ? What is R ?

If $\boldsymbol{N}(A)$ = line through $\boldsymbol{v} = (2,1,0,1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

32 If the special solutions to $R\boldsymbol{v} = \boldsymbol{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R :

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \quad \end{bmatrix} \text{ (empty 3 by 1).}$$

Any zero rows come after these rows: $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) What are the five 2 by 2 reduced echelon matrices R whose entries are all 0's and 1's?

(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R?

(a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's!

34 Explain why A and $-A$ always have the same reduced echelon form R.

One reason that R is the same for A and $-A$: They have the same nullspace. They also have the same column space, but that is not required for two matrices to share the same R. (R tells us the nullspace and row space.)

Challenge Problems

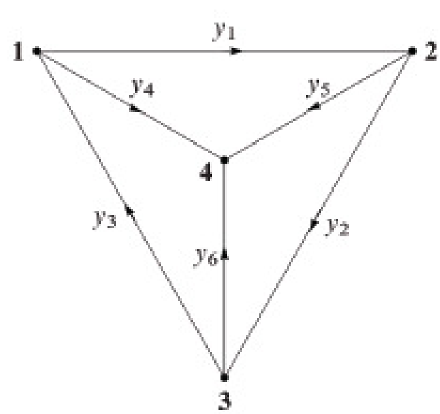
35 If A is 4 by 4 and invertible, describe all vectors in the nullspace of the 4 by 8 matrix $B = [A \;\; A]$.

The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{y}\\-\boldsymbol{y}\end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 How is the nullspace $\boldsymbol{N}(C)$ related to the spaces $\boldsymbol{N}(A)$ and $\boldsymbol{N}(B)$, if $C = \begin{bmatrix}A\\B\end{bmatrix}$?

If $C\boldsymbol{v} = \mathbf{0}$ then $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

37 Kirchhoff's Law says that *current in* = *current out* at every node. This network has six currents $y_1, \ldots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $A\boldsymbol{y} = \mathbf{0}$ for Kirchhoff's Law at the four nodes. Reduce to $U\boldsymbol{y} = \mathbf{0}$. Find three special solutions in the nullspace of A.



Currents: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 5.3, Page 280

1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$$

$A\boldsymbol{v} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $s_1 = (-1, -1, 1, 0)$ and $s_2 = (2, -2, 0, 1)$; $v_{complete} = v_p + c_1 s_1 + c_2 s_2$;

$$[R \quad d] = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ gives the particular solution } v_p = (4, -1, 0, 0).$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $A\boldsymbol{v} = \boldsymbol{b}$ to be solvable. Together these two conditions put $\boldsymbol{b}$ into the _____ space.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix} \quad \text{Then } [R \quad \boldsymbol{d}] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$A\boldsymbol{v} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A)$ = line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{v}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{v}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

Questions 3–15 are about the solution of $\boldsymbol{Av} = \boldsymbol{b}$. Follow the steps in the text to $\boldsymbol{v_p}$ and $\boldsymbol{v_n}$. Start from the augmented matrix $[\,\boldsymbol{A} \quad \boldsymbol{b}\,]$.

3 Write the complete solution as $\boldsymbol{v}_p$ plus any multiple of $\boldsymbol{s}$ in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

$\boldsymbol{v}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. The matrix is singular but the equations are still solvable; $\boldsymbol{b}$ is in the column space. Our particular solution has free variable $y = 0$.

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$\boldsymbol{v}_{\text{complete}} = \boldsymbol{v}_p + \boldsymbol{v}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + v_2(-3, 1, 0, 0) + v_4(0, 0, -2, 1).$

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include $\boldsymbol{b}$ as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.
Back-substitution gives the particular solution to $A\boldsymbol{v} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{v} = \boldsymbol{0}$: $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find $\boldsymbol{v}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{v}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{v} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination y_1(row 1) + y_2(row 2) + y_3(row 3) gives the zero row?

$\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[\,0\;\;0\;\;0\;\;0\,] =$ row 3 − 2 (row 2) + 4(row 1) **provided $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$**.

8 Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

(a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$.

(a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) We need $b_3 = 2b_2$, because $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$.

9 In Worked Example **5.3 A**, combine the pivot columns of A with the numbers -9 and 3 in the particular solution $\boldsymbol{v}_p$. What is that linear combination and why?

$$L\,[U \quad \boldsymbol{c}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix}$$

$= [A \quad b]$; particular $\boldsymbol{v}_p = (-9, 0, 3, 0)$ means $-9(1,2,3) + 3(3,8,7) = (0,6,-6)$. This is $A\boldsymbol{v}_p = \boldsymbol{b}$.

10 Construct a 2 by 3 system $A\boldsymbol{v} = \boldsymbol{b}$ with particular solution $\boldsymbol{v}_p = (2,4,0)$ and null (homogeneous) solution $\boldsymbol{v}_n =$ any multiple of $(1,1,1)$.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2,4,0)$ and $\boldsymbol{x}_{\text{null}} = (c,c,c)$.

11 Why can't a 1 by 3 system have $\boldsymbol{v}_p = (2,4,0)$ and $\boldsymbol{v}_n =$ any multiple of $(1,1,1)$?

A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{v} = \boldsymbol{b}$ has two solutions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, find two solutions to $A\boldsymbol{v} = \mathbf{0}$.

(b) Then find another solution to $A\boldsymbol{v} = \boldsymbol{b}$.

(a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 Explain why these are all false:

(a) The complete solution is any linear combination of $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$.

(b) A system $A\boldsymbol{v} = \boldsymbol{b}$ has at most one particular solution.

(c) The solution $\boldsymbol{v}_p$ with all free variables zero is the shortest solution (minimum length $\|\boldsymbol{v}\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution $\boldsymbol{v}_n$ in the nullspace.

(a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$

(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)

(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 Suppose column 5 has no pivot. Then $\boldsymbol{v}_5$ is a _____ variable. The zero vector (is) (is not) the only solution to $A\boldsymbol{v} = \mathbf{0}$. If $A\boldsymbol{v} = \boldsymbol{b}$ has a solution, then it has _____ solutions.

If column 5 has no pivot, v_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 Suppose row 3 has no pivot. Then that row is _____. The equation $R\boldsymbol{v} = \boldsymbol{d}$ is only solvable provided _____. The equation $A\boldsymbol{v} = \boldsymbol{b}$ (*is*) (*is not*) (*might not be*) solvable.

If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $\boldsymbol{c}_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

Questions 16–21 are about matrices of "full rank" $r = m$ or $r = n$.

16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ (*always exists*) (*is unique*). The column space of A is _____. An example is $A =$ _____.

The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

17 The largest possible rank of a 6 by 4 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ (*always exists*) (*is unique*). The nullspace of A is _____. An example is $A =$ _____.

The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The solution is *unique*. The nullspace contains only the zero *vector*. An example is $A = R = [\,I \quad F\,]$ for any 4 by 2 matrix F.

18 Find by elimination the rank of A and also the rank of A^{T}:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \text{ (rank depends on } q\text{)}.$$

Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 Find the rank of A and also of $A^{\mathrm{T}}A$ and also of AA^{T}:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Both matrices A have rank 2. Always $A^{\mathrm{T}}A$ and AA^{T} have **the same rank** as A.

20 Reduce A to its echelon form U. Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

$$A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}; A = LU \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathbf{2} & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}.$$

21 Find the complete solution in the form $\boldsymbol{v}_p + \boldsymbol{v}_n$ to these full rank systems:

(a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution.

22 If $A\boldsymbol{v} = \boldsymbol{b}$ has infinitely many solutions, why is it impossible for $A\boldsymbol{v} = \boldsymbol{B}$ (new right side) to have only one solution ? Could $A\boldsymbol{v} = \boldsymbol{B}$ have no solution ?

If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3 :

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

24 Give examples of matrices A for which the number of solutions to $A\boldsymbol{v} = \boldsymbol{b}$ is

(a) 0 or 1, depending on $\boldsymbol{b}$

(b) ∞, regardless of $\boldsymbol{b}$

(c) 0 or ∞, depending on $\boldsymbol{b}$

(d) 1, regardless of $\boldsymbol{b}$.

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\, x \,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b \,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

25 Write down all known relations between r and m and n if $A\boldsymbol{v} = \boldsymbol{b}$ has

(a) no solution for some $\boldsymbol{b}$

(b) infinitely many solutions for every $\boldsymbol{b}$

(c) exactly one solution for some $\boldsymbol{b}$, no solution for other $\boldsymbol{b}$

(d) exactly one solution for every $\boldsymbol{b}$.

(a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix *R*.

26 Continue elimination from U to R. Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R :

$$U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R = I.$$

27 Suppose U is square with n pivots (an invertible matrix). *Explain why* $R = I$.

If U has n pivots, then R has n pivots **equal to 1**. Zeros above and below those pivots make $R = I$.

28 Apply Gauss-Jordan elimination to $U\boldsymbol{v} = \mathbf{0}$ and $U\boldsymbol{v} = \boldsymbol{c}$. Reach $R\boldsymbol{v} = \mathbf{0}$ and $R\boldsymbol{v} = \boldsymbol{d}$:

$$\begin{bmatrix} U & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $R\boldsymbol{v} = \mathbf{0}$ to find $\boldsymbol{v}_n$ (its free variable is $\boldsymbol{v}_2 = 1$). Solve $R\boldsymbol{v} = \boldsymbol{d}$ to find $\boldsymbol{v}_p$ (its free variable is $\boldsymbol{v}_2 = 0$).

$$\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix};\ \boldsymbol{v}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix};\ \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}.$$

Free $v_2 = 0$ gives $\boldsymbol{v}_p = (-1, 0, 2)$ because the pivot columns contain I.

29 Apply Gauss-Jordan elimination to reduce to $R\boldsymbol{v} = \mathbf{0}$ and $R\boldsymbol{v} = \boldsymbol{d}$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $U\boldsymbol{v} = \mathbf{0}$ or $R\boldsymbol{v} = \mathbf{0}$ to find $\boldsymbol{v}_n$ (free variable $= 1$). What are the solutions to $R\boldsymbol{v} = \boldsymbol{d}$?

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \text{ leads to } \boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}:$$

no solution because of the 3rd equation

30 Reduce to $U\boldsymbol{v} = \boldsymbol{c}$ (Gaussian elimination) and then $R\boldsymbol{v} = \boldsymbol{d}$ (Gauss-Jordan):

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \boldsymbol{b}.$$

Find a particular solution $\boldsymbol{v}_p$ and all homogeneous (null) solutions $\boldsymbol{v}_n$.

$$\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}; \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}; \boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

31 Find matrices A and B with the given property or explain why you can't:

(a) The only solution of $A\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$.

(b) The only solution of $B\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$.

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 Reduce $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ to $\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$ and find the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \boldsymbol{0}$ and $U\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since $\boldsymbol{b} = (1, 3, 6, 5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0, 0, 1)$ and the complete solution is $\boldsymbol{x} = (0, 0, 1) + c\boldsymbol{s}$. (Or use the particular solution $\boldsymbol{x}_p = (7, -2, 0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1, 0, 0, 0)$ elimination leads to $U\boldsymbol{x} = (1, -1, 0, 1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

33 The complete solution to $A\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + c \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$. Find A.

If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

Challenge Problems

34 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as the only special solution to $A\boldsymbol{v} = \boldsymbol{0}$.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{v} = \boldsymbol{0}$?

(b) What is the exact row reduced echelon form R of A ? Good question.

(c) How do you know that $A\boldsymbol{v} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$?

(a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \boldsymbol{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (line of solution!). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solve for all $\boldsymbol{b}$, because A and R have *full row rank* $r = 3$.

35 If you have this information about the solutions to $A\boldsymbol{v} = \boldsymbol{b}$ for a specific $\boldsymbol{b}$, what does that tell you about the *shape* of A (m and n) ? And possibly about $\boldsymbol{r}$ and $\boldsymbol{b}$.

1. There is exactly one solution.

2. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \left[\begin{smallmatrix}\mathbf{2}\\\mathbf{1}\end{smallmatrix}\right] + c\left[\begin{smallmatrix}\mathbf{1}\\\mathbf{1}\end{smallmatrix}\right]$.

3. There are no solutions.

4. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \left[\begin{smallmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{0}\end{smallmatrix}\right] + c\left[\begin{smallmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{1}\end{smallmatrix}\right]$

5. There are infinitely many solutions.

1. $r = n$ (no special solutions) and $\boldsymbol{b}$ is in the column space

2. $n - r = 1$ (one special solution)

3. $\boldsymbol{b}$ is not in the column space (so $r < m$)

4. Same conclusion as part **2**

5. $r < n$ (there are special solutions) and $\boldsymbol{b}$ is in the column space

36 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $C\boldsymbol{v} = \boldsymbol{b}$ have the same (complete) solutions for every $\boldsymbol{b}$. Is it true that $A = C$?

If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b}$ = column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 5.4, page 295

Questions 1–10 are about linear independence and linear dependence.

1 Show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are independent but $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are dependent:

$$\boldsymbol{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \boldsymbol{u}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \boldsymbol{u}_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}.$$

Solve $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 + c_4\boldsymbol{u}_4 = \mathbf{0}$ or $A\boldsymbol{c} = \mathbf{0}$. The $\boldsymbol{u}$'s go in the columns of A. $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix}1&1&1&2\\0&1&1&3\\0&0&1&4\end{bmatrix}\begin{bmatrix}\boldsymbol{c}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{u}_1 + \boldsymbol{u}_2 - 4\boldsymbol{u}_3 + \boldsymbol{u}_4 = \mathbf{0}$ (dependent).

2 (Recommended) Find the largest possible number of independent vectors among

$$\boldsymbol{u}_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \boldsymbol{u}_2 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \boldsymbol{u}_3 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \boldsymbol{u}_4 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \boldsymbol{u}_5 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \boldsymbol{u}_6 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

$\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{u} = 0$ so no four of these six vectors can be independent.

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $U\boldsymbol{v} = \mathbf{0}$ is $\boldsymbol{v} = \mathbf{0}$. Then the upper triangular U has independent columns.

$U\boldsymbol{v} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$. A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, columns add to $\mathbf{0}$.

6 Choose three independent columns of U and A. Then make two other choices.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

7 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero. Which singular matrix gives $[\, \boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3 \,] = [\, \boldsymbol{w}_1 \;\; \boldsymbol{w}_2 \;\; \boldsymbol{w}_3 \,]\, A$?

The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the difference are *dependent* and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

8 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the sums $\boldsymbol{v}_1 = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 + \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ are *independent*. (Write $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$ in terms of the $\boldsymbol{w}$'s. Find and solve equations for the c's, to show they are zero.)

If $c_1(\boldsymbol{w}_2+\boldsymbol{w}_3)+c_2(\boldsymbol{w}_1+\boldsymbol{w}_3)+c_3(\boldsymbol{w}_1+\boldsymbol{w}_2) = \mathbf{0}$ then $(c_2+c_3)\boldsymbol{w}_1+(c_1+c_3)\boldsymbol{w}_2+(c_1+c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2+c_3 = c_1+c_3 = c_1+c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

9 Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because _____ .
(b) The two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ will be dependent if _____ .
(c) The vectors $\boldsymbol{u}_1$ and $(0, 0, 0)$ are dependent because _____ .

(a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\, \boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* "$\boldsymbol{u}_2$ is a multiple of $\boldsymbol{u}_1$" —since $\boldsymbol{u}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{u}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{u}_1 + 3(0, 0, 0) = \mathbf{0}$.

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

The plane is the nullspace of $A = [1 \;\; 2 - 3 - 1]$. Three free variables give three solutions $(x, y, z, t) = (2, -1 - 0 - 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

Questions 11–14 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors, to find the space they span.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$
(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$
(c) all vectors in $\mathbf{R}^3$ with whole number components
(d) all vectors with positive components.

(a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when _____ has a solution. The vector $\boldsymbol{c}$ is in the row space of A when _____ has a solution.

True or false : If the zero vector is in the row space, the rows are dependent.

$\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False*. The zero vector is always in the row space.

13 Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A (b) column space of U (c) row space of A (d) row space of U :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The column space and row space of A and U all have the same dimension = 2. *The row spaces of A and U are the same*, because the rows of U are combinations of the rows of A (and vice versa!).

14 $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ are combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Write $\boldsymbol{v}$ and $\boldsymbol{w}$ as combinations of $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$. The two pairs of vectors _____ the same space. When are they a basis for the same space?

$\boldsymbol{v}=\frac{1}{2}(\boldsymbol{v}+\boldsymbol{w})+\frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$ and $\boldsymbol{w}=\frac{1}{2}(\boldsymbol{v}+\boldsymbol{w})-\frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$. The two pairs *span* the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent*.

Questions 15–25 are about the requirements for a basis.

15 If $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are linearly independent, the space they span has dimension _____. These vectors are a _____ for that space. If the vectors are the columns of an m by n matrix, then m is _____ than n. If $m=n$, that matrix is _____.

The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$.

16 Suppose $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do) (do not) (might not) span $\mathbf{R}^4$.

(b) Those vectors (are) (are not) (might be) linearly independent.

(c) Any four of those vectors (are) (are not) (might be) a basis for $\mathbf{R}^4$.

(a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent
(c) Any four *might be* a basis.

17 Find three different bases for the column space of $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$. Then find two different bases for the row space of U.

The column space of $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1+$ row 2) and (row 1 and $-$ row 2) are bases for the row spaces of U.

18 Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space and the nullspace of I $(4 \text{ by } 4)$.

These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\boldsymbol{N}(I)=\{\text{zero vector}\}$.

19 The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then? *Looking ahead*: The rank r counts the number of _____ columns.

n-independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ *rank* $=m=n$. The rank counts the number of *independent* columns.

20 Find a basis for the plane $x-2y+3z=0$ in $\mathbf{R}^3$. Find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $A\boldsymbol{v}=\mathbf{0}$ has only the solution $\boldsymbol{v}=\mathbf{0}$ because _____.

(b) If $\boldsymbol{b}$ is in $\mathbf{R}^5$ then $A\boldsymbol{v}=\boldsymbol{b}$ is solvable because the basis vectors _____ $\mathbf{R}^5$.

Conclusion : A is invertible. Its rank is 5. Its rows are also a basis for $\mathbf{R}^5$.

(a) The only solution to $A\boldsymbol{v}=\mathbf{0}$ is $\boldsymbol{v}=\mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{v}=\boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 Suppose **S** is a 5-dimensional subspace of $\mathbf{R}^6$. True or false (example if false) :

(a) Every basis for **S** can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for **S** by removing one vector.

(a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in **S**.

23 U comes from A by subtracting row 1 from row 3 :

$$A=\begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces.

24 True or false (give a good reason) :

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

(a) *False* $A=[\,1\ \ 1\,]$ has dependent columns, independent row (b) *False* column space $\neq$ row space for $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ (c) *True*: Both dimensions $=2$ if A is invertible, dimensions $=0$ if $A=0$, otherwise dimensions $=1$ (d) *False*, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

25 For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

Questions 26–28 are about spaces where the "vectors" are matrices.

26 Find a basis (and the dimension) for these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$

These are simple bases (among many others) for (a) diagonal matrices (b) skew-symmetric matrices. The dimensions are $3, 6, 3$.

27 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$. What space of 3 by 3 matrices do they span?

$I, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; echelon matrices do *not* form a subspace; they ***span*** the upper triangular matrices (not every U is echelon).

The echelon matrices span all upper traingular matrices. (How could you produce the matrix with $a_{22} = 1$ as its only nanzero entry?)

28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Questions 29–32 are about spaces where the "vectors" are functions.

29 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

(a) $y(x) =$ constant C (b) $y(x) = 3x$ this is one basis for the 2 by 3 matrices with $(2, 1, 1)$ in their nullspace (4-dim subspace). (c) $y(x) = 3x + C = y_p + y_n$ solves $dy/dx = 3$.

30 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace $\boldsymbol{S}$ with $y(0) = 0$. What is the dimension of $\boldsymbol{S}$?

$y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

31 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$ (b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

(a) $y(x) = e^{2x}$ is a basis for, all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

32 Suppose y_1, y_2, y_3 are three different functions of x. The space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

$y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

33 Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a + c + d = 0$ and also for the space $\mathbf{T}$ with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

Basis for $\mathbf{S}$: $(1, 0, -1, 0), (0, 1, 0, 0), (1, 0, 0, -1)$; basis for $\mathbf{T}$: $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3, -3, 2, 1) =$ nullspace for 3 equation in $\mathbf{R}^4$ has dimension 1.

34 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$

(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$

(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$

(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$

(a) No, 2 vectors don't span $\mathbf{R}^3$ (b) No, 4 vectors in $\mathbf{R}^3$ are dependent (c) Yes, a basis (d) No, these three vectors are dependent

35 Suppose A is 5 by 4 with rank 4. Show that $A\boldsymbol{v} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\, A \;\; \boldsymbol{b} \,]$ is invertible. Show that $A\boldsymbol{v} = \boldsymbol{b}$ is solvable when $[\, A \;\; \boldsymbol{b} \,]$ is singular.

If the 5 by 5 matrix $[\, A \;\; \boldsymbol{b} \,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\, A \;\; \boldsymbol{b} \,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{v} = \boldsymbol{b}$ has a solution.

36 (a) Find a basis for all solutions to $d^4y/dx^4 = y(x)$.

(b) Find a particular solution to $d^4y/dx^4 = y(x) + 1$. Find the complete solution.

(a) The functions $y = \sin x$, $y = \cos x$, $y = e^x$, $y = e^{-x}$ are a basis for solutions to $d^4y/dx^4 = y(x)$.

(b) A particular solution to $d^4y/dx^4 = y(x)+1$ is $y(x) = -\mathbf{1}$. The complete solution is $y(x) = -1 + c, \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$ (or use another basis for the nullspace of the 4th derivative).

Challenge Problems

37 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices ! Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 =$ zero matrix, and prove that each $c_i = 0$.)

$$\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}. \quad \begin{matrix}\text{The six } P\text{'s} \\ \text{are dependent}\end{matrix}.$$

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

38 Intersections and sums have $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$. Start with a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ for the intersection $\mathbf{V} \cap \mathbf{W}$. Extend with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ to a basis for $\mathbf{V}$, and separately with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$ to a basis for $\mathbf{W}$. Prove that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s and $\boldsymbol{w}$'s together are ***independent***. The dimensions have $(r+s)+(r+t) = (r)+(r+s+t)$ as desired.

The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients = zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

39 Inside $\mathbf{R}^n$, suppose dimension $(\mathbf{V})$ + dimension $(\mathbf{W}) > n$. Why is some nonzero vector in both $\mathbf{V}$ and $\mathbf{W}$? Start with bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_q$, $p + q > n$.

If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

40 Suppose A is 10 by 10 and $A^2 = 0$ (zero matrix): A times each column of A is $\mathbf{0}$. This means that the column space of A is contained in the _____. If A has rank r, those subspaces have dimension $r \le 10 - r$. So the rank of A is $r \le 5$, if $A^2 = 0$.

If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$, and we must have $r \le 10 - r$ and $r \le 5$.

Problem Set 5.5, page 308

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}}))$ $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space basis = row 1 = $(1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space basis = column1 = $(1, 2)$; left nullspace $(-2, 1)$. B: Row space basis = both rows = $(1, 2, 4)$ and $(2, 5, 8)$; column space basis = two columns = $(1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$.

3 Row space basis = rows of $U = (0,1,2,3,4)$ and $(0,0,0,1,2)$; column space basis = pivot columns (of A not U) $=$ $(1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1)$ = last row of E^{-1}!

4 (a) $\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $[1 \quad 1]$ (d) $\begin{bmatrix}-9&-3\\3&1\end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A = \begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}$ has those rows spanning its row space $B = [1 \quad -2 \quad 1]$ has the same rows spanning its nullspace and $BA^{\mathrm{T}} = 0$.

6 A: $\dim$ $\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: $\dim$ $\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = [A \quad A]$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $[I \quad 0]$ and $[I \quad I; \quad 0 \quad 0]$ and $[0] = 3$ by 2 have *row space dimensions* $= 3,3,0 =$ column space dimensions; *nullspace dimensions* $2,3,2$; left *nullspace dimensions* $0,2,3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank= 3, nullspace and left nullspace contain only $(0,0,0)$. For **rand** $(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here. (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix}1&1\\0&2\\1&0\end{bmatrix}\begin{bmatrix}1&0&1\\1&2&0\end{bmatrix} = \begin{bmatrix}2&2&1\\2&4&0\\1&0&1\end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space (same dimension!) (b) *True*: A and $-A$ have the same four subspaces (c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

17 Row space = yz plane; column space = xy plane; nullspace = x axis; left nullspace = z axis. For $I + A$: Row space = column space = $\mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row $3-2$ row $2+$ row $1 =$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x} = \mathbf{0}$ leads to $0 = b_3 - b_2 - b_1$ so $(-1,-1,1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3 - 2b_1 = 0$ and $b_4 + b_2 - 4b_1 = 0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace. *Why*? Those vectors multiply the matrix to give *zero rows*. Section 4.1 will show another approach: $A\boldsymbol{x} = \boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{w}\end{bmatrix}\begin{bmatrix}\boldsymbol{v}^{\mathrm{T}} & \boldsymbol{z}^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 2\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}3 & 2\\ 4 & 2\\ 5 & 1\end{bmatrix}$ has column space spanned by $\boldsymbol{u}$ and $\boldsymbol{w}$, row space spanned by $\boldsymbol{v}$ and $\boldsymbol{z}$.

23 As in Problem 22: Row space basis $(3,0,3),(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A = \begin{bmatrix}1 & 0\end{bmatrix}$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A) = \boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}} = A$ or $A^{\mathrm{T}} = -A$ they are also the same for A^{T})

26 The rows of $C = AB$ are combinations of the rows of B. So rank $C \leq$ rank B. Also rank $C \leq$ rank A, because the columns of C are combinations of the columns of A.

27 Choose $d = bc/a$ to make $\left[\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right]$ a rank-1 matrix. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$. Those two vectors are perpendicular !

28 B and C (checkers and chess) both have rank 2 if $p \neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero; $N(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $N(C)$ is a challenge.

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1.$

30 There are vectors along the floor and along a wall that are *not perpendicular*. In fact the vectors where the wall meets the floor are in both subspaces (and not perpendicular to themselves).

31 Every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ has $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Each row of A^{T} (= each column of A) has a *zero* dot product with $\boldsymbol{y}$—those dot products are the zeros on the right hand side of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

32 The plane $\boldsymbol{P}$ is exactly the nullspace of the matrix $A = [1\ 1\ 1\ 1]$. Then $P^{\perp}$ is the row space of A, and the vector $v = (1,1,1,1)$ is a basic for P^{+}.

33 The vector $(1,4,5)$ in the row space of A would have to be orthogonal to $(4,5,1)$ in the nullspace—and it's not. So no matrix A.

34 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

35 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

36 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

37 If a vector $\boldsymbol{v}$ is in the subspace S, then $\boldsymbol{v}$ is perpendicular to every vector in $S^\perp$. Therefore $\boldsymbol{v}$ belongs to $(S^\perp)^\perp$. Those lines show that S is **contained in** $(S^\perp)^\perp$. But if S has dimension d, $S^\perp$ will have dimension $n - d$ and $(S^\perp)^\perp$ will have dimension $n - (n - d) = d$.

If the d-dimensional space S is contained in the d-dimensional space $(S^\perp)^\perp$, the two spaces must be the same ! (Why is that true ?)

38 This problem shows that A and $A^{\mathrm{T}}A$ have the same nullspace (a very important fact, proved again on page 391). The proof here starts from $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{0}$, which puts $A\boldsymbol{v}$ in the nullspace of A^{T}. But $A\boldsymbol{v}$ is also in the column space of A ($A\boldsymbol{v}$ is always a combination of the columns, by matrix multiplication). So $A\boldsymbol{v}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A)$, perpendicular to itself and therefore $A\boldsymbol{v} = 0$.

Conclusion : $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{0}$ leads to $A\boldsymbol{v} = \boldsymbol{0}$. And certainly $A\boldsymbol{v} = \boldsymbol{0}$ leads to $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{0}$ (just multiply by A). So $\boldsymbol{N}(A^{\mathrm{T}}A) = \boldsymbol{N}(A)$.

Problem Set 5.6, page 319

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ for $\boldsymbol{y} = (1, -1, 1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination leads to

$$\begin{aligned} -v_1 + v_2 &= b_1 \\ -v_2 + v_3 &= b_2 - b_1 \\ -v_2 + v_3 &= b_3 \end{aligned} \quad \text{and then} \quad \begin{aligned} -v_1 + v_2 &= b_1 \\ -v_2 + v_3 &= b_2 - b_1 \\ \boldsymbol{0} &= \boldsymbol{b_3 - b_2 + b_1} \end{aligned}$$

The two nonzero rows of R are $1\ {-1}\ 0$ and $0\ 1\ {-1}$ (signs were reversed to make the pivot $= +1$). Row 3 of R is zero. The tree has edges from node 1 to 2 and node 2 to 3.

4 The equations in 5.6.3 can be solved when $b_3 - b_2 + b_1 = 0$ (this is actually Kirchhoff's Voltage Law). These are exactly all the vectors $\boldsymbol{b}$ that are orthogonal to $\boldsymbol{y} = (1, -1, 1)$. (If $\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{b} \neq 0$, then KVL fails and $A\boldsymbol{v} = \boldsymbol{b}$ has no solution.)

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1, -1, 0)$ and not solvable for $\boldsymbol{f} = (1, 0, 0)$; $\boldsymbol{f}$ must be orthogonal to $(1, 1, 1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{v} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{v} = 1, -1, 0$ and currents $-A\boldsymbol{v} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 The triangle graph has $A^{\mathrm{T}}A$ = graph Laplacian :

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

All vectors (c, c, c) are in nullspace of A = nullspace of $A^{\mathrm{T}}A$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

9 Elimination on $A\boldsymbol{v} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

10 The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The nonzero rows of U keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 (a) The diagonal $2, 3, 3, 2$ counts edges that go in or out of nodes $1, 2, 3, 4$ on the graph. When A^{T} multiplies A, those diagonal entries are dot products (row i of A^{T}) $\cdot$ (column i of A) $= ||$column $i||^2$ = number of -1's or 1's in column i = degree of node i.

(b) Column i (from node i) overlays column j (from node j) only when an edge connects nodes i and j. Then the row of A for that edge has -1 and 1 in those columns—those numbers multiply to give -1.

12 The nullspace of $A^{\mathrm{T}}A$ contains $(1, 1, 1, 1)$ just like $\boldsymbol{N}(A)$. The rank is $4 - 1 = 3$. A vector $\boldsymbol{f}$ is in the column space of $A^{\mathrm{T}}A$ (= row space by symmetry) exactly when $\boldsymbol{f}$ is orthogonal to the nullspace—which means that $f_1 + f_2 + f_3 + f_4 = 0$. If you add up the 4 equations $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{f}$, you see this again.

13 The n by n *adjacency matrix* for the 4 node graph is

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad W^2 = \begin{bmatrix} 2 & \mathbf{1} & 1 & 2 \\ \mathbf{1} & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

You can check that the i, j entry of W^2 is the number of 2-*step paths* from i to j. When $i = j$ those paths go out and back. Only one 2-step path connects nodes 1 and 2, going through node 3.

14 The number of loops in this connected graph is $n - m + 1 = \mathbf{7} - \mathbf{7} + \mathbf{1} = \mathbf{1}$. What answer if the graph has two separate components (no edges between)?

15 Start from (4 nodes) $-$ (6 edges) $+$ (3 loops) $= 1$. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.

16 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

17 A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \cdots + (n-1) = n(n-1)/2$. Tree has 5 edges.

18 $\boldsymbol{N}(A)$ contains all multiples of $(1, 1, \ldots, 1)$ and *no other vectors*. The equations $A\boldsymbol{v} = \mathbf{0}$ tell you that $v_i = v_j$ when nodes i and j are connected by an edge. Then *every* $v_i = v_j$ whenever the graph is connected—just go from node i to node j using edges in the graph.

19 (a) With n nodes and all edges, $A^{\mathrm{T}}A$ will have $n-1$ along its diagonal (the degree of every edge). It will gave -1 in every off-diagonal entry (a complete graph has an edge between every pair of nodes i and j).

(b) If the edge connecting nodes 1 and 3 is removed, this reduces by 1 the degrees $(A^{\mathrm{T}}A)_{11}$ and $(A^{\mathrm{T}}A)_{33}$ on the diagonal: those degrees are now $n-2$. And $(A^{\mathrm{T}}A)_{13} = (A^{\mathrm{T}}A)_{31} = 0$ because that edge is gone.

20 With batteries b_1 to b_5 in the 5 edges of the square graph, the equation $A^{\mathrm{T}}(A\boldsymbol{v} - \boldsymbol{b}) = \mathbf{0}$ gives the voltages v_1, v_2, v_3, v_4 at the 4 nodes. Here $b = (1, 1, 1, 1, 1)$.

$$A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b} \text{ is } \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Notice that adding the 4 equations gives $0 = 0$: good. The solution $\boldsymbol{v}$ gives voltages

$$\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n = \begin{bmatrix} -2 \\ -5/4 \\ -3/4 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where the particular solution was chosen to have $v_4 = 0$.

Chapter 5 Notes, page 321

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$.
The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

6 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1+c_2)\boldsymbol{f} = \boldsymbol{f}((c_1+c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.

Problem Set 6.1, page 333

1 A has eigenvalues 1 and $\frac{1}{2}$, A^2 has eigenvalues 1 and $(\frac{1}{2})^2 = \frac{1}{4}$, A^∞ has eigenvalues 1 and 0 (notice $(\frac{1}{2})^\infty = 0$).

(a) Exchange the rows of A to get B:

$$B = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix} \text{ has eigenvalues 1 and } -\tfrac{1}{2}.$$

B is still a Markov matrix, so $\lambda = 1$ is still an eigenvalue. The sum down the main diagonal (the "trace") is now $.5$ so the second eigenvalue must be $-.5$. Then trace $= .2 + .3 = 1 - .5$.

Zero eigenvalues remain zero after elimination because the matrix remains singular and its determinant remains zero.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to $\mathbf{0}$ and $\mathbf{6}$. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $x_1 = (3, -2)$ and $x_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3. $A + B$ has $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

7 U is triangular so its eigenvalues are the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$. (This is because det $(U - \lambda I)$ will be just the product $(u_{11} - \lambda)(u_{22} - \lambda) \ldots (u_{nn} - \lambda)$ from the main diagonal.)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ with } \lambda = 2 \text{ and } 0 \qquad U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 1 \text{ and } 0.$$

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$ (b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1, 2)$ and $\boldsymbol{x}_2 = (1, -1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 With $\lambda = 0, 1, 2$ the rank is **2**. The eigenvalues of B^2 are $0, 1, 4$. The eigenvalues of $(B^2 + I)^{-1}$ are $(0 + 1)^{-1} = 1, (1 + 1)^{-1} = \frac{1}{2}, (4 + 1)^{-1} = \frac{1}{5}$.

12 The projection matrix P has $\lambda = 1, 0, 1$ with eigenvectors $(1,2,0), (2,-1,0), (0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda = 1$. *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$ (c) $\boldsymbol{x}_1 = (-1,1,0,0)$, $\boldsymbol{x}_2 = (-3,0,1,0)$, $\boldsymbol{x}_3 = (-5,0,0,1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$.

14 Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$ (more generally $c\boldsymbol{x}_1$, and $d\boldsymbol{x}_2$ with $cd \neq 0$).

15 These matrices all have $\lambda_1 = 0$ and $\lambda_2 = 0$ (which we can see from trace $= 0$ and determinant $= 0$):

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } A^2 = 0 \qquad A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \text{ has } A^2 = 0.$$

16 $\lambda = 0, 0, 6$ (*notice rank* 1 *and trace* 6) with $\boldsymbol{x}_1 = (0,-2,1)$, $\boldsymbol{x}_2 = (1,-2,0)$, $\boldsymbol{x}_3 = (1,2,1)$.

17 $\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ so $\lambda_1 = 6$. Then $\lambda_2 = 1$ to make trace $= 5 + 2 = 6 + 1$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

The other eigenvalue is $d - b$ to make trace $= a + d = (a+b) + (d-b)$.

18 These 3 matrices have $\lambda = 4$ and 5, trace 9, det 20: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

19 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

20 (a) $A = \begin{bmatrix} 0 & -1 \\ \mathbf{-28} & \mathbf{11} \end{bmatrix}$ has trace 11 and determinant 28, so $\lambda = 4$ and 7.

(b) $A = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$ has trace $\lambda_1 + \lambda_2$ and determinant $\lambda_1\lambda_2$ so its eigenvalues must be λ_1 and λ_2. This is a typical **companion matrix**.

21 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have *different eigenvectors*.

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 If you know n independent eigenvectors and their eigenvalues, you know the matrix A. In Section 6.2, the $\boldsymbol{x}$'s and $\boldsymbol{\lambda}$'s go into V and Λ, and the matrix must be $A = V\Lambda V^{-1}$. In this section, Problem 23 suggests that $A\boldsymbol{v} = B\boldsymbol{v}$ for every vector $\boldsymbol{v}$ (which proves $A = B$) because

$$\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n \qquad A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n = B\boldsymbol{v}.$$

24 The block matrix has $\lambda = 1, 2$ from B and 5, 7 from D. All entries of C are multiplied by zeros in $\det(A - \lambda I)$, so C has no effect on the eigenvalues.

25 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

26 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

27 Triangular matrix: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; Rank-1 matrix: $\lambda(C) = 0, 0, 6$.

28 $\det \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix} = -\lambda^3 + 1 = 0$ for $\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3}$.

Those complex eigenvalues λ_2, λ_3 are $\cos 120° \pm i \sin 120° = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

The trace of P is $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

$\det \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{bmatrix} = -\lambda^3 + \lambda^2 + \lambda - 1 = 0$ for $\lambda = 1, 1, -1$. The trace is $1 + 1 - 1 = \mathbf{1}$. Three eigenvectors are $(1, 1, 1)$ and $(1, 0, 1)$ and $(1, 0, -1)$. Since P is symmetric we could have chosen orthogonal eigenvectors—change the first to $(0, 1, 0)$.

29 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2) \cdots (\lambda_n)$.

30 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{\quad})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

Problem Set 6.2, page 345

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and V.

1 (a) Factor these two matrices into $A = V\Lambda V^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = V\Lambda V^{-1}$ then $A^3 = (V)(\Lambda^3)(V^{-1})$ and $A^{-1} = (V)(\Lambda^{-1})(V^{-1})$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, use $V\Lambda V^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

Put the eigenvectors in V and eigenvalues in Λ. $A = V\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 Suppose $A = V\Lambda V^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (V)(\Lambda + 2I)(V)^{-1}$.

If $A = V\Lambda V^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still V. $\quad V(\Lambda + 2I)V^{-1} = V\Lambda V^{-1} + V(2I)V^{-1} = A + 2I$.

4 True or false: If the columns of V (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable
(c) V is invertible (d) V is diagonalizable.

(a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of V

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix V is triangular, then V^{-1} is triangular. Prove that A is also triangular.

With $V = I, A = V\Lambda V^{-1} = \Lambda$ is a diagonal matrix. If V is triangular, then V^{-1} is triangular, so $V\Lambda V^{-1}$ is also triangular.

6 Describe all matrices V that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

The columns of V are nonzero multiples of $(2,1)$ and $(0,1)$: in either order. The same matrices V will diagonize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} /2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for any a and b.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing V^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

Do the multiplication $V\Lambda^k V^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $V\Lambda^k V^{-1} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find A and its eigenvalues and eigenvectors.
(b) Find the limit as $n \to \infty$ of the matrices $A^n = V\Lambda^n V^{-1}$.
(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

(a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1,1),\ \boldsymbol{x}_2 = (1,-2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, ...

Questions 11–14 are about diagonalizability.

11 True or false : If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

(a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 True or false : If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $V\Lambda V^{-1}$.

(a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $A\boldsymbol{x} = 5\boldsymbol{x}$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

$A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The matrix $A = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change ?

The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (A will have unequal eigenvalues, so eigenvectors are independent.)

Questions 15–19 are about powers of matrices.

15 $A^k = V\Lambda^k V^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

$A^k = V\Lambda^k V^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; $A_1^k \to A_1^\infty, A_2^k \to 0$.

16 (Recommended) Find Λ and V to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $V\Lambda^k V^{-1}$? In the columns of this limiting matrix you see the _____.

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $V\Lambda^k V^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 Find Λ and V to diagonalize A_2 in Problem 15. What is $(A_2)^{10}\boldsymbol{u}_0$ for these $\boldsymbol{u}_0$?

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

$\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $A_2^{10}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 Diagonalize A and compute $V\Lambda^k V^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{has} \qquad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Multiply those last three matrices to get $A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 Diagonalize B and compute $V\Lambda^k V^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

$B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 Suppose $A = V\Lambda V^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1\lambda_2\cdots\lambda_n$. This quick proof only works when A can be _____.

$\det A = (\det V)(\det \Lambda)(\det V^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof works when A is *diagonalizable*.

21 Show that trace VT = trace TV, by adding the diagonal entries of VT and TV:

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Choose T as ΛV^{-1}. Then $V\Lambda V^{-1}$ has the same trace as $\Lambda V^{-1}V = \Lambda$. The trace of A equals the trace of Λ, which is certainly the sum of the eigenvalues.

trace $VT = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TV. Diagonalizable trace of $V\Lambda V^{-1}$ = trace of $(\Lambda V^{-1})V$ = trace of Λ: *sum of the* λ's.

22 $AB - BA = I$ is impossible since the left side has trace $=$ _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

$AB - BA = I$ is impossible since trace AB $-$ trace BA $= zero \neq$ trace I. $AB - BA = C$ is possible when trace $(C) = 0$.
$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has $EE^{\mathrm{T}} - E^{\mathrm{T}}E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

23 If $A = V\Lambda V^{-1}$, diagonalize the block matrix $B = \left[\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{2A} \end{smallmatrix}\right]$. Find its eigenvalue and eigenvector (block) matrices.

If $A = V\Lambda V^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$. So B has the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix V. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same V). What is this subspace when $V = I$? What is its dimension?

The A's form a subspace since cA and $A_1 + A_2$ all have the same V. When $V = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So every matrix with $A^2 = A$ can be diagonalized.

If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 (Recommended) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$. If $\lambda = 0$ then $\boldsymbol{x}$ is in the nullspace. If $\lambda \neq 0$ then $\boldsymbol{x}$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = V\sqrt{\Lambda}\,V^{-1}$. Why is there no real matrix square root of B?

$R = V\sqrt{\Lambda}V^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real.
Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|\boldsymbol{A^{1024}}\| > \mathbf{10^{700}} \quad \boldsymbol{B^{1024} = I} \quad \boldsymbol{C^{1024} = -C} \quad \|\boldsymbol{D^{1024}}\| < \mathbf{10^{-78}}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

29 If A and B have the same λ's with the same full set of independent eigenvectors, their factorizations into _____ are the same. So $A = B$.

The factorizations of A and B into $V\Lambda V^{-1}$ are the same. So $A = B$. (This is the same as Problem 6.1.25, expressed in matrix form.)

30 Suppose the same V diagonalizes both A and B. They have the same eigenvectors in $A = V\Lambda_1 V^{-1}$ and $B = V\Lambda_2 V^{-1}$. Prove that $AB = BA$.

$A = V\Lambda_1 V^{-1}$ and $B = V\Lambda_2 V^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from $V\Lambda_1 V^{-1} V\Lambda_2 V^{-1} = V\boldsymbol{\Lambda_1\Lambda_2}V^{-1} = V\boldsymbol{\Lambda_2\Lambda_1}V^{-1}$. This is $V\Lambda_2 V^{-1} V\Lambda_1 V^{-1} = BA$.

31 (a) If $A = \left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{0} & \mathbf{d} \end{smallmatrix}\right]$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ *zero matrix*.

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right]$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

(a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A-aI)(A-dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

32 Substitute $A = V\Lambda V^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

When $A = V\Lambda V^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = V(\Lambda - \lambda_j I)V^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. In the product $p(A) = (A - \lambda_1 I)\cdots(A - \lambda_n I)$, each inside V^{-1} cancels V. This leaves V times (*product of diagonal matrices* $\Lambda - \lambda_j I$) times V^{-1}. That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen the following reasoning but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.3 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A-\lambda I)\ \mathrm{cof}\ (A-\lambda I)^{\mathrm{T}} = \det(A-\lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for λ. If other matrices B are substituted, does the identity remain true? If $AB \neq BA$, even the order of multiplication seems unclear . . .

Challenge Problems

33 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = V\Lambda V^{-1}$. The eigenvectors (columns of V) are $(1,i)$ and $(i,1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and $\det = 1$). Their eigenvectors are $(1,-i)$ and $(1,i)$:

$$A^n = V\Lambda^n V^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} /2i$$

$$= \begin{bmatrix} (e^{in\theta}+e^{-in\theta})/2 & \cdots \\ (e^{in\theta}-e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

34 The transpose of $A = V\Lambda V^{-1}$ is $A^{\mathrm{T}} = (V^{-1})^{\mathrm{T}}\Lambda V^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(V^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***.

How do you multiply three matrices $V\Lambda V^{-1}$ to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = V\Lambda V^{-1} = \lambda_1\boldsymbol{x}_1\boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n\boldsymbol{x}_n\boldsymbol{y}_n^{\mathrm{T}}.$$

Columns of V times rows of ΛV^{-1} will give r rank-1 matrices ($r =$ rank of A).

35 The inverse of $A = \mathbf{eye}(n) + \mathbf{ones}(n)$ is $A^{-1} = \mathbf{eye}(n) + C * \mathbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

Note that $\mathsf{ones}(n) * \mathsf{ones}(n) = n * \mathsf{ones}(n)$. This leads to $C = 1/(n+1)$.

$$\begin{aligned} AA^{-1} &= (\mathsf{eye}(n) + \mathsf{ones}(n)) * (\mathsf{eye}(n) + C * \mathsf{ones}(n)) \\ &= \mathsf{eye}(n) + (1 + C + Cn) * \mathsf{ones}(n) = \mathsf{eye}(n). \end{aligned}$$

Problem Set 6.3, page 357

1 Find all solutions $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ to $\boldsymbol{y'} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\boldsymbol{y}$. Which solution starts from $\boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = (2,2)$?

The eigenvalues come from $\det(A - \lambda I) = 0$. This is

$$\lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6) = 0 \text{ so } \lambda = \mathbf{2, 6}$$

Eigenvectors: $(A - 2I)\boldsymbol{x}_1 = \mathbf{0}$ and $(A - 6I)\boldsymbol{x}_2 = 0$ give $\boldsymbol{x}_1 = (1, -1)$ and $\boldsymbol{x}_2 = (1, 3)$

Solutions are $\boldsymbol{y}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Constants c_1, c_2 come from $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \boldsymbol{y(0)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Then $\boldsymbol{c_1 = c_2 = 1}$.

2 Find two solutions of the form $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ to $\boldsymbol{y'} = \begin{bmatrix} 3 & 10 \\ 2 & 4 \end{bmatrix}\boldsymbol{y}$.

The eigenvalues come from $\lambda^2 - 7\lambda - 8 = 0$. Factor into $(\lambda - 8)(\lambda + 1)$ to see $\lambda = \mathbf{8}$, and $\mathbf{-1}$.

$$(A - 8I)\boldsymbol{x}_1 = \begin{bmatrix} -5 & 10 \\ 2 & -1 \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0} \text{ gives } \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)\boldsymbol{x}_2 = \begin{bmatrix} 4 & 10 \\ 2 & 5 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0} \text{ gives } \boldsymbol{x}_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

The two solutions are $y(t) = e^{8t}\boldsymbol{x}_1$ and $e^{-t}\boldsymbol{x}_2$

3 If $a \neq d$, find the eigenvalues and eigenvectors and the complete solution to $\boldsymbol{y'} = A\boldsymbol{y}$. This equation is stable when a and d are ______.

$$\boldsymbol{y'} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\boldsymbol{y}.$$

The eigenvalues are $\lambda = a$ and $\lambda = d$. The eigenvectors come from

$$(A - aI)\,\boldsymbol{x}_1 = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0}. \quad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - dI)\,\boldsymbol{x}_2 = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}. \quad \boldsymbol{x}_2 = \begin{bmatrix} b \\ d-a \end{bmatrix}$$

Two solutions are $y = e^{at}\boldsymbol{x}_1$ and $y = e^{dt}\boldsymbol{x}_2$. Stability for **negative** a and d.

4 If $a \neq -b$, find the solutions $e^{\lambda_1 t}\boldsymbol{x}_1$ and $e^{\lambda_2 t}\boldsymbol{x}_2$ to $y' = A\boldsymbol{y}$:

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}. \quad \text{Why is } \boldsymbol{y'} = A\boldsymbol{y} \text{ not stable?}$$

A is singular so $\lambda_1 = 0$. Trace is $a + b$ so $\lambda_2 = a + b$. $(A - 0I)\,\boldsymbol{x}_1 = \mathbf{0}$ gives

$$\boldsymbol{x}_1 = \begin{bmatrix} b \\ -a \end{bmatrix} \qquad (A-(a+b)I)\,\boldsymbol{x}_2 = \begin{bmatrix} -b & b \\ a & -a \end{bmatrix}\boldsymbol{x}_2 = 0 \text{ gives } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The system is not stable because $\lambda = 0$ is an eigenvalue. If $\lambda_2 = a+b$ is negative, the system is "neutral" and the solution approaches a steady state (a multiple of $\boldsymbol{x}_1$).

5 Find the eigenvalues λ_1, λ_2, λ_3 and the eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ of A. Write $\boldsymbol{y}(0) = (0,1,0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = V\boldsymbol{c}$ and solve $\boldsymbol{y}' = A\boldsymbol{y}$. What is the limit of $\boldsymbol{y}(t)$ as $t \to \infty$ (the steady state) ? *Steady states come from* $\lambda = 0$.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Calculation gives $\det(A - \lambda I) = -(\lambda+1)\lambda(\lambda+3)$ and eigenvalues $\lambda = 0, -1, -3$.

$$\lambda = 0 \text{ has eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = -1 \text{ has } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \lambda = -3 \text{ has } \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Notice: Those eigenvectors are orthogonal (because A is symmetric). Then $\boldsymbol{y}(0)$ is

$$(0,1,0) = \frac{1}{3}(\boldsymbol{x}_1 - \boldsymbol{x}_3) \text{ so } \boldsymbol{y}(t) = \frac{1}{3}e^{0t}\boldsymbol{x}_1 - \frac{1}{3}e^{-3t}\boldsymbol{x}_2 \text{ approaches } y(\infty) = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

6 The simplest 2 by 2 matrix without two independent eigenvectors has $\lambda = 0, 0$:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ has a first solution } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{0t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find a second solution to these equations $y_1{}' = y_2$ and $y_2{}' = 0$. That second solution starts with t times the first solution to give $y_1 = t$. What is y_2 ?

Note A complete discussion of $\boldsymbol{y}' = A\boldsymbol{y}$ for all cases of repeated λ's would involve the *Jordan form* of A : too technical. Section 6.4 shows that a triangular form is sufficient, as Problems 6 and 8 confirm. We can solve for y_2 and then y_1.

The first solution to $y_1' = y_2$ and $y_2' = 0$ is $(y_1(t), y_2(t)) = (1,0) =$ eigenvector.

A second solution has $(y_1, y_2) = (t, 1)$. The factor t appears when there is no $\boldsymbol{x}_2$.

7 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}\boldsymbol{y}.$$

What combination $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{y}(0) = (5, -2)$?

$$\boldsymbol{y}_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{y}_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \text{ If } \boldsymbol{y}(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \text{ then } \boldsymbol{y}(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 Solve Problem 7 for $\boldsymbol{y} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \text{ Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^t. $\lambda = 4$ and 1. $z(t) = -2e^t$.

Then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 7.

9 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{y'} = A\boldsymbol{y}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$:

$$\text{Solve} \quad \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{y} \quad \text{with} \quad \boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{y}(\infty)\,?$$

(a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3, 2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1, -1)$. Then the usual 3 steps:

1. Write $\boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2$
2. Follow those eigenvectors by $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{y}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3, 2)$.

10 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

$d(v + w)/dt = (w - v) + (v - w) = 0$, so the total $v + w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

has $\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -2 \end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} & v(\infty) = 20 \\ w(1) = 20 - 10e^{-2} & w(\infty) = 20 \end{matrix}$

11 Reverse the diffusion of people in Problem 10 to $d\boldsymbol{z}/dt = -A\boldsymbol{z}$:

$$\frac{dv}{dt} = v - w \quad \text{and} \quad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

$\dfrac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has $\lambda = 0$ and $+2$: $v(t) = 20 + 10e^{2t} \to \infty$ as $t \to \infty$.

12 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \quad (a \text{ and } b \text{ are real})$$

Find the stability conditions on a and b so that all solutions of $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and $d\boldsymbol{z}/dt = B\boldsymbol{z}$ approach zero as $t \to \infty$.

$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$, and the solutions of $\boldsymbol{y}' = A\boldsymbol{y}$ approach zero. $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$, and all solutions of $\boldsymbol{z}' = B\boldsymbol{z}$ approach zero.

13 Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbf{R}^2$. Its eigenvalues are 1 and 0 with eigenvectors $(1,1)$ and $(1,-1)$. If $d\boldsymbol{y}/dt = -P\boldsymbol{y}$ (notice minus sign) can you find the limit of $\boldsymbol{y}(t)$ at $t = \infty$ starting from $\boldsymbol{y}(0) = (3,1)$?

A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1,1)$. Eigenvectors $P\boldsymbol{x} = \mathbf{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1,-1)$. For the solution to $\boldsymbol{y}' = -P\boldsymbol{y}$,

$$\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{y}(t) = e^{-t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

14 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

$\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

15 (a) Write $(4,0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ starting from $(4,0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ to find $\boldsymbol{y}(t)$.

(a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ i \end{bmatrix} + 2\begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $\boldsymbol{y}(t) = 2e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

Questions 16–19 reduce second-order equations to first-order systems for (y, y').

16 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{y} = (y, y')$:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

$\dfrac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\lambda = \frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

17 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too*. Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

$A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$.

18 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

Find $\boldsymbol{y}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{y}(0) = (3, 0)$.

(a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{y}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

19 If c is not an eigenvalue of A, substitute $\boldsymbol{y} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A?

Substituting $\boldsymbol{y} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b}$ = particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

20 A particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - \boldsymbol{b}$ is $\boldsymbol{y}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ give $\boldsymbol{y}_n$. Find the complete solution $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_n$:

(a) $\dfrac{dy}{dt} = y - 4$ (b) $\dfrac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{y} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

$\boldsymbol{y}_p = 4$ and $\boldsymbol{y}(t) = ce^t + 4$; $\quad \boldsymbol{y}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{y}(t) = c_1 e^t \begin{bmatrix} 1 \\ t \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

21 Find a matrix A to illustrate each of the unstable regions in the stability picture:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. These show the unstable cases

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$

22 Which of these matrices are stable ? Then Re $\lambda < 0$, trace < 0, and det > 0.

$$A_1 = \begin{bmatrix} -2 & -3 \\ -4 & -5 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \quad A_3 = \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix}.$$

A_1 is unstable (trace $= -7$ but determinant $= -2$; $\lambda_1 < 0$ but $\lambda_2 > 0$).

A_2 is unstable (singular so $\lambda_1 = 0$).

A_3 is stable (trace $= -7$ and determinant 12; $\lambda_1 < 0$ and $\lambda_2 < 0$).

23 For an n by n matrix with trace $(A) = T$ and det $(A) = D$, find the trace and determinant of $-A$. Why is $\boldsymbol{z}' = -A\boldsymbol{z}$ unstable whenever $\boldsymbol{y}' = A\boldsymbol{y}$ is stable ?

If trace $(A) = T$ then trace $(-A) = -T$

If determinant $(A) = D$ then determinant $(-A) = (-1)^n D$

The eigenvalues of $-A$ are $-$(eigenvalues of A).

24 (a) For a real 3 by 3 matrix with stable eigenvalues (Re $\lambda < 0$), show that trace < 0 and det < 0. Either three real negative λ or else $\lambda_2 = \overline{\lambda}_1$ and λ_3 is real.

(b) The trace and determinant of a 3 by 3 matrix do not determine all three eigenvalues ! Show that A is unstable even with trace < 0 and determinant < 0:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{bmatrix}.$$

(a) If all three real parts are negative (stability), trace $=$ sum of real parts < 0.

Also $\det = \lambda_1 \lambda_2 \lambda_3 < 0$ from 3 negative λ's or from $(a+ib)(a-ib)\lambda_3 = (a^2+b^2)\lambda_3 < 0$.

If a real matrix has a complex eigenvalue $\lambda = a + ib$, then $\overline{\lambda} = a - ib$ is also an eigenvalue. The third eigenvalue must be real to make the trace real.

(b) The triangular matrix A has $\lambda = 1, 1, -5$ even with trace $= -3$ and $\det = -5$. There must be a third test for 3 by 3 matrices and that test must fail for this matrix.

25 You might think that $\boldsymbol{y}' = -A^2\boldsymbol{y}$ would always be stable because you are squaring the eigenvalues of A. But why is that equation unstable for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

This real matrix A has $\lambda = i$ and $-i$. Then $\lambda^2 = -1$ and -1. So $\boldsymbol{y}' = -A^2\boldsymbol{y}$ has eigenvalues 1 and 1 (unstable).

26 Find the three eigenvalues of A and the three roots of $s^3 - s^2 + s - 1 = 0$ (including $s = 1$). The equation $y''' - y'' + y' - y = 0$ becomes

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \quad \text{or } \boldsymbol{z}' = A\boldsymbol{z}.$$

Each eigenvalue λ has an eigenvector $\boldsymbol{x} = (1, \lambda, \lambda^2)$.

$s^3 - s^2 + s - 1 = 0$ comes from substituting $y = e^{st}$ into $y''' - y'' + y' - y = 0$.

$\lambda^3 - \lambda^2 + \lambda - 1 = 0$ comes from computing $\det(A - \lambda I)$ for the 3 by 3 matrix.

One root is $s = 1$ (and $\lambda = 1$). The full cubic polynomial is

$s^3 - s^2 + s - 1 = (s - 1)(s^2 + 1)$ with roots $\mathbf{1}, \boldsymbol{i}, -\boldsymbol{i}$.

Eigenvectors $(1, \lambda, \lambda^2) = (1, 1, 1), (1, i, -1), (1, -i, -1)$ for this companion matrix.

27 Find the two eigenvalues of A and the double root of $s^2 + 6s + 9 = 0$:

$$y'' + 6y' + 9y = 0 \text{ becomes } \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \quad \text{or } \boldsymbol{z}' = A\boldsymbol{z}.$$

The repeated eigenvalue gives only one solution $\boldsymbol{z} = e^{\lambda t}\boldsymbol{x}$. Find a second solution $\boldsymbol{z}$ from the second solution $y = te^{\lambda t}$.

The matrix has $\det(A - \lambda I) = \lambda^2 + 6\lambda + 9$. This is $\boldsymbol{(\lambda + 3)^2}$ so eigenvalues λ = roots $s = \mathbf{-3, -3}$. The two solutions are $y = e^{-3t}$ and $y = te^{-3t}$. Those translate to $\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^{-3t} \begin{bmatrix} t \\ 1 - 3t \end{bmatrix}$

28 Explain why a 3 by 3 companion matrix has eigenvectors $\boldsymbol{x} = (\mathbf{1}, \boldsymbol{\lambda}, \boldsymbol{\lambda^2})$.

First Way: If the first component is $x_1 = 1$, the first row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the second component $x_2 =$ _____. Then the second row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the third component $x_3 = \lambda^2$.

Second Way: $\boldsymbol{y}' = A\boldsymbol{y}$ starts with $y_1' = y_2$ and $y_2' = y_3$. $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves those equations. At $t = 0$ the equations become $\lambda x_1 = x_2$ and _____.

$$A\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -D & -C & -B \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$ because rows 1 and 2 are true and row 3 is $-D - C\lambda - B\lambda^2 = \lambda^3$. That is $\lambda^3 + B\lambda^2 + C\lambda + D = 0$ corresponding to $y''' + By'' + Cy' + Dy = 0$.

29 Find A to change the scalar equation $y'' = 5y' - 4y$ into a vector equation for $\boldsymbol{z} = (y, y')$:

$$\frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

What are the eigenvalues of the companion matrix A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' - 4y$.

$$\frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 5y' - 4y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

The eigenvalues come from $\lambda^2 - 5\lambda + 4 = 0$. Then $\lambda = \mathbf{1}$ and $\mathbf{4}$. Unstable because $y'' - 5y' + 4y$ has negative damping.

30 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{z}' = A\boldsymbol{z}$:

$$\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

Find $\boldsymbol{z}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{z}(0) = (3, 0)$.

(a) $y_1 = \cos 3t$ and $y_2 = \sin 3t$ and their combinations solve $y'' = -9y$. The initial conditions $y(0) = 3, y'(0) = 0$ are satisfied by $y = \mathbf{3\cos 3t}$.

(b) The matrix A has $\det \begin{bmatrix} -\lambda & 1 \\ -9 & -\lambda \end{bmatrix} = \lambda^2 + 9 = 0$ and $\lambda = \mathbf{3i}, \mathbf{-3i}$. Eigenvectors $(\mathbf{1}, \mathbf{3i}), (\mathbf{1}, \mathbf{-3i})$.

$\boldsymbol{z}(t) = c_1 e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + c_2 e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix}$ gives $c_1 + c_2 = 3$ and $3ic_1 - 3ic_2 = 0$ at $t = 0$.

Then $c_1 = c_2 = \dfrac{3}{2}$ gives $\begin{bmatrix} y \\ y' \end{bmatrix} = \dfrac{3}{2} e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + \dfrac{3}{2} e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} \mathbf{3\cos 3t} \\ \mathbf{-9\sin 3t} \end{bmatrix}$.

31 (a) Change the third order equation $y''' - 2y'' - y' + 2y = 0$ to a first order system $\boldsymbol{z}' = A\boldsymbol{z}$ for the unknown $\boldsymbol{z} = (y, y', y'')$. The companion matrix A is 3 by 3.

(b) Substitute $y = e^{\lambda t}$ and also find $\det(A - \lambda I)$. Those lead to the same λ's.

(c) One root is $\lambda = 1$. Find the other roots and these complete solutions :

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \qquad \boldsymbol{z} = C_1 e^{\lambda_1 t} \boldsymbol{x}_1 + C_2 e^{\lambda_2 t} \boldsymbol{x}_2 + C_3 e^{\lambda_3 t} \boldsymbol{x}_3.$$

(a) $\boldsymbol{z}' = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = A\boldsymbol{z}$

(b) The characteristic equation is $\det(A - \lambda I) = -(\lambda^3 - 2\lambda^2 - \lambda + 2) = 0$.

(c) $\lambda = 1$ is a root so we can factor out $(\lambda - 1)$:

$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda^2 - \lambda - 2) = (\lambda - 1)(\lambda - 2)(\lambda + 1)$ has roots $\mathbf{1}, \mathbf{2}, \mathbf{-1}$.

The complete solution is $y = c_1 e^t + c_2 e^{2t} + c_3 e^{-t}$.

This vectorizes into $\boldsymbol{z} = C_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

32 These companion matrices have $\lambda = 2, 1$ and $\lambda = 4, 1$. Find their eigenvectors:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix} \qquad \text{Notice trace and determinant!}$$

A has $\lambda^2 - 3\lambda + 2 = 0 = (\lambda - 2)(\lambda - 1)$. $\lambda = \mathbf{2}, \mathbf{1}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

B has $\lambda^2 - 5\lambda + 4 = 0 = (\lambda - 4)(\lambda - 1)$. $\lambda = \mathbf{4}, \mathbf{1}$ with eigenvectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem Set 6.4, page 369

1 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, find an eigenvalue and an eigenvector of e^{At} and also of $-e^{-At}$.

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{e^{At}x} = \boldsymbol{e^{\lambda t}x}$ and $-e^{-At}\boldsymbol{x} = -e^{-\lambda t}\boldsymbol{x}$. Use the infinite series:

$$e^{At}\boldsymbol{x} = (I + At + \tfrac{1}{2}(At)^2 + \cdots)\boldsymbol{x}$$
$$= (I + \lambda t + \tfrac{1}{2}(\lambda t)^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}.$$

2 (a) From the infinite series $e^{At} = I + At + \cdots$ show that its derivative is Ae^{At}.

(b) The series for e^{At} ends quickly if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Find e^{At} and take its derivative (which should agree with Ae^{At}).

(a) The time derivative of the matrix e^{At} is Ae^{At}:

$\frac{d}{dt}(I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots) = A + A^2t + \frac{1}{2}A^3t^2 + \cdots) = Ae^{At}$.

(b) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^2 = 0$ and $e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

The derivative of $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which agrees with Ae^{At}.

This derivative also agrees with A itself but that is an accident.

3 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with eigenvectors in $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, compute $e^{At} = Ve^{\Lambda t}V^{-1}$.

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e^t} & \boldsymbol{e^{2t} - e^t} \\ \mathbf{0} & e^{2t} \end{bmatrix}.$$

Check $e^{At} = I$ at $t = 0$.

4 Why is $e^{(A + 3I)t}$ equal to e^{At} multiplied by e^{3t}?

If $AB = BA$ then $e^{(A+B)t} = e^{At}e^{Bt}$. (This usually fails if $AB \neq BA$.)

Here $B = 3I$ always gives $AB = BA$ so $e^{(A+3I)t} = e^{At}e^{3It} = e^{At}e^{3t}$ is **true**.

5 Why is $e^{A^{-1}}$ *not* the inverse of e^A? What is the correct inverse of e^A?

The correct inverse of e^A is e^{-A}. In general $e^{At}e^{AT} = e^{A(t+T)}$. Choose $t = 1, T = -1$.

The matrix $e^{A^{-1}}$ is a series of powers of A^{-1} and $(e^A)(e^{A^{-1}}) = e^{A+A^{-1}}$: not wanted.

6 Compute $A^n = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^n$. Add the series to find $e^{At} = \begin{bmatrix} e^t & c(e^t-1) \\ 0 & 1 \end{bmatrix}$.

Start by assuming $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & nc \\ 0 & 0 \end{bmatrix}$ (certainly true for ($n = 1$).

Then by induction $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & nc \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & (n+1)\,c \\ 0 & 0 \end{bmatrix}$.

The first equation is true for $n = 1$. Then the second equation says that every matrix multiplication adds c to the off-diagonal entry. So the first equation is true for $n = 2, 3, 4, \ldots$

Now add up the series for e^{At} :

$$I + At + \tfrac{1}{2}(At)^2 + \cdots = \begin{bmatrix} 1 + t + \frac{1}{2}t^2 + \cdots & 0 + ct + \frac{1}{2}2ct^2 + \cdots \\ 0 & 1 + 0 + 0 + \cdots \end{bmatrix} = \begin{bmatrix} e^t & c(e^t-1) \\ 0 & 1 \end{bmatrix}$$

7 Find e^A and e^B by using Problem 6 for $c = 4$ and $c = -4$. Multiply to show that the matrices $e^A e^B$ and $e^B e^A$ and e^{A+B} are all different.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

With $t = 1$ in Problem 6 , $A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ has $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ has $e^B = \begin{bmatrix} e & -4(e-1) \\ 0 & 1 \end{bmatrix}$

Then $e^A e^B = \begin{bmatrix} e^2 & (-4e+4)(e-1) \\ 0 & 1 \end{bmatrix}$ and $e^B e^A = \begin{bmatrix} e^2 & (4e-4)(e-1) \\ 0 & 1 \end{bmatrix}$ and $e^{A+B} = \begin{bmatrix} e^2 & 0 \\ 0 & 1 \end{bmatrix}$. Those three off-diagonal entries are different because AB and BA have off-diagonals -4 and 4.

8 Multiply the first terms $I + A + \frac{1}{2}A^2$ of e^A by the first terms $I + B + \frac{1}{2}B^2$ of e^B. Do you get the correct first three terms of e^{A+B} ? *Conclusion*: e^{A+B} is not always equal to $(e^A)(e^B)$. The exponent rule only applies when $AB = BA$.

$\left(I + A + \frac{1}{2}A^2\right)\left(I + B + \frac{1}{2}B^2\right) = I + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 + \cdots$

The correct three terms of e^{A+B} are $I + A + B + \frac{1}{2}A^2 + \frac{1}{2}AB + \frac{1}{2}BA + \frac{1}{2}B^2$.

Then AB agrees with $\frac{1}{2}AB + \frac{1}{2}BA$ **only if** $\boldsymbol{AB = BA}$.

9 Write $A = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$ in the form $V\Lambda V^{-1}$. Find e^{At} from $Ve^{\Lambda t}V^{-1}$.

This is Problem 6 using diagonalization $A = V\Lambda V^{-1}$ by the eigenvector matrix V :

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 4(e^t-1) \\ 0 & 1 \end{bmatrix}$$

10 Starting from $\boldsymbol{y}(0)$ the solution at time t is $e^{At}\boldsymbol{y}(0)$. Go an additional time t to reach $e^{At}\,e^{At}\boldsymbol{y}(0)$. Conclusion: e^{At} times e^{At} equals _____.

The conclusion is that e^{At} times e^{At} equals e^{2At}. No problem with $AB \neq BA$ because here B is the same as A.

11 Diagonalize A by V and confirm this formula for e^{At} by using $Ve^{\Lambda t}V^{-1}$:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{2t} & 4(e^{3t} - e^{2t}) \\ 0 & e^{3t} \end{bmatrix} \quad \text{At } t = 0 \text{ this matrix is } \_\_\_\_\_ .$$

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = V\Lambda V^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 4(e^{3t} - e^{2t}) \\ 0 & e^{3t} \end{bmatrix} = \boldsymbol{I} \textbf{ at } \boldsymbol{t = 0}.$$

12 (a) Find A^2 and A^3 and A^n for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with repeated eigenvalues $\lambda = 1, 1$.

(b) Add the infinite series to find e^{At}. (The $Ve^{\Lambda t}V^{-1}$ method won't work.)

(a) $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. (b) $e^{At} =$

$$\begin{bmatrix} 1 + t + \frac{1}{2}t^2 + \cdots & t + \frac{1}{2}2t^2 + \frac{1}{6}3t^3 + \cdots \\ 0 & 1 + t + \frac{1}{2}t^2 + \cdots \end{bmatrix} = \begin{bmatrix} e^t & t(1 + t + \frac{1}{2}t^2 + \cdots) \\ 0 & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t & \boldsymbol{te^t} \\ 0 & e^t \end{bmatrix}$$

Notice the factor t appearing as usual when there are equal roots (or equal eigenvalues).

13 (a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$ as a combination of eigenvectors of this matrix A :

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y} \quad \text{with } \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(b) Write the equations as $y_1' = y_2$ and $y_2' = y_1$. Find an equation for y_1'' with y_2 eliminated. Solve for $y_1(t)$ and compare with part (a).

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\lambda = 1$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -1$ with $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then $\boldsymbol{y}(0) = 4\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{y}(t) = 4e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) If $y_1' = y_2$ and $y_2' = y_1$ then $y_1'' = y_2' = y_1$.

The second order equation $y_1'' = y_1$ has $y_1 = c_1e^t + c_2e^{-t}$.

The initial conditions produce the solution of part (a).

14 Similar matrices A and $B = V^{-1}AV$ have the *same eigenvalues* if V is invertible.

$$\textit{Second proof} \qquad \det(V^{-1}AV - \lambda I) = (\det V^{-1})\,(\det(A - \lambda I))\,(\det V).$$

Why is this equation true? Then both sides are zero when $\det(A - \lambda I) = 0$.

We use the rule $\det ABC = (\det A)(\det B)(\det C)$.

Here $A = V^{-1}$ and $C = V$ have $(\det A)(\det C) = 1$.

This only leaves $\mathbf{det}\,\boldsymbol{B}$ which is $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I})$.

Conclusion: $\boldsymbol{V^{-1}AV}$ **has the same eigenvalues as** $\boldsymbol{A}$. Similar matrices!

15 If B is *similar* to A, the growth rates for $\boldsymbol{z}' = B\boldsymbol{z}$ are the same as for $\boldsymbol{y}' = A\boldsymbol{y}$. That equation converts to the equation for $\boldsymbol{z}$ when $B = V^{-1}AV$ and $\boldsymbol{z} =$ _____.

If $\boldsymbol{y}' = A\boldsymbol{y}$ just set $\boldsymbol{y} = V\boldsymbol{z}$ to get $V\boldsymbol{z}' = AV\boldsymbol{z}$ which is $\boldsymbol{z}' = V^{-1}AV\boldsymbol{z}$. Similar matrices come from a change of variable in the differential equation.

16 If $A\boldsymbol{x} = \lambda\boldsymbol{x} \neq \mathbf{0}$, what is an eigenvalue and eigenvector of $(e^{At} - I)A^{-1}$?

The same $\boldsymbol{x}$ is an eigenvector, with eigenvalue in

$$(e^{At} - I)A^{-1}\boldsymbol{x} = \frac{1}{\lambda}(e^{At} - I)\boldsymbol{x} = \frac{e^{\lambda t} - 1}{\lambda}\boldsymbol{x}.$$

17 The matrix $B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

$$e^{Bt} = I + Bt + 0 = \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix}. \text{ The derivative is } \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}.$$

The derivative is always Be^{Bt}; here it also equals B.

18 Starting from $\boldsymbol{y}(0) = \mathbf{0}$, solve $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}$ as a combination of the eigenvectors. Suppose the source is $\boldsymbol{q} = q_1\boldsymbol{x}_1 + \cdots + q_n\boldsymbol{x}_n$. Solve for one eigenvector at a time, using the solution $y(t) = (e^{at} - 1)q/a$ to the scalar equation $y' = ay + q$.

Then $\boldsymbol{y}(t) = (e^{At} - I)A^{-1}\boldsymbol{q}$ is a combination of eigenvectors when all $\lambda_i \neq 0$.

For each eigenvector $\boldsymbol{x}$, a solution to $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{x}$ is $\boldsymbol{y}(t) = \dfrac{e^{\lambda t} - 1}{\lambda}\boldsymbol{x}$ by Problem 16.

Then by linearity $\boldsymbol{y}(t) = \Sigma \dfrac{e^{\lambda_i t} - 1}{\lambda_i} q_i\boldsymbol{x}_i$ is the solution when $\boldsymbol{q} = q_1\boldsymbol{x}_1 + \cdots + q_n\boldsymbol{x}_n$.

This is the same as $\boldsymbol{y}_p(t) = (e^{At} - I)A^{-1}\boldsymbol{q}$.

19 Solve for $\boldsymbol{y}(t)$ as a combination of the eigenvectors $\boldsymbol{x}_1 = (1, 0)$ and $\boldsymbol{x}_2 = (1, 1)$:

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q} \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{with} \quad \begin{matrix} y_1(0) = 0 \\ y_2(0) = 0 \end{matrix}$$

Write $\boldsymbol{q} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as a combination $3\boldsymbol{x}_1 + \boldsymbol{x}_2$ of the eigenvectors of A. By Problem 18,

$$\boldsymbol{y}_p(t) = \frac{e^t - 1}{1}\,3\boldsymbol{x}_1 + \frac{e^{2t} - 1}{2}\,\boldsymbol{x}_2.$$

20 Solve $\boldsymbol{y}' = A\boldsymbol{y} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$ in three steps. First find the λ's and $\boldsymbol{x}$'s.

(**1**) Write $\boldsymbol{y}(0) = (3, 1)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$
(**2**) Multiply c_1 and c_2 by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.
(**3**) Add the solutions $c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$.

Th eigenvalues come from $\det \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$. Then $\lambda = \mathbf{4}$ and $\mathbf{-1}$.

The eigenvectors are found to be $\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (1) $\quad \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{4}{5}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (2) $\quad$ Two solutions $\frac{4}{5}e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\frac{3}{5}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (3) $\quad \boldsymbol{y}(t) = \frac{4}{5}e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{3}{5}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

21 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{y}(0)$ solves $d\boldsymbol{y}/dt = A\boldsymbol{y}$.

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \frac{1}{24}(At)^4 + \cdots$$
$$\frac{d}{dt}(e^{At} = A + A^2 t + \frac{1}{2}A^3 t^2 + \frac{1}{4}A^4 t^3 + \cdots = \boldsymbol{Ae^{At}}.$$

Problems 22-25 are about time-varying systems $\boldsymbol{y}' = \boldsymbol{A(t)y}$. Success then failure.

22 Suppose the constant matrix C has $C\boldsymbol{x} = \lambda\boldsymbol{x}$, and $p(t)$ is the integral of $a(t)$. Substitute $\boldsymbol{y} = e^{\lambda p(t)}\boldsymbol{x}$ to show that $d\boldsymbol{y}/dt = a(t)C\boldsymbol{y}$. Eigenvectors still solve this special time-varying system: constant matrix C multiplied by the scalar $a(t)$.

Here the time-varying coefficient matrix has the special form $a(t)C$, with the matrix C constant in time. Its eigenvalues and eigenvectors are $a(t)\lambda$ and $\boldsymbol{x}$ (main point: λ and $\boldsymbol{x}$ are constant). Then we can solve $\boldsymbol{y}' = a(t)C\boldsymbol{y}$ starting with an eigenvector:

$$\boldsymbol{y}(t) = e^{\int a(t)\lambda\, dt}\boldsymbol{x} \qquad \text{solves} \qquad \frac{d\boldsymbol{y}}{dt} = a(t)\lambda\boldsymbol{y} = a(t)C\boldsymbol{y}.$$

A combination of these solutions is also a solution—and can match $\boldsymbol{y}(0)$.

23 Continuing Problem 22, show from the series for $M(t) = e^{p(t)C}$ that $dM/dt = a(t)CM$. Then M is the fundamental matrix for the special system $\boldsymbol{y}' = a(t)C\boldsymbol{y}$. If $a(t) = 1$ then its integral is $p(t) = t$ and we recover $M = e^{Ct}$.

This question puts together the "fundamental matrix" $M(t)$ from Problem 22. Write $p(t) = \int a(t)\, dt$.

$$M = e^{p(t)C} = I + p(t)C + \frac{1}{2}p^2(t)C^2 + \cdots \qquad \text{and } \frac{dp}{dt} = a(t) \text{ give}$$
$$\frac{dM}{dt} = a(t)C + a(t)C^2 p(t) + \cdots = \boldsymbol{a(t)CM}.$$

24 The integral of $A = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix}$ is $P = \begin{bmatrix} t & t^2 \\ 0 & 0 \end{bmatrix}$. The exponential of P is $e^P = \begin{bmatrix} e^t & t(e^t-1) \\ 0 & 1 \end{bmatrix}$. From the chain rule we might hope that the derivative of $e^{P(t)}$ is $P' e^{P(t)} = Ae^{P(t)}$. Compute the derivative of $e^{P(t)}$ and compare with the wrong answer $Ae^{P(t)}$. (One reason this feels wrong: Writing the chain rule as $(d/dt)e^P = e^P dP/dt$ would give $e^P A$ instead of $A\,e^P$. That is wrong too.)

Now the matrix $A(t)$ does not have the special form $A = a(t)C$ of problems 22–23. The problem shows that the simple formula doesn't solve $\boldsymbol{y}' = A(t)\boldsymbol{y}$. We can't just integrate $A(t)$ and use the matrix $e^{\int A(t)dt}$.

$$P = \int A(t)\,dt = \begin{bmatrix} t & t^2 \\ 0 & 0 \end{bmatrix} \quad \text{has} \quad P^2 = \begin{bmatrix} t^2 & t^3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P^n = \begin{bmatrix} t^n & t^{n+1} \\ 0 & 0 \end{bmatrix}$$

Then $\dfrac{dP}{dt} = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix} = A$ and $e^P = I + P + \dfrac{1}{2}P^2 + \cdots = \begin{bmatrix} e^t & te^t - t \\ 0 & 1 \end{bmatrix}$.

But the derivative of e^P **is not** $e^P \frac{dP}{dt}$. This matrix $e^{P(t)}$ is not solving $\boldsymbol{y}' = A(t)\boldsymbol{y}$.

25 Find the solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$ in Problem 24 by solving for y_2 and then y_1:

$$\text{Solve} \quad \begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ starting from } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}.$$

Certainly $y_2(t)$ stays at $y_2(0)$. Find $y_1(t)$ by "undetermined coefficients" A, B, C: $y_1' = y_1 + 2ty_2(0)$ is solved by $y_1 = y_p + y_n = At + B + Ce^t$.

Choose A, B, C to satisfy the equation and match the initial condition $y_1(0)$.

The wrong answer in Problem 24 included the incorrect factor te^t in $e^{P(t)}$.

To solve $\boldsymbol{y}' = A(t)\boldsymbol{y}$ in Problem 24 we can start with its second equation:

$$\boldsymbol{y}' = A(t)\boldsymbol{y} \quad \text{is} \quad \begin{aligned} dy_1/dt &= y_1 + 2ty_2 \\ dy_2/dt &= 0 \end{aligned}$$

Then $y_2(t) = y_2(0) =$ constant and the first equation becomes $dy_1/dt = y_1 + 2ty_2(0)$. A particular solution has the form $y_1 = At + B$. Substitute this y_1 to find A and B:

$$\frac{dy_1}{dt} = y_1 + 2ty_2(0) \text{ gives } A = At + B + 2ty_2(0) \text{ and then } A = -2y_2(0) = B.$$

Now add a null solution Ce^t to start from $y_1(0)$:

$$y_1(t) = (y_1(0) + 2y_2(0))e^t - 2y_2(0)t - 2y_2(0).$$

This correct solution has no factor te^t.

Problem Set 6.5, page 379

Problems 1–14 are about eigenvalues. Then come differential equations.

1 Which of A, B, C have two real λ's ? Which have two independent eigenvectors ?

$$A = \begin{bmatrix} 7 & -11 \\ -11 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -11 \\ 11 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & -11 \\ 0 & 7 \end{bmatrix}$$

A is symmetric: Real λ's with a full set of two eigenvectors.

$B = 7I+$ antisymmetric: Complex $\lambda = 7 \pm 11i$, full set of (complex) eigenvectors.

$C = 7I - 11 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: Eigenvalues $7, 7$ but only one eigenvector.

2 Show that A has real eigenvalues if $b \geq 0$ and nonreal eigenvalues if $b < 0$:

$$A = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$ have $\lambda^2 - b = 0$. Then $\lambda = \pm\sqrt{b}$ if $b \geq 0$.

$\begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}$ has $\lambda = \mathbf{1} \pm \sqrt{\boldsymbol{b}}$.

3 Find the eigenvalues and the unit eigenvectors of the symmetric matrices

$$\text{(a)} \ S = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{(b)} \quad S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

(a) $\det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} = (2-\lambda)\lambda^2 + 4\lambda + 4\lambda = -\lambda^3 + 2\lambda^2 + 8\lambda$

$= -\lambda(\lambda - 4)(\lambda + 2)$. $\quad \lambda = \mathbf{0}, \mathbf{4}, \mathbf{-2}$.

Unit (orthonormal!) eigenvectors $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \dfrac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

(b) $\det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{bmatrix} = \lambda(1-\lambda^2) + 4(1+\lambda) - 4(1-\lambda) = 9\lambda - \lambda^3$

$= -\lambda(\lambda - 3)(\lambda + 3)$.

$\lambda = \mathbf{0}, \mathbf{3}, \mathbf{-3}$ with orthonormal eigenvectors $\dfrac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \dfrac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \dfrac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

4 Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} \mathbf{-2} & \mathbf{6} \\ \mathbf{6} & \mathbf{7} \end{bmatrix}$. What is Λ?

The eigenvalues from $\lambda^2 - 5\lambda - 50 = 0 = (\lambda - 10)(\lambda + 5)$ are $\lambda_1 = \mathbf{10}$ and $\lambda_2 = \mathbf{5}$. The unit eigenvectors are in Q :

$$Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{with} \quad \Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}.$$

5 Show that this A (**symmetric but complex**) has only one line of eigenvectors :

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable. Its eigenvalues are } 0 \text{ and } 0.$$

$A^{\mathrm{T}} = A$ is not so special for complex matrices. *The good property is* $\overline{A}^{\mathrm{T}} = A$.

$\det(A - \lambda I) = \lambda^2$ gives $\lambda = \mathbf{0}, \mathbf{0}$. But $A - \lambda I = A$ has **rank 1** : Only one line of eigenvectors in its nullspace.

6 Find *all* orthogonal matrices from all $\boldsymbol{x}_1, \boldsymbol{x}_2$ to diagonalize $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

$\lambda^2 - 25\lambda = 0$ gives eigenvalues **0** and **25**. The (real) eigenvectors in Q can be

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} -4 & -3 \\ 3 & -4 \end{bmatrix}.$$

7 (a) Find a symmetric matrix $S = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know that S must have a negative pivot?

(c) How do you know that S can't have two negative eigenvalues?

The determinant of S is negative if $\boldsymbol{b^2} > \mathbf{1}$. This determinant is (pivot 1)(pivot 2). Also $\det S = \lambda_1$ times λ_2. So exactly one eigenvalue is negative if $b^2 > 1$.

8 If $A^2 = 0$ then the eigenvalues of A must be _____. Give an example with $A \neq 0$. But if A is symmetric, diagonalize it to prove that the matrix is $A = 0$.

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$. Here $A^2 = 0$ so λ must be zero.

Nonsymmetric example : $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

The only symmetric example is $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $A = Q\Lambda Q^{\mathrm{T}}$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue. (If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then also $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$.) Prove that every real 3 by 3 matrix has at least one real eigenvalue.

A real 3 by 3 matrix has $\det(A - \lambda I) = -\lambda^3 + c_2\lambda^2 + c_1\lambda + c_2 = 0$. If λ_1 satisfies this equation so does $\lambda_2 = \overline{\lambda_1}$ (take the conjugate of every term). But the sum $\lambda_1 + \lambda_2 + \lambda_3 =$ trace of $A =$ real number. So λ_3 must be real.

10 Here is a quick "proof" that the eigenvalues of *all* real matrices are real:

$$\textbf{False proof} \quad A\boldsymbol{x} = \lambda \boldsymbol{x} \quad \text{gives} \quad \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} = \lambda \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} \quad \text{so} \quad \lambda = \frac{\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[\,0 \;\; -1; \;\; 1 \;\; 0\,]$ with $\lambda = i$ and $\boldsymbol{x} = (i, 1)$.

The flaw is to expect that $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ and $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$ are real and $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} > 0$. When complex numbers are involved, it is $\overline{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{x}$ that is real and positive for every vector $\boldsymbol{x} \neq \mathbf{0}$.

11 Write A and B in the form $\lambda_1 \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}}$ of the spectral theorem $Q \Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad (\text{keep } \|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1).$$

A has $\lambda = \mathbf{4}, \mathbf{2}$ with unit eigenvectors in Q. Multiply columns times rows:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = Q \Lambda Q^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & \\ & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

B has $\lambda = 0, 25$ with these unit eigenvectors in Q:

$$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 0 & \\ & 25 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} = 0 + 25 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} [\,3/4 \quad 4/5\,].$$

12 What number b in $\left[\begin{smallmatrix} 2 & b \\ 1 & 0 \end{smallmatrix}\right]$ makes $A = Q \Lambda Q^{\mathrm{T}}$ possible? What number makes $A = V \Lambda V^{-1}$ impossible? What number makes A^{-1} impossible?

$b = \mathbf{1}$ makes A symmetric and then $A = Q \Lambda Q^{\mathrm{T}}$. $b = \mathbf{-1}$ makes $\lambda = 1, \ 1$ with only one eigenvector. $b = \mathbf{0}$ makes the matrix singular.

13 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\ ?\ ]$$

What is the dot product of the two unit eigenvectors ? A small angle !

The unit eigenvector for $\lambda = 1 + 10^{-15}$ is $\dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The two eigenvectors are at a $45\,°$ angle, far from orthogonal (even if A is nearly symmetric).

14 (Recommended) This matrix M is skew-symmetric and also orthogonal. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. They can only be i or $-i$. Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

The four eigenvalues must be $\lambda = \boldsymbol{i}, \boldsymbol{i}, -\boldsymbol{i}, -\boldsymbol{i}$ to produce trace = zero.

15 The complete solution to equation (8) for two oscillating springs (Figure 6.3) is

$$\boldsymbol{y}(t) = (A_1 \cos t + B_1 \sin t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (A_2 \cos \sqrt{3}t + B_2 \sin \sqrt{3}t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the numbers A_1, A_2, B_1, B_2 if $\boldsymbol{y}(0) = (3, 5)$ and $\boldsymbol{y}'(0) = (2, 0)$.

The numbers A_1, A_2 come from $y(0) = (3, 5)$ since $\cos 0 = 1$:

$$A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \text{gives} \qquad A_1 = 4 \text{ and } A_2 = -1.$$

The numbers B_1, B_2 come from $y'(0) = (2, 0)$ since $(\sin t)' = 1$ at $t = 0$ and $(\sin \sqrt{3}t)' = \sqrt{3}$ at $t = 0$:

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sqrt{3}B_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \text{gives} \qquad B_1 = B_2 = \frac{1}{\sqrt{3}}.$$

16 If the springs in Figure 6.3 have different constants k_1, k_2, k_3 then $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is

$$\begin{array}{ll} \text{Upper mass} & y_1'' + k_1 y_1 - k_2(y_2 - y_1) = 0 \\ \text{Lower mass} & y_2'' + k_2(y_2 - y_1) + k_3 y_2 = 0 \end{array} \qquad S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

For $k_1 = 1, k_2 = 4, k_3 = 1$ find the eigenvalues $\lambda = \omega^2$ of S and the complete sine/cosine solution $\boldsymbol{y}(t)$ in equation (7).

The matrix $S = \begin{bmatrix} 1+4 & -4 \\ -4 & 4+1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1 = \omega_1^2$ and $\lambda_2 = 9 = \omega_2^2$.
The complete solution to $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is

$$\boldsymbol{y}(t) = (A_1 \cos t + B_1 \sin t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (A_2 \cos 3t + B_2 \sin 3t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

17 Suppose the third spring is removed ($k_3 = 0$ and nothing is below mass 2). With $k_1 = 3, k_2 = 2$ in Problem 16, find S and its real eigenvalues and orthogonal eigenvectors. What is the sine/cosine solution $\boldsymbol{y}(t)$ if $\boldsymbol{y}(0) = (1, 2)$ gives the cosines and $\boldsymbol{y}'(0) = (2, -1)$ gives the sines ?

When $k_1 = 3, k_2 = 2, k_3 = 0$, the matrix S becomes $S = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ with $\lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6) = 0$.

The eigenvector for $\lambda_1 = \omega_1^2 = 1$ is $\boldsymbol{x}_1 = (1, 2)$. The orthogonal eigenvector for $\lambda_2 = \omega_2^2 = 6$ is $\boldsymbol{x}_2 = (2, -1)$. Then $A_1 = 1$ and $A_2 = 0, B_1 = 0$ and $B_2 = 1/\sqrt{6}$ come from $\boldsymbol{y}(0) = \boldsymbol{x}_1$ and $\boldsymbol{y}'(0) = \boldsymbol{x}_2$. The solution to $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is $\boldsymbol{y}(t) = (\cos t)\boldsymbol{x}_1 + (\sin \sqrt{6}t)\boldsymbol{x}_2/\sqrt{6}$.

18 Suppose the top spring is also removed ($k_1 = 0$ and also $k_3 = 0$). S is singular ! Find its eigenvalues and eigenvectors. If $\boldsymbol{y}(0) = (1, -1)$ and $\boldsymbol{y}' = (0, 0)$ find $\boldsymbol{y}(t)$. If $\boldsymbol{y}(0)$ changes from $(1, -1)$ to $(1, 1)$ what is $\boldsymbol{y}(t)$?

$S = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ has $\lambda_1 = 0$ with $\boldsymbol{x}_1 = (1, 1)$ and $\lambda_2 = 2k_2$ with $\boldsymbol{x}_2 = (1, -1)$.

$\boldsymbol{y}(0) = (1, -1)$ and $\boldsymbol{y}'(0) = (0, 0)$ give $\boldsymbol{y}(t) = (\cos\sqrt{2k_2}\,t)\,\boldsymbol{x}_2$.

$\boldsymbol{y}(0) = (1, \;\; 1)$ and $\boldsymbol{y}'(0) = (0, 0)$ give $\boldsymbol{y}(t) = \boldsymbol{x}_1 = (1, 1)$: no movement!

There is no force from springs 1 and 3 and no initial velocity $\boldsymbol{y}'(0)$.

19 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$). Energy is conserved.

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \boldsymbol{y} \qquad \text{or} \qquad \begin{aligned} y_1' &= cy_2 - by_3 \\ y_2' &= ay_3 - cy_1 \\ y_3' &= by_1 - ay_2. \end{aligned}$$

The derivative of $\|\boldsymbol{y}(t)\|^2 = y_1^2 + y_2^2 + y_3^2$ is $2y_1y_1' + 2y_2y_2' + 2y_3y_3'$. Substitute y_1', y_2', y_3' to get *zero*. The energy $\|\boldsymbol{y}(t)\|^2$ stays equal to $\|\boldsymbol{y}(0)\|^2$.

$y_1y_1' + y_2y_2' + y_3y_3' = y_1(cy_2 - by_3) + y_2(ay_3 - cy_1) + y_3(by_1 - ay_2) = \mathbf{0}$.

Then $\|\boldsymbol{y}(t)\|^2$ stays constant, equal to $\|\boldsymbol{y}(0)\|^2$.

20 *When $A = -A^{\mathrm{T}}$ is skew-symmetric*, e^{At} is ***orthogonal***. Prove $(e^{At})^{\mathrm{T}} = e^{-At}$ from the series $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$.

$A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{y}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

21 The mass matrix M can have masses $m_1 = 1$ and $m_2 = 2$. Show that the eigenvalues for $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ are $\lambda = 2 \pm \sqrt{2}$, starting from $\det(K - \lambda M) = 0$:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \text{ are positive definite.}$$

Find the two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Show that $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 \neq 0$ but $\boldsymbol{x}_1^{\mathrm{T}}M\boldsymbol{x}_2 = 0$.

$K\boldsymbol{x} = \lambda M\boldsymbol{x}$ is $(K - \lambda M)\boldsymbol{x} = \mathbf{0}$ and we need the determinant of $K - \lambda M$ to be 0:

$$\det\begin{bmatrix} 2-\lambda & -2 \\ -2 & 4-2\lambda \end{bmatrix} = 2(\lambda^2 - 4\lambda + 2) = 0 \qquad \lambda = \frac{4 \pm \sqrt{16-8}}{2} = \mathbf{2 \pm \sqrt{2}}.$$

The eigenvectors $\boldsymbol{x}_1 = (\sqrt{\mathbf{2}}, \mathbf{-1})$ and $\boldsymbol{x}_2 = (\sqrt{\mathbf{2}}, \mathbf{1})$ come from

$$(K - \lambda_1 M)\boldsymbol{x}_1 = \begin{bmatrix} -\sqrt{2} & -2 \\ -2 & -2\sqrt{2} \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0} \text{ and } (K - \lambda_2 M)\boldsymbol{x}_2 = \begin{bmatrix} \sqrt{2} & -2 \\ -2 & 2\sqrt{2} \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}.$$

Notice that $\boldsymbol{x}_1$ is **not** orthogonal to $\boldsymbol{x}_2$—it is "M-orthogonal":

$$\boldsymbol{x}_1^{\mathrm{T}}M\boldsymbol{x}_2 = \begin{bmatrix} \sqrt{2} & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = 0.$$

22 What difference equation would you use to solve $\boldsymbol{y}'' = -S\boldsymbol{y}$?

$y'' = -Sy$ is well approximated by $y_{n+1} - 2y_n + y_{n-1} = -(\Delta t)^2 S y_n$. The initial conditions come in as $y_0 = y(0)$ and $y_1 = y(0) + \Delta t y'(0)$ (but that is only a first order accurate approximation to the true $y(\Delta t)$).

23 The second order equation $\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$ reduces to a first order system $\boldsymbol{y}_1{}' = \boldsymbol{y}_2$ and $\boldsymbol{y}_2{}' = -S\boldsymbol{y}_1$. If $S\boldsymbol{x} = \omega^2\boldsymbol{x}$ show that the companion matrix $A = [0\ I\ ;\ -S\ 0]$ has eigenvalues $i\omega$ and $-i\omega$ with eigenvectors $(\boldsymbol{x}, i\omega\boldsymbol{x})$ and $(\boldsymbol{x}, -i\omega\boldsymbol{x})$.

The first-order equation with *block* companion matrix for $y'' = -Sy$ is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

For the eigenvalues: If $S\boldsymbol{x} = \omega^2\boldsymbol{x}$ then

$$\begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \pm\, i\omega\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \pm\, i\omega\boldsymbol{x} \\ -\,\omega^2\boldsymbol{x} \end{bmatrix} = \pm\, i\omega \begin{bmatrix} \boldsymbol{x} \\ \pm\, i\omega\boldsymbol{x} \end{bmatrix}.$$

So the block companion matrix A has eigenvalues $i\omega$ and $-i\omega$. Then we can compute and use the exponential e^{At} (if we want to).

24 Find the eigenvalues λ and eigenfunctions $y(x)$ for the differential equation $y'' = \lambda y$ with $y(0) = y(\pi) = 0$. There are infinitely many !

This is an important problem in function space—instead of eigenvectors in $\mathbf{R}^n$ we look for functions of x between $x = 0$ and $x = \pi$:

$$\frac{d^2y}{dt^2} = \lambda y(x) \text{ with boundary conditions } y(0) = y(\pi) = 0.$$

This equation is satisfied by $y(x) = a\cos\left(\sqrt{\lambda}\,x\right) + b\sin\left(\sqrt{\lambda}\,x\right)$.

The boundary condition $y(0) = 0$ makes $a = 0$.

The condition $y(\pi) = \sin\left(\sqrt{\lambda}\,\pi\right) = 0$ makes $\sqrt{\lambda} = \mathbf{1}$ or $\mathbf{2}$ or $\mathbf{3}$ or ... Then

$$\lambda = \mathbf{1^2} \text{ or } \mathbf{2^2} \text{ or any } \boldsymbol{n^2} \qquad y(x) = \sin(\sqrt{\lambda}\,x).$$

Problem Set 7.1, page 393

1 Suppose your pulse is measured at $b_1 = 70$ beats per minute, then $b_2 = 120$, then $b_3 = 80$. The least squares solution to three equations $v = b_1, v = b_2, v = b_3$ with $A^{\mathrm{T}} = [1\ \ 1\ \ 1]$ is $\widehat{v} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} =$ _____. Use calculus and projections:

(a) Minimize $E = (v-70)^2 + (v-120)^2 + (v-80)^2$ by solving $dE/dv = 0$.

Solution (a) $\frac{dE}{dv} = 2(v-70) + 2(v-120) + 2(v-80) = 0$ at the minimizing $\widehat{v}$.

Cancel the 2's: $3v = 70 + 120 + 80 = 270$ so $\widehat{v} = v_{\text{average}} = \mathbf{90}$

(b) Project $\boldsymbol{b} = (70, 120, 80)$ onto $\boldsymbol{a} = (1, 1, 1)$ to find $\widehat{v} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Solution (b) The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is $\boldsymbol{p} = \boldsymbol{a}\widehat{v}$:

$$\boldsymbol{b} = \begin{bmatrix} 70 \\ 120 \\ 80 \end{bmatrix} \qquad \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \widehat{v} = \frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \frac{270}{3} = \mathbf{90}.$$

2 Suppose $Av = \boldsymbol{b}$ has m equations $a_i v = b_i$ in *one unknown* v. For the sum of squares $E = (a_1 v - b_1)^2 + \cdots + (a_m v - b_m)^2$, find the minimizing $\widehat{v}$ by calculus. Then form $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ with one column in A, and reach the same $\widehat{v}$.

Solution To minimize E we solve $dE/dv = 0$. For $m = 3$ equations $a_i v = b_i$,

$$\frac{dE}{dv} = 2a_1(a_1 v - b_1) + 2a_2(a_2 v - b_2) + 2a_3(a_3 v - b_3) = 0 \text{ is zero when}$$

$$v = \widehat{v} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{a_1^2 + a_2^2 + a_3^2} = \frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}}.$$

When A has one column, $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ is the same as $(\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a})\widehat{v} = (\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b})$.

3 With $\boldsymbol{b} = (4, 1, 0, 1)$ at the points $x = (0, 1, 2, 3)$ set up and solve the normal equation for the coefficients $\widehat{\boldsymbol{v}} = (C, D)$ in the nearest line $C + Dx$. Start with the four equations $A\boldsymbol{v} = \boldsymbol{b}$ that would be solvable if the points fell on a line.

Solution The unsolvable equation has $m = 4$ points on a line: only $n = 2$ unknowns.

$$A\boldsymbol{v} = \boldsymbol{b} \text{ is } \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ leading to } A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}:$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \text{ gives } \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \frac{1}{2a} \begin{bmatrix} 60 \\ -20 \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-1} \end{bmatrix}$$

The closest line to the four points is $\boldsymbol{b} = \mathbf{3} - \boldsymbol{x}$.

4 In Problem 3, find the projection $\boldsymbol{p} = A\boldsymbol{v}$. Check that those four values lie on the line $C + Dx$. Compute the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and verify that $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$.

Solution The projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ is

$$\boldsymbol{p} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ with error } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

The best line $C + Dx = 3 - x$ does produce $\boldsymbol{p} = (3, 2, 1, 0)$ at the four points $x = 0, 1, 2, 3$.

Multiply this $\boldsymbol{e}$ by A^{T} to get $A^{\mathrm{T}}\boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as expected.

5 (Problem 3 by calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares : the last one is $(1 - C - 3D)^2$. Find the derivative equations $\partial E/\partial C = \partial E/\partial D = 0$. Divide by 2 to obtain $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution Minimize $E = (4 - C)^2 + (1 - C - D)^2 + (-C - 2D)^2 + (1 - C - 3D)^2$.

The partial derivatives are $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ at the minimum :

$$-2(4 - C) - 2(1 - C - D) - 2(-C - 2D) - 2(1 - C - 3D) = 0$$

$$-2(1 - C - D) - 4(-C - 2D) - 6(1 - C - 3D) = 0$$

Factoring out -2 and collecting terms this is the same equation $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$!

$$\begin{aligned} 6 - 4C - \ \ 6D &= 0 \\ 4 - 6C - 14D &= 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

6 For the closest parabola $C + Dt + Et^2$ to the same four points, write down 4 unsolvable equations $A\boldsymbol{v} = \boldsymbol{b}$ for $\boldsymbol{v} = (C, D, E)$. Set up the normal equations for $\widehat{\boldsymbol{v}}$. If you fit the best cubic $C + Dt + Et^2 + Ft^3$ to those four points (thought experiment), what is the error vector $\boldsymbol{e}$?

Solution The parabola $C + Dt + Et^2$ fits the 4 points exactly if $A\boldsymbol{v} = \boldsymbol{b}$:

$$\begin{array}{ll} t = 0 & C + 0D + 0E = 4 \\ t = 1 & C + 1D + 1E = 1 \\ t = 2 & C + 2D + 4E = 0 \\ t = 3 & C + 3D + 9E = 1 \end{array} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \qquad .\phi A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 4+1+0+1 \\ 0+1+0+3 \\ 0+1+0+9 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}.$$

The cubic $C + Dt + Et^2 + Ft^3$ can fit 4 points exactly, with **error = zero vector**.

7 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

8 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 7. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$.

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \mathbf{0}$.

9 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 8. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \mathbf{0}$.

Solution If $\boldsymbol{b} =$ previous error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection of $\boldsymbol{b}$ is $\boldsymbol{p} = \mathbf{0}$.

10 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line $\boldsymbol{e}$. The error is $\boldsymbol{e} = \mathbf{0}$ because this $\boldsymbol{b}$ is _____.

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \mathbf{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

11 Find the best line $C + Dt$ to fit $\boldsymbol{b} = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Solution The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm{T}}A$

12 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. At those 4 points, the equations $C + Dx + Ey = b$ are $A\boldsymbol{v} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{v} = (C, D, E)$.

Solution $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$.

The solution $(C, D, E) = (2, -1, \frac{3}{2})$ gives the best plane $2 - x - \frac{3}{2}y$.

13 With $\boldsymbol{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line $C + Dt$, find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

Solution $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$ $\quad \widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

14 (By calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$. These normal equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

15 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains p ? Which contains $\widehat{\boldsymbol{v}}$?

Solution The error $\boldsymbol{e}$ is contained in the nullspace $\boldsymbol{N}(A^{\mathrm{T}})$, since $A^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{0}$. The projection $\boldsymbol{p}$ is contained in the column space $\boldsymbol{C}(A)$. The vector $\widehat{\boldsymbol{v}}$ of coefficients can be any vector in $\mathbf{R}^{\boldsymbol{n}}$.

16 Find the height C of the best *horizontal line* to fit $\boldsymbol{b} = (0, 8, 8, 20)$. An exact fit would solve the four unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$ and $A^{\mathrm{T}} = [\,1\ \ 1\ \ 1\ \ 1\,]$.

$A^{\mathrm{T}}A = [\,4\,]$. $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = \mathbf{9} =$ best C. $\boldsymbol{e} = (-9, -1, -1, 11)$.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \boldsymbol{0}$?

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

Solution If $\boldsymbol{b} =$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \boldsymbol{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____ ?

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

Questions 21–26 ask for projections onto lines. Also errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and matrices $\boldsymbol{P}$.

21 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

Solution (a) The projection $\boldsymbol{p}$ is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{6}{3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ perpendicular to } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution (b) In this case the projection is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}\frac{-11}{11} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

22 Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{v}\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$

Solution (b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

23 In Problem 22 find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to find the projection $\boldsymbol{p}$.

Solution $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{p} = P_1\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$. $P_2 = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\boldsymbol{p} = P_2\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

24 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 22. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

Solution The projection matrices P_1 and P_2 (note correction P_2 not $P - 2$) are

$$P_1 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad P_2 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It is *not true* that $(P_1 + P_2)^2 = P_1 + P_2$. The sum of projection matrices is **not usually** a projection matrix.

25 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those two matrices P_1P_2 and explain the answer.

Solution $P_1 = \dfrac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \dfrac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$.

$P_1P_2 =$ *zero matrix because* $\boldsymbol{a_1}$ is perpendicular to $\boldsymbol{a_2}$.

26 Continuing Problem 25, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal !

Solution $P_1 + P_2 + P_3 = \dfrac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \dfrac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \dfrac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$

We *can* add projections onto *orthogonal vectors*. This is important.

27 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.

Solution The projections of $(1, 1)$ onto the lines through $(1, 0)$ and $(1, 2)$ are $\boldsymbol{p}_1 = (1, 0)$ and $\boldsymbol{p}_2 = (3/5, 6/5) = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

28 (Quick and recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $P =$ square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix}$.

29 If A is doubled, then $P = 2A(4A^{\mathrm{T}}A)^{-1}2A^{\mathrm{T}}$. This is the same as $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The column space of $2A$ is the same as _____. Is $\widehat{\boldsymbol{v}}$ the same for A and $2A$?

Solution $2A$ has the same column space as A. Same $\boldsymbol{p}$. But $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

30 What linear combination of $(1,2,-1)$ and $(1,0,1)$ is closest to $\boldsymbol{b} = (2,1,1)$?

Solution $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So $\boldsymbol{b}$ is in the plane: no error $\boldsymbol{e}$. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

31 (*Important*) If $P^2 = P$ show that $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto which fundamental subspace ?

Solution If $P^2 = P$ then $\boldsymbol{(I-P)^2} = (I-P)(I-P) = I-PI-IP+P^2 = \boldsymbol{I-P}$. When P projects onto the column space, $I-P$ projects onto the *left nullspace*.

32 If P is the 3 by 3 projection matrix onto the line through $(1,1,1)$, then $I-P$ is the projection matrix onto _____.

Solution $I-P$ is the projection onto the plane $x_1+x_2+x_3 = 0$, perpendicular to the direction $(1,1,1)$:

$$I-P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

33 Multiply the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is _____.

Solution $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. Geometric reason: $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$ for every $\boldsymbol{b}$. So $P^2 = P$.

34 If A is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. Then $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$ is true. *When A is invertible, why is* $P = I$ and $\boldsymbol{e} = \boldsymbol{0}$?

Solution If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \boldsymbol{0}$.

35 An important fact about $A^{\mathrm{T}}A$ is this: ***If*** $\boldsymbol{A^{\mathrm{T}}Ax = 0}$ ***then*** $\boldsymbol{Ax = 0}$. *New proof*: The vector $A\boldsymbol{x}$ is in the nullspace of _____. $A\boldsymbol{x}$ is always in the column space of _____. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

Solution If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is in the *nullspace of* A^{T}. But $A\boldsymbol{x}$ is always in the *column space of* A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace*.

Notes on mean and variance and test grades

If all grades on a test are 90, the mean is $m = 90$ and the variance is $\sigma^2 = 0$. Suppose the expected grades are $g_1, \ldots, g_N$. Then σ^2 comes from *squaring distances to the mean*:

$$\textbf{Mean} \quad m = \frac{g_1 + \cdots + g_N}{N} \qquad \textbf{Variance} \quad \sigma^2 = \frac{(g_1-m)^2 + \cdots + (g_N-m)^2}{N}$$

After every test my class wants to know m and σ. My expectations are usually way off.

36 Show that σ^2 also equals $\frac{1}{N}(g_1^2 + \cdots + g_N^2) - m^2$.

Solution Each term $(g_i - m)^2$ equals $g_i^2 - 2g_i m + m^2$, so

$$\sigma^2 = \frac{(\text{sum of } g_i^2) - 2m(\text{sum of } g_i) + Nm^2}{N} = \frac{(\text{sum of } g_i^2) - 2mNm + Nm^2}{N}$$
$$= \frac{(\text{sum of } g_i^2)}{N} - m^2.$$

37 If you flip a fair coin N times (1 for heads, 0 for tails) what is the expected number m of heads ? What is the variance σ^2 ?

Solution For a fair coin you expect $\boldsymbol{N/2}$ heads in N flips. The variance σ^2 turns out to be $\boldsymbol{N/4}$.

Problem Set 7.2, page 402

1 For a 2 by 2 matrix, suppose the 1 by 1 and 2 by 2 determinants a and $ac - b^2$ are positive. Then $c > b^2/a$ is also positive.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals ______.

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals ______.

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

Solution Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$.

(i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2 = \det = ac - b^2 > 0$.

(ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

2 Which of S_1, S_2, S_3, S_4 has two positive eigenvalues? Use a and $ac - b^2$, don't compute the λ's. Find an $\boldsymbol{x}$ with $\boldsymbol{x}^{\mathrm{T}} S_1 \boldsymbol{x} < 0$, confirming that A_1 fails the test.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

Solution Only $S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 10^2$.

$\boldsymbol{x}^{\mathrm{T}} S_1 \boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has $\det = 0$.

3 For which numbers b and c are these matrices positive definite?

$$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

Solution

Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$

Positive definite for $c > b$ $\quad L = \begin{bmatrix} 1 & 1 \\ -b/c & 0 \end{bmatrix} \quad D = \begin{bmatrix} c & 0 \\ 0 & c-b/c \end{bmatrix} \quad S = LDL^{\mathrm{T}}.$

4 What is the energy $q = ax^2 + 2bxy + cy^2 = \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ for each of these matrices? Complete the square to write q as a sum of squares $d_1(\quad)^2 + d_2(\quad)^2$.

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

Solution $f(x,y) = x^2 + 4xy + 9y^2 = (x+2y)^2 + 5y^2;\; x^2 + 6xy + 9y^2 = (x+3y)^2.$

5 $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2x_1x_2$ certainly has a saddle point and not a minimum at $(0,0)$. What symmetric matrix S produces this energy? What are its eigenvalues?

Solution $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2x_1x_2$ comes from $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which has eigenvalues 1 and -1: S is **indefinite**.

6 Test to see if $A^{\mathrm{T}}A$ is positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution The first and second matrices have independent columns in A, so $A^{\mathrm{T}}A$ is positive definite. The third matrix has dependent columns so $A^{\mathrm{T}}A$ is only *positive semidefinite*.

7 Which 3 by 3 symmetric matrices S and T produce these quadratic energies?

$\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right)$. Why is S positive definite?

$\boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right)$. Why is T semidefinite?

Solution

$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive *definite*—its determinants are $D_1 = 2, D_2 = 3, D_3 = 4$.

$$T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{array}{l} \text{is positive } \textit{semidefinite} \text{ with} \\ \text{determinants } D_1 = 2, D_2 = 3, D_3 = 0 \end{array}.$$

The energy $\boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} = 0$ when $\boldsymbol{x} = (1, 1, 1)$.

8 Compute the three upper left determinants of S to establish positive definiteness. (The first is 2.) Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

Solution The upper left determinants of S are $2, 6, 30$. The pivots are $2, 3, 5$ (ratios of determinants). Notice that the product of pivots is $\mathbf{30}$.

9 For what numbers c and d are S and T positive definite? Test the 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Solution For $c = 1$, the matrix S has eigenvalues $3, 0, 0$. For any c, the eigenvalues all add $c - 1$. So S is positive definite for $c > 1$. (Same answer using determinants.) For T the determinants are $1, d - 4, -4d + 12$. If $d > 4$ then $-4d + 12$ is negative! So T is **never** positive definite for any d.

10 *If S is positive definite then S^{-1} is positive definite.* Best proof: The eigenvalues of S^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } S^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \_\_\_\_\_ .$$

Solution Positive definite $\Rightarrow$ all eigenvalues $\lambda > 0 \Rightarrow$ all eigenvalues $1/\lambda$ of S^{-1} are positive. Also for 2×2: the determinant tests are passed.

11 *If S and T are positive definite, their sum $S + T$ is positive definite*. Pivots and eigenvalues are not convenient for $S + T$. Better to prove $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} > 0$.

Solution Energy $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} > 0 + 0$

12 A positive definite matrix *cannot have a zero* (or even worse, a negative number) *on its diagonal*. Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & \mathbf{0} & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

Solution $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ is **zero** when $\boldsymbol{x} = (0, 1, 0)$.

13 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a _____ on the main diagonal.

Solution If a_{jj} is smaller than all eigenvalues, then $A - a_{jj}I$ would have positive eigenvalues. But this matrix has a zero on the diagonal. But Problem 13, it can't be positive definite. So A_{jj} can't be smaller than all eigenvalues!

14 Show that *if all* $\lambda > 0$ *then* $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$. We must do this for *every* nonzero $\boldsymbol{x}$, not just the eigenvectors. So write $\boldsymbol{x}$ as a combination of the eigenvectors and *explain why all "cross terms" are* $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$. Then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is

$$(c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n)^{\mathrm{T}}(c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n x_n) = c_1^2\lambda_1\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_1 + \cdots + c_n^2\lambda_n\boldsymbol{x}_n^{\mathrm{T}}\boldsymbol{x}_n > 0.$$

Solution The "cross terms" have the form $(c_i x_i)^{\mathrm{T}}(c_j\lambda_j\boldsymbol{x}_j)$. This is zero because symmetric matrices S have orthogonal eigenvectors.

15 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.
(b) The only positive definite projection matrix is $P = I$.
(c) A diagonal matrix with positive diagonal entries is positive definite.
(d) A symmetric matrix with a positive determinant might not be positive definite!

Solution

(a) All $\lambda_i > 0$ so zero is not an eigenvalue and S is invertible
(b) All projection matrices except $P = I$ are singular
(c) The energy for a positive diagonal matrix is $\boldsymbol{x}^{\mathrm{T}}D\boldsymbol{x} = d_1x_1^2 + \cdots + d_nx_n^2 > 0$ when $\boldsymbol{x} \neq \boldsymbol{0}$
(d) $S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has $\det S = 1$ but S is **negative** definite

16 With positive pivots in D, the factorization $S = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $A = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $S = A^{\mathrm{T}}A$ which is "symmetrized LU":

From $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ find S. From $S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$ find $A =$ **chol**(S).

Solution If $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ then $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$ = positive definite S.

$$S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 0 & 9 \end{bmatrix} = LDL^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & \\ & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

so $A = \sqrt{D}\,L^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

17 Without multiplying $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of S (b) the eigenvalues of S

(c) the eigenvectors of S (d) a reason why S is symmetric positive definite.

Solution $\det S = 10$, $\lambda(S) = 2$ and 5, eigenvectors $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$, S has positive eigenvalues.

18 For $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$ find the second derivative matrices H_1 and H_2 :

$$\textbf{Test for minimum} \quad H = \begin{bmatrix} \partial^2F/\partial x^2 & \partial^2F/\partial x\partial y \\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2 \end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up ($=$ convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where first derivatives are zero).

Solution $F_1 = \frac{1}{4}x^4 + x^2y + y^2$ has $\partial F_1/dx = x^3 + 2xy$ and $\partial F_1/dy = x^2 + 2y$. Then the 2nd derivatives are

$$H_1 = \begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}. \quad F_2 = x^3 + xy - x \text{ has } H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}.$$

19 The graph of $z = x^2 + y^2$ is a bowl opening upward. The graph of $z = x^2 - y^2$ is *a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0, 0)$?

Solution $ax^2 + 2bxy + cy^2$ has a saddle point $(0, 0)$ if $\partial z/\partial x = \partial z/\partial y = 0$ (which is true) and if $H = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is positive definite.

20 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

Solution The matrix for this problem is $S = \begin{bmatrix} 4 & 6 \\ 6 & c \end{bmatrix}$ and this has a saddle for $c < 9$. Then $\lambda_1 > 0 > \lambda_2$ because the determinants are $4 > 0$ and $4c - 3b < 0$.

21 When S and T are symmetric positive definite, ST might not even be symmetric. But its eigenvalues are still positive. Start from $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ and take dot products with $T\boldsymbol{x}$. Then prove $\lambda > 0$.

Solution If $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(T\boldsymbol{x})^{\mathrm{T}}ST\boldsymbol{x} = \lambda(T\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x}$. Left side > 0 because S is positive definite, right side has $\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0$ because T is positive definite. Therefore $\lambda > 0$.

22 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \mathbf{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \mathbf{0}$ whenever $\boldsymbol{x} \neq \mathbf{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $A^{\mathrm{T}}CA$ is positive definite : *the crucial matrix in engineering*.

Solution $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ because $\boldsymbol{y} = A\boldsymbol{x}$ is only zero when $\boldsymbol{x}$ is zero (A has independent columns).

23 Find the eigenvalues and unit eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$. Then find $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \text{ and } AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that $\boldsymbol{u}_1$ is a unit eigenvector of AA^{T}. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathbf{T}}.$$

Solution $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has eigenvalues 50 and 0. Its eigenvectors are $\boldsymbol{v}_1 = (1,2)/\sqrt{5}$ and $\boldsymbol{v}_2 = (-2,1)/\sqrt{5}$. Then $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sqrt{50} = (50,100)/\sqrt{250}$.
The SVD is $\dfrac{\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

24 Write down orthonormal bases for the four fundamental subspaces of this A.

Solution $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has bases $\begin{bmatrix} 1 \\ 3 \end{bmatrix}/\sqrt{10}$ for $\mathbf{C}(A)$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}/\sqrt{5}$ for row space $\mathbf{C}(A^{\mathrm{T}})$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}/\sqrt{5}$ for $\mathbf{N}(A)$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}/\sqrt{10}$ for $\mathbf{N}(A^{\mathrm{T}})$.

25 (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2 ?

(b) For every rank-one matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2 ?

Solution The diagonal entries of $A^{\mathrm{T}}A$ are $||\text{column } 1||^2$ to $||\text{column } n||^2$. The sum of those is the sum of all a_{ij}^2. The trace of $A^{\mathrm{T}}A$ is always the sum of all σ_i^2 and for a rank one matrix that sum is only σ_1^2.

26 Find the eigenvalues and unit eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. Keep each $A\boldsymbol{v} = \sigma\boldsymbol{u}$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^{\mathrm{T}}$.

Solution A is symmetric with $A^{\mathrm{T}}A = A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with eigenvalues x from $x^2 - 3x + 1 = 0$ and $x = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$. Then $\sigma = \sqrt{x} = \frac{1}{2}\left(\sqrt{5} \pm 1\right)$.

27 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ *has the same shape as* A.

Solution $A^{\mathrm{T}}A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 3 and 1, so A has singular values $\sqrt{3}$ and 1. The unit eigenvectors are $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$. $AA^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has eigenvalues 3 and 1 and 0 and eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ divided by $\sqrt{6}, \sqrt{2}, \sqrt{3}$.

28 Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ and $\boldsymbol{u} = \frac{1}{3}(2,2,1)$. Its only singular value is $\sigma_1 =$ _____.

Solution $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ has $A\boldsymbol{v} = 12\boldsymbol{u}$ for that unit vector $\boldsymbol{v}$. The only singular value is $\sigma_1 = 12$. (Other A are also possible.)

29 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by ***as small a matrix as possible*** to produce a singular matrix A_0. Hint : U and V do not change.

From $\quad A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}}$ find the nearest A_0.

Solution The nearest singular matrix is $A_0 = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^{\mathrm{T}}$. Since U and V are orthogonal matrices, the size of $A - A_0$ is only σ_2. In other words, $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$ is the closest rank 1 matrix to A.

30 The SVD for $A + I$ doesn't use $\Sigma + I$. *Why is* $\sigma(A + I)$ *not just* $\sigma(A) + I$?

Solution The SVD of $A + I$ uses the eigenvectors of $(A + I)^{\mathrm{T}}(A + I)$. Those are not the eigenvectors of $A^{\mathrm{T}}A$ (or $A^{\mathrm{T}}A + I$).

31 Multiply $A^{\mathrm{T}}A\boldsymbol{v} = \sigma^2\boldsymbol{v}$ by A. Put in parentheses to show that $A\boldsymbol{v}$ is an eigenvector of AA^{T}. We divide by its length $||A\boldsymbol{v}|| = \sigma$ to get the unit eigenvector $\boldsymbol{u}$.

Solution A times $A^{\mathrm{T}}A\boldsymbol{v} = \sigma^2\boldsymbol{v}$ is $(AA^{\mathrm{T}})A\boldsymbol{v} = \sigma^2(A\boldsymbol{v})$. So $A\boldsymbol{v}$ is an eigenvector of AA^{T}.

32 My favorite example of the SVD is when $Av(x) = dv/dx$, with the endpoint conditions $v(0) = 0$ and $v(1) = 0$. We are looking for orthogonal functions $v(x)$ so that their derivatives $Av = dv/dx$ are also orthogonal. The perfect choice is $v_1 = \sin \pi x$ and $v_2 = \sin 2\pi x$ and $v_k = \sin k\pi x$. Then each u_k is a cosine.

The derivative of v_1 is $Av_1 = \pi \cos \pi x = \pi u_1$. The singular values are $\sigma_1 = \pi$ and $\sigma_k = k\pi$. Orthogonality of the sines (and orthogonality of the cosines) is the foundation for Fourier series.

You may object to $AV = U\Sigma$. The derivative $A = d/dx$ is not a matrix ! The orthogonal factor V has functions $\sin k\pi x$ in its columns, not vectors. The matrix U has cosine functions $\cos k\pi x$. Since when is this allowed ? One answer is to refer you to the **chebfun** package on the web. This extends linear algebra to matrices whose columns are functions—not vectors.

Another answer is to replace d/dx by a first difference matrix A. Its shape will be $N+1$ by N. A has 1's down the diagonal and -1's on the diagonal below. Then $AV = U\Sigma$ has discrete sines in V and discrete cosines in U. For $N = 2$ those will be sines and cosines of $30°$ and $60°$ in $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$.

$**$ Can you construct the difference matrix A (3 by 2) and $A^{\mathrm{T}}A$ (2 by 2)? The discrete sines are $\boldsymbol{v}_1 = (\sqrt{3}/2, \sqrt{3}/2)$ and $\boldsymbol{v}_2 = (\sqrt{3}/2, -\sqrt{3}/2)$. Test that $A\boldsymbol{v}_1$ is orthogonal to $A\boldsymbol{v}_2$. What are the singular values σ_1 and σ_2 in Σ ?

Solution The sines and cosines are perfect examples of the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s for the operator (infinite-dimensional matrix) $A =$ derivative d/dx. The sines $\boldsymbol{v}_k = \sin \pi kx$ are orthogonal, the cosines $\boldsymbol{u}_k = \cos \pi kx$ are orthogonal, and $A\boldsymbol{v}_k = \sigma_k \boldsymbol{u}_k$. (The derivative of a sine is a cosine with $\sigma_k = \pi k$.) For differences instead of derivatives, we can try the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

Problem Set 7.3, page 413

1 *Transpose the derivative with integration by parts*: $(dy/dx, g) = -(y, dg/dx)$. Ay is dy/dx with boundary conditions $y(0) = 0$ and $y(1) = 0$. Why is $\int y' g dx$ equal to $-\int y g' dx$? Then A^{T} (which is normally written as A^*) is $A^{\mathrm{T}}g = -dg/dx$ with **no** boundary conditions on g. $A^{\mathrm{T}}Ay$ is $-y''$ with $y(0) = 0$ and $y(1) = 0$.

Solution Integration by parts for $0 \le x \le 1$ produces boundary terms at $x = 0$ and 1:

$$\int_0^1 \frac{dy}{dx}\, g(x)\, dx = -\int_0^1 y(x)\, \frac{dg}{dx}\, dx + y(x)\, g(x)\, \Big|_{x=0}^{x=1}$$

The boundary terms are zero if $y(0) = y(1) = 0$. Then the adjoint (or transpose) of d/dx is $-d/dx$, with no boundary condition on g when there are 2 boundary conditions on y (fixed-fixed).

Problems 2-6 have boundary conditions at $x = 0$ and $x = 1$: no initial conditions.

2 Solve this boundary value problem in two steps. Find the complete solution $y_p + y_n$ with two constants in y_n, and find those constants from the boundary conditions:

Solve $-y'' = 12x^2$ with $y(0) = 0$ and $y(1) = 0$ and $y_p = -x^4$.

Solution $y_p = -x^4$ solves $-y_p'' = 12x^2$. It has $y_p(0) = 0$ and $y_p = -1$. We need to add the solution to $-Y'' = 0$ with $Y(0) = 0$ and $Y(1) = 1$. Then $Y = A + Bx$ has $A = 0$ and $B = 1$. The complete solution is $y = -x^4 + x$.

3 Solve the same equation $-y'' = 12x^2$ with $y(0) = 0$ and $y'(1) = 0$ (zero slope).

Solution Changing $y(1) = 0$ to $y'(1) = 0$ will change the solution to $y = -x^4 + Bx$ with $y' = -4x^3 + B$. For $y'(1) = 0$ we need $B = 4$.

4 Solve the same equation $-y'' = 12x^2$ with $y'(0) = 0$ and $y(1) = 0$. Then try for both slopes $y'(0) = 0$ and $y'(1) = 0$: *this has no solution* $y = -x^4 + Ax + B$.

Solution With $y'(0) = 0$ the solution we want is $y = -x^4 + A$. The constant A is determined by $y(1) = -1 + A = 0$. We cannot have $y'(1) = 0$ because $y' = -4x^3$.

5 Solve $-y'' = 6x$ with $y(0) = 2$ and $y(1) = 4$. Boundary values need not be zero.

Solution $-y'' = 6x$ leads to $y = -x^3 + A + Bx$. The boundary conditions are $y(0) = A = 2$ and $y(1) = -1 + 2 + B = 4$. Then $B = 3$ and $y = -x^3 + 2 + 3x$.

6 Solve $-y'' = e^x$ with $y(0) = 5$ and $y(1) = 0$, starting from $y = y_p + y_n$.

Solution $-y'' = e^x$ leads to $y = -e^x + A + Bx$. The first boundary condition is $y(0) = -1 + A = 5$ so that $A = \mathbf{6}$. Then $y(1) = -e + 6 + B = 0$ and $B = e - 6$.

Problems 7-11 are about the LU factors and the inverses of second difference matrices.

7 The matrix T with $T_{11} = 1$ factors perfectly into $LU = A^{\mathrm{T}}A$ (all its pivots are 1).

$$\boldsymbol{T} = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU.$$

Each elimination step adds the pivot row to the next row (and L subtracts to recover T from U). The inverses of those difference matrices L and U are **sum matrices**. Then the inverse of $T = LU$ is $U^{-1}L^{-1}$:

$$T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} = U^{-1}L^{-1}.$$

Compute T^{-1} for $N = 4$ (as shown) and for any N.

Solution $T^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ T is fixed-free second difference matrix. For any N, T^{-1} has the same pattern with first row $N \quad N-1 \quad \cdots \quad 2 \quad 1$

8 The matrix equation $TY = (0,1,0,0) =$ *delta vector* is like the differential equation $-y'' = \delta(x-a)$ with $a = 2\Delta x = \frac{2}{5}$. The boundary conditions are $y'(0) = 0$ and $y(1) = 0$. Solve for $y(x)$ and graph it from 0 to 1. Also graph $Y =$ second column of T^{-1} at the points $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. The two graphs are ramp functions.

Solution Two integrations of the delta function $\delta(x)$ will produce the unit ramp $R(x) = 0$ for $x \le 0, R(x) = x$ for $x \ge 0$. Shifting $\delta(x)$ to $\delta\left(x - \frac{2}{5}\right)$ will shift the solution to $y = -R\left(x - \frac{2}{5}\right) + A + Bx$. Then $y'(0) = -1 + B$ gives $B = 1$, and $y(1) = 0$ gives $-\frac{3}{5} + A + 1 = 0$ and $A = -\frac{2}{5}$.

9 The matrix B has $B_{11} = 1$ (like $T_{11} = 1$) and also $B_{NN} = 1$ (where $T_{NN} = 2$). Why does B have the same pivots 1, 1, ... as T, except for zero in the last pivot position? The early pivots don't know $B_{NN} = 1$.

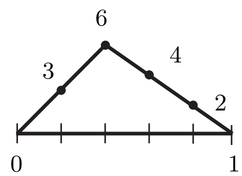
Then B is not invertible: $-y'' = \delta(x - a)$ has no solution with $y'(0) = y'(1) = 0$.

Solution B starts with the pivots $1, 1, 1, \ldots$ (as T did) but reducing the N, N entry by 1 will reduce the last pivot by 1. So we have last pivot = zero and B is not invertible. The analog for differential equations is $y' = 0$ at both endpoints: No ramp function except $y = 0$ can meet those boundary conditions.

10 When you compute K^{-1}, multiply by $\det K = N + 1$ to get nice numbers:

Column 2 of $5K^{-1}$ solves the equation $K\boldsymbol{v} = \boldsymbol{5\delta}$ when the delta vector is $\boldsymbol{\delta} =$ _____
We know from $\boldsymbol{KK^{-1} = I}$ that $\boldsymbol{K}$ times each column of $\boldsymbol{K^{-1}}$ is a delta vector.

$$\mathbf{5}K^{-1} = \begin{bmatrix} 4 & \mathbf{3} & 2 & 1 \\ 3 & \mathbf{6} & 4 & 2 \\ 2 & \mathbf{4} & 6 & 3 \\ 1 & \mathbf{2} & 3 & 4 \end{bmatrix}$$



graph of column 2

Solution Column 2 of $5K^{-1}$ is like the solution to $-y'' = 5\delta\left(x - \frac{2}{5}\right)$. The column of $5K^{-1}$ has a max in row 2 and the solution $y(x)$ has a max at $x = \frac{2}{5}$.

11 K comes with two boundary conditions. T only has $y(1) = 0$. B has no boundary conditions on y. Verify that $K = A^{\mathrm{T}}A$. Then remove the first row of A to get $T = A_1^{\mathrm{T}}A_1$. Then remove the last row to get dependent rows: $B = A_0^{\mathrm{T}}A_0$.

The backward first difference $A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$ gives $K = A^{\mathrm{T}}A$.

Solution A is the matrix in Problem 7 with 1's on the main diagonal and -1's on the diagonal above. $A^{\mathrm{T}}A$ is the symmetric second difference matrix with three nonzero diagonals. Those diagonals contain -1's and 2's and -1's. Then removing the top row of A gives a rectangular A_1 with $A_1^{\mathrm{T}}A_1 = T$ as in Problem 7 ($T_{11} = 1$ not 2). Removing the last row gives A_2 with $A_2^{\mathrm{T}}A_2 = B$ and $B_{NN} = 1$ not 2.

12 Multiply K_3 by its eigenvector $\boldsymbol{y}_n = (\sin n\pi h, \sin 2n\pi h, \sin 3n\pi h)$ to verify that the eigenvalues λ_1, λ_2, λ_3 are $\lambda_n = 2 - 2\cos\frac{n\pi}{4}$ in $K\boldsymbol{y}_n = \lambda_n\boldsymbol{y}_n$. This uses the trigonometric identity $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

Solution The eigenvectors of K are "sine vectors" just as the eigenfunctions of $-y'' = \lambda y$ with $y(0) = 0 = y(1)$ are sine functions.

13 Those eigenvalues of K_3 are $2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$. Those add to 6, which is the trace of K_3. Multiply those eigenvalues to get the determinant of K_3.

Solution Multiplying $2 - \sqrt{2}$ times $2 + \sqrt{2}$ gives $4 - 2 = 2$. Then multiplying by 2 gives 4. This is the determinant (and $2 - \sqrt{2}, 2 + \sqrt{2}, 2$ are the eigenvalues) of 3 by 3 matrix K_3.

14 The slope of a ramp function is a step function. The slope of a step function is a delta function. Suppose the ramp function is $r(x) = -x$ for $x \le 0$ and $r(x) = x$ for $x \ge 0$ (so $r(x) = |x|$). *Find dr/dx and d^2r/dx^2.*

Solution For the down-up ramp function $r(\boldsymbol{x}) = |x|$ = absolute value of x, the derivatives are $dr/dx = -1$ then $+1$ and $d^2r/dx^2 = 2\delta(x)$ because dr/dx jumps by 2 at $x = 0$.

15 Find the second differences $y_{n+1} - 2y_n + y_{n-1}$ of these infinitely long vectors $\boldsymbol{y}$:

Constant	$(\ldots, 1, 1, 1, 1, 1, \ldots)$
Linear	$(\ldots, -1, 0, 1, 2, 3, \ldots)$
Quadratic	$(\ldots, 1, 0, 1, 4, 9, \ldots)$
Cubic	$(\ldots, -1, 0, 1, 8, 27, \ldots)$
Ramp	$(\ldots, 0, 0, 0, 1, 2, \ldots)$
Exponential	$(\ldots, e^{-i\omega}, e^0, e^{i\omega}, e^{2i\omega}, \ldots)$.

It is amazing how closely those second differences follow second derivatives for $y(x) = 1, x, x^2, x^3, \max(x, 0)$, and $e^{i\omega x}$. From $e^{i\omega x}$ we also get $\cos\omega x$ and $\sin\omega x$.

Solution The six second differences are: zero vector, zero vector, constant vector of 2's, 6 times the linear vector, (for ramp: delta vector with $\delta_0 = 1$), $e^{i\omega} - 2 + e^{-i\omega} = 2\cos\omega - 2$ times the exponential vector. **Like 2nd derivatives** of $1, x, x^2, x^3$, ramp, $e^{i\omega x}$.

Problem Set 7.4, page 422

1 What solution to Laplace's equation completes "degree 3" in the table of pairs of solutions? We have one solution $u = x^3 - 3xy^2$, and we need another solution.

Solution Start with $s = -y^3$. Then $s_{yy} = -6y$, and therefore we need $s_{xx} = 6y$. Integrating twice with respect to x gives $3y^2x$. Therefore the second function is $s(x, y) = -y^3 + 3x^2y$.

2 What are the two solutions of degree 4, the real and imaginary parts of $(x + iy)^4$? Check $u_{xx} + u_{yy} = 0$ for both solutions.

Solution Expanding $(x + iy)^4$ gives

$$(x + iy)^4 = x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i$$

Therefore the two solutions would be:

$$u(x, y) = \boldsymbol{x^4 - 6x^2y^2 + y^4} \quad \text{and} \quad s(x, y) = \boldsymbol{4x^3y - 4xy^3}$$

Checking the first solution:

$$\frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial x^2} + \frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial y^2} = (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$$

Checking the second solution:

$$\frac{\partial^2(4x^3y - 4xy^3)}{\partial x^2} + \frac{\partial^2(4x^3y - 4xy^3)}{\partial y^2} = (24xy - 0) + (0 - 24xy) = 0$$

3 What is the second x-derivative of $(x+iy)^n$? What is the second y-derivative ? Those cancel in $u_{xx}+u_{yy}$ because $i^2=-1$.

Solution The second x-derivative of $(x+iy)^n$ is :

$$\frac{\partial^2(x+iy)^n}{\partial x^2}=n(n-1)(x+iy)^{n-2}$$

The second y-derivative of $(x+iy)^n$ cancels that because

$$\frac{\partial^2(x+iy)^n}{\partial y^2}=i\cdot i\cdot n(n-1)(x+iy)^{n-2}=-n(n-1)(x+iy)^{n-2}$$

4 For the solved 2×2 example inside a 4×4 square grid, write the four equations (9) at the four interior nodes. Move the known boundary values 0 and 4 to the right hand sides of the equations. You should see K2D on the left side multiplying the correct solution $\boldsymbol{U}=(U_{11},U_{12},U_{21},U_{22})=(1,2,2,3)$.

Solution The equations at the interior node would be :

$$\begin{aligned}4U_{1,1}-U_{2,1}-U_{0,1}-U_{1,2}-U_{1,0}&=0\\4U_{1,2}-U_{2,2}-U_{0,2}-U_{1,3}-U_{1,1}&=0\\4U_{2,1}-U_{3,1}-U_{1,1}-U_{2,2}-U_{2,0}&=0\\4U_{2,2}-U_{3,2}-U_{1,2}-U_{2,3}-U_{2,1}&=0\end{aligned}$$

Substituting the known boundary values leaves :

$$\begin{aligned}4U_{1,1}-U_{2,1}-U_{1,2}&=4\\4U_{1,2}-U_{2,2}-U_{1,1}&=8\\4U_{2,1}-U_{1,1}-U_{2,2}&=0\\4U_{2,2}-U_{1,2}-U_{2,1}&=4\end{aligned}$$

Writing this in matrix form gives :

$$\begin{bmatrix}4&-1&0&-1\\-1&4&-1&0\\0&-1&4&-1\\-1&0&-1&4\end{bmatrix}\begin{bmatrix}U_{1,1}\\U_{1,2}\\U_{2,1}\\U_{2,2}\end{bmatrix}=\begin{bmatrix}4\\8\\0\\4\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}U_{1,1}\\U_{1,2}\\U_{2,1}\\U_{2,2}\end{bmatrix}=\begin{bmatrix}2\\3\\1\\2\end{bmatrix}$$

5 Suppose the boundary values on the 4×4 grid change to $U=0$ on three sides and $U=8$ on the fourth side. Find the four inside values so that each one is the average of its neighbors.

Solution The values at the 16 nodes will be

$$\begin{matrix}0&0&0&0\\0&\frac{1}{2}&\frac{1}{2}&0\\0&\frac{3}{2}&\frac{3}{2}&0\\0/4&4&4&0/4\end{matrix}$$

Notice that the corner boundary values **do not enter** the 5-point equations around interior points. Every interior value must be the average of its four neighbors. By symmetry the two middle columns must be the same.

6 (MATLAB) Find the inverse $(K2\text{D})^{-1}$ of the 4 by 4 matrix displayed for the square grid.

Solution The circulant matrix $K2$D on page 422 has a circulant inverse:

$$(K2\text{D})^{-1} = \frac{1}{24}\begin{bmatrix} 7 & 2 & 1 & 2 \\ 2 & 7 & 2 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 1 & 2 & 7 \end{bmatrix}.$$

7 Solve this Poisson finite difference equation (right side $\neq 0$) for the inside values $U_{11}, U_{12}, U_{21}, U_{22}$. All boundary values like U_{10} and U_{13} are zero. The boundary has i or j equal to 0 or 3, the interior has i *and* j equal to 1 or 2:

$$4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{i,j-1} - U_{i,j+1} = \mathbf{1} \text{ at four inside points.}$$

Solution The interior solution to the Poisson equation (on this small grid) is

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

On a larger grid U_{ij} will not be constant in the interior.

8 A 5×5 grid has a 3 by 3 interior grid: 9 unknown values U_{11} to U_{33}. Create the 9×9 difference matrix $K2$D.

Solution Order the points by rows to get $U_{11}, U_{12}, U_{13}, U_{21}, U_{22}, U_{23}, U_{31}, U_{32}, U_{33}$. Then $K2D$ is symmetric with 3 by 3 blocks:

$$K2D = \begin{bmatrix} A & -I & 0 \\ -I & A & -I \\ 0 & -I & A \end{bmatrix} \qquad A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

9 Use eig$(K2\text{D})$ to find the nine eigenvalues of $K2$D in Problem 8. Those eigenvalues will be positive! The matrix $K2$D is symmetric positive definite.

Solution eig$(K2\text{D})$ in Problem 8 produces 9 eigenvalues between 0 and 4:

The eigenvalues come from eig$(K2\text{D})$ and explicitly from equation (11). Notice that pairs of eigenvalues add to 8. The eigenvalue distribution is symmetric around $\lambda = 4$:

$$1.1716 \quad 2.5828 \quad 2.5828 \quad 4.0 \quad 4.0 \quad 4.0 \quad 5.4142 \quad 5.4142 \quad 6.8284$$

10 If $u(x)$ solves $u_{xx} = 0$ and $v(y)$ solves $v_{yy} = 0$, verify that $u(x)v(y)$ solves Laplace's equation. Why is this only a 4-dimensional space of solutions? Separation of variables does not give all solutions—only the solutions with separable boundary conditions.

Solution If $\dfrac{\partial^2 u}{\partial x^2} = 0$ and $\dfrac{\partial^2 v}{\partial y^2} = 0$ then

$$\frac{\partial^2 u(x)v(y)}{\partial x^2} + \frac{\partial^2 u(x)v(y)}{\partial y^2} = v(y)\frac{\partial^2 u(x)}{\partial x^2} + u(x)\frac{\partial^2 v(y)}{\partial y^2}$$
$$= v \cdot 0 + u \cdot 0 = 0$$

Therefore $u(x)v(y)$ solves Laplace's equation. But the only solutions found this way are $u(x)v(y) = (A + Bx)(C + Dy)$.

Problem Set 7.5, page 428

Problems 1 — 5 are about complete graphs. Every pair of nodes has an edge.

1 With $n = 5$ nodes and all edges, find the diagonal entries of $A^{\mathrm{T}}A$ (the degrees of the nodes). All the off-diagonal entries of $A^{\mathrm{T}}A$ are -1. Show the reduced matrix R without row 5 and column 5. Node 5 is "grounded" and $v_5 = 0$.

Solution The complete graph (all edges included) has no zeros in $A^{\mathrm{T}}A$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Singular!}$$

The grounded matrix would be

$$(A^{\mathrm{T}}A)_{\text{reduced}} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Invetible!}$$

2 Show that the *trace* of $A^{\mathrm{T}}A$ (sum down the diagonal = sum of eigenvalues) is $n^2 - n$. What is the trace of the reduced (and invertible) matrix R of size $n-1$?

Solution $A^{\mathrm{T}}A$ is n by n and each diagonal entry is $n-1$. Therefore the trace is $n(n-1) = \boldsymbol{n^2 - n}$. The reduced matrix R has $n-1$ diagonal entries, each still equal to $n-1$. Therefore the trace is $(n-1)(n-1) = n^2 - 2n + 1$.

3 For $n = 4$, write the 3 by 3 matrix $R = (A_{\text{reduced}})^{\mathrm{T}}(A_{\text{reduced}})$. Show that $RR^{-1} = I$ when R^{-1} has all entries $\frac{1}{4}$ off the diagonal and $\frac{2}{4}$ on the diagonal.

Solution

$$\textbf{Reduced matrix } R = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

R by its proposed inverse gives

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

4 For every n, the reduced matrix R of size $n-1$ is *invertible*. Show that $RR^{-1} = I$ when R^{-1} has all entries $1/n$ off the diagonal and $2/n$ on the diagonal.

Solution

$$\frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-1-1 & 3-2-1 & 3-1-2 \\ -2+3-1 & -1+6-1 & -1+3-2 \\ -2-1+3 & -1-2+3 & -1-1+6 \end{bmatrix} = I.$$

5 Write the 6 by 3 matrix $M = A_{\text{reduced}}$ when $n = 4$. The equation $M\boldsymbol{v} = \boldsymbol{b}$ is to be solved by least squares. The vector $\boldsymbol{b}$ is like scores in 6 games between 4 teams (team 4 always scores zero; it is grounded). Knowing the inverse of $R = M^{\mathrm{T}}M$, what is the least squares ranking $\widehat{v}_1$ for team 1 from solving $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$?

Solution Remove column 4 of A when node 4 is grounded ($x_4 = 0$).

$$M = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ has independent columns}$$

The least squares solution $\widehat{v}$ to $M\boldsymbol{v} = \boldsymbol{b}$ comes from $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$. This $\widehat{\boldsymbol{v}}$ gives the predicted point spreads when all teams play all other teams. The first component $\widehat{v}_1$ would come from the first row of $(M^{\mathrm{T}}M)^{-1}$ multiplying by $M^{\mathrm{T}}\boldsymbol{b}$. Note that

$$M^{\mathrm{T}}M = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \text{ and } (M^{\mathrm{T}}M)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

6 For the tree graph with 4 nodes, $A^{\mathrm{T}}A$ is in equation (1). What is the 3 by 3 matrix $R = (A^{\mathrm{T}}A)_{\text{reduced}}$? How do we know it is positive definite?

Solution The reduced form of $A^{\mathrm{T}}A$ removes row 4 and column 4:

$$\text{Singular } A^{\mathrm{T}}A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ reduces to invertible } \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The first is positive semidefinite (A has dependent columns). the second is positive definite (the reduced A has 3 independent columns).

7 (a) If you are given the matrix A, how could you reconstruct the graph?

Solution Each row of A tells you an edge in the graph.

(b) If you are given $L = A^{\mathrm{T}}A$, how could you reconstruct the graph (no arrows)?

Solution Each nonzero off the main diagonal of $A^{\mathrm{T}}A$ tells you an edge.

(c) If you are given $K = A^{\mathrm{T}}CA$, how could you reconstruct the weighted graph?

Solution Each nonzero off the main diagonal tells you the weight of that edge.

8 Find $K = A^{\mathrm{T}}CA$ for a line of 3 resistors with conductances $c_1 = 1$, $c_2 = 4$, $c_3 = 9$. Write K_{reduced} and show that this matrix is positive definite.

Solution A **circle** of three resistors has 3 edges and 3 nodes:

$$A^{\mathrm{T}}CA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 & -1 \\ -4 & 13 & -9 \\ -1 & -9 & 10 \end{bmatrix} \text{ is only } \textbf{semidefinite} \qquad .$$

$$(A^{\mathrm{T}}CA)_{\text{reduced}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{13} \end{bmatrix}$$

The determinant tests $5 > 0$ and $(5)(13) > 4^2$ are passed.

9 A 3 by 3 square grid has $n = 9$ nodes and $m = 12$ edges. Number nodes by rows.

(a) How many nonzeros among the 81 entries of $L = A^{\mathrm{T}}A$?

Solution The 9 nodes ordered by rows have $2, 3, 2, 3, 4, 3, 2, 3, 2$ neighbors around them. Those add to 24 nonzeros off the diagonal. The 9 diagonal entries make 33 nonzeros out of $9^2 = 81$ entries in $L = A^{\mathrm{T}}A$.

(b) Write down the 9 diagonal entries in the degree matrix D: they are not all 4.

Solution Those 9 numbers are the degrees of the 9 nodes (= diagonal entries in $A^{\mathrm{T}}A$).

(c) Why does the middle row of $L = D - W$ have four -1's ? Notice $L = K2\mathrm{D}$!

Solution The middle node in the grid has 4 **neighbors**.

10 Suppose all conductances in equation (5) are equal to c. Solve equation (6) for the voltages v_2 and v_3 and find the current I flowing out of node 1 (and into the ground at node 4). What is the "system conductance" I/V from node 1 to node 4 ?

This overall conductance I/V should be larger than the individual conductances c.

Solution The reduced equation (6) with conductances $= c$ is

$$\begin{bmatrix} 3c & -c \\ -c & 2c \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} cV \\ cV \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.6V \\ 0.8V \end{bmatrix}.$$

Then the flows on the five edges in Figure 7.6 use A in equation (2). Remember the minus sign:

$$-cA\boldsymbol{v} = -c\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} V \\ 0.6V \\ 0.8V \\ 0 \end{bmatrix} = cV\begin{bmatrix} 0.4 \\ 0.2 \\ -0.2 \\ 1.0 \\ 0.6 \end{bmatrix}$$

The total flow (on edges $1{+}2{+}4$ out of node 1, or on edges $3{+}4$ into the grounded node 4, is $I = 1.6cV$. The overall system conductance is $1.6c$, greater than the individual conductance c on each edge.

11 The multiplication $A^{\mathrm{T}}A$ can be columns of A^{T} times rows of A. For the tree with $m = 3$ edges and $n = 4$ nodes, each (column times row) is $(4 \times 1)(1 \times 4) = 4 \times 4$. Write down those three column-times-row matrices and add to get $L = A^{\mathrm{T}}A$.

Solution Suppose the 3 tree edges go out of node 1 to nodes $2, 3, 4$. (The problem allows to choose other trees, including a line of 4 nodes.) Then

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \text{ sum of (columns of} A^{\mathrm{T}})(\text{ rows of } A)$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}.$$

12 A graph with two separate 3-node trees is *not connected*. Write its 6 by 4 incidence matrix A. Find *two* solutions to $A\boldsymbol{v} = \mathbf{0}$, not just one solution $\boldsymbol{v} = (1, 1, 1, 1, 1, 1)$. To reduce $A^{\mathrm{T}}A$ we must ground *two* nodes and remove two rows and columns.

Solution The incidence matrix for two 3-node trees is

$$A = \begin{bmatrix} A_{\text{tree}} & 0 \\ 0 & A_{\text{tree}} \end{bmatrix} \quad \text{with} \quad A_{\text{tree}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{(for example)}$$

The columns of A_{tree} add to zero so we have 2 independent solutions to $A\boldsymbol{v} = \mathbf{0}$:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{come from} \quad A_{\text{tree}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

13 "Element matrices" from column times row appear in the **finite element method**. Include the numbers c_1, c_2, c_3 in the element matrices K_1, K_1, K_3.

$$K_i = (\text{row } i \text{ of } A)^{\mathrm{T}} \ (\boldsymbol{c_i}) \ (\text{row } i \text{ of } A) \qquad K = A^{\mathrm{T}}CA = \boldsymbol{K_1 + K_2 + K_3}.$$

Write the element matrices that add to $A^{\mathrm{T}}A$ in (1) for the 4-node line graph.

$$A^{\mathrm{T}}A = \begin{bmatrix} \begin{bmatrix} K_1 & \\ & \end{bmatrix} & & \\ & \begin{bmatrix} K_2 & \\ & \end{bmatrix} & \\ & & \begin{bmatrix} K_3 \end{bmatrix} \end{bmatrix} = \begin{array}{l} \text{assembly of the nonzero} \\ \text{entries of } K_1 + K_2 + K_3 \\ \text{from edges } 1, 2, \text{ and } 3 \end{array}$$

Solution The three "element matrices" for the three edges come from multiplying the three columns of A^{T} by the three rows of A. Then $A^{\mathrm{T}}A$ equals

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}.$$

When the diagonal matrix C is included, those are multiplied by c_1, c_2, and c_3. Those products produce 2 by 2 blocks of nonzeros in 4×4 matrices :

$$K_1 = c_1 \begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} \quad K_2 = c_2 \begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix} \quad K_3 = c_3 \begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

Then $\boldsymbol{A^{\mathrm{T}}CA = K_1 + K_2 + K_3}$. This 'assembly" of the element stiffness matrices just requires placing the nonzeros correctly into the final matrix $A^{\mathrm{T}}CA$.

14 An n by n grid has n^2 nodes. How many edges in this graph ? How many interior nodes ? How many nonzeros in A and in $L = A^{\mathrm{T}}A$? *There are no zeros in* L^{-1} !

Solution An n by n grid has n horizontal rows ($n-1$ edges on each row) and n vertical columns ($n - 1$ edges down each column). Altogether $\boldsymbol{2n(n-1)}$ edges. There are

$\boldsymbol{(n-2)^2}$ interior nodes—a square grid with the boundary nodes removed to reduce n to $n-2$.

Every edge produces 2 nonzeros (-1 and $+1$) in A. Then A has $\boldsymbol{4n(n-1)}$ nonzeros. The matrix $A^{\mathrm{T}}A$ has size n^2 with n^2 diagonal nonzeros—and off the diagonal of $A^{\mathrm{T}}A$ there are two -1's for each edge: altogether $n^2+4n(n-1)=\boldsymbol{5n^2-4n}$ nonzeros out of n^4 entries. For $n=2$, this means 12 nonzeros in a 4 by 4 matrix.

15 When only $\boldsymbol{e}=C^{-1}\boldsymbol{w}$ is eliminated from the 3-step framework, equation (**??**) shows

$$\textbf{Saddle-point matrix}\quad \textbf{Not positive definite}\qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first block row by $A^{\mathrm{T}}C$ and subtract from the second block row:

$$\textbf{After block elimination}\qquad \begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

After m positive pivots from C^{-1}, why does this matrix have negative pivots? The two-field problem for $\boldsymbol{w}$ and $\boldsymbol{v}$ is finding a saddle point, not a minimum.

Solution The three equations $e=b-Av$ and $w=Ce$ and $A^{\mathrm{T}}w=f$ reduce to two equations when e is replaced by $C^{-1}w$:

$$\begin{aligned} C^{-1}\boldsymbol{w}&=\boldsymbol{b}-Av \\ A^{\mathrm{T}}\boldsymbol{w}&=\boldsymbol{f} \end{aligned}\qquad \text{become}\qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first equation by $A^{\mathrm{T}}C$ to get $A^{\mathrm{T}}\boldsymbol{w}=A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}$. Subtract from the second equation $A^{\mathrm{T}}\boldsymbol{w}=\boldsymbol{f}$, to eliminate w:

$$A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}=\boldsymbol{f}.$$

This gives the second row of the block matrix after elimination:

$$\begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

The pivots of that matrix on the left side start with $1/c_1, 1/c_2, \ldots, 1/c_m$. Then we get the n pivots of $-A^{\mathrm{T}}CA$ which are **negative**, because this matrix is negative definite.

Altogether we are finding a saddle point $(\boldsymbol{v},\boldsymbol{w})$ of the energy (quadratic function). The derivative of that quadratic gives our linear equations. The block matrix in those equations has m positive eigenvalues and n negative eigenvalues.

16 The least squares equation $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$ comes from the projection equation $A^{\mathrm{T}}\boldsymbol{e}=\boldsymbol{0}$ for the error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$. Write those two equations in the symmetric saddle point form of Problem 7 (with $\boldsymbol{f}=\boldsymbol{0}$).

In this case $\boldsymbol{w}=\boldsymbol{e}$ because the weighting matrix is $C=I$.

Solution Ordinary least squares for $A\boldsymbol{v}=\boldsymbol{b}$ separates the data vector $\boldsymbol{b}$ in two perpendicular parts:

$$\boldsymbol{b}=(A\widehat{\boldsymbol{v}})+(\boldsymbol{b}-A\widehat{\boldsymbol{v}})=(\text{projection of }\boldsymbol{b})+(\text{error in }\boldsymbol{b}).$$

The error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$ satisfies $A^{\mathrm{T}}\boldsymbol{e}=A^{\mathrm{T}}\boldsymbol{b}-A^{\mathrm{T}}A\boldsymbol{v}=\boldsymbol{0}$ (which means that $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$, the key equation). That equation $d^{\mathrm{T}}\boldsymbol{e}=\boldsymbol{0}$ is Kirchhoff's Current Law for flows in a network. It is a candidate for the "most important equation in applied mathematics"—the conservation equation or continuity equation "flow in = flow out."

In the form of Problem 15 (with $C = I$) the equations are

$$\begin{bmatrix} I & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \quad \text{or} \quad \begin{matrix} \boldsymbol{e} + A\boldsymbol{v} = \boldsymbol{b} \\ A^{\mathrm{T}}\boldsymbol{e} \quad\;\; = \boldsymbol{0}\,. \end{matrix}$$

17 Find the three eigenvalues and three pivots and the determinant of this saddle point matrix with $C = I$. One eigenvalue is negative because A has one column:

$$m = 2, n = 1 \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution The eigenvalues come from $\det(M - \lambda I) = 0$:

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2 - 2(1-\lambda) = 0\,.$$

Then $(1-\lambda)(\lambda^2 - \lambda - 2) = 0$ and $(1-\lambda)(\lambda - 2)(\lambda + 1) = 0$ and the eigenvalues are $\lambda = \mathbf{1}, \mathbf{2}, \mathbf{-1}$. Check the sum $1 + 2 - 1 = \mathbf{2}$ equal to the trace (sum down the main diagonal $1 + 1 + 0 = \mathbf{2}$).

The determinant is the product $\lambda_1\lambda_2\lambda_3 = (1)(2)(-1) = -2$. Notice $m = 2$ positive λ's and $n = 1$ negative eigenvalue.

Elimination finds the three pivots (which also multiply to give $\det M = -2$):

$$\begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{-2} \end{bmatrix}.$$

Problem Set 8.1, page 443

1 (a) To prove that $\cos nx$ is orthogonal to $\cos kx$ when $k \neq n$, use $(\cos nx)\ (\cos kx) = \frac{1}{2}\cos\,(n+k)x + \frac{1}{2}\cos\,(n-k)x$. Integrate from $x=0$ to $x=\pi$. What is $\int \cos^2 kx\,dx$?

(b) **Correction** From 0 to π, $\boldsymbol{\cos x}$ **is not orthogonal to** $\boldsymbol{\sin 2x}$ (the book wrongly proposed $\int_0^\pi \cos x\ \sin x\,dx$, but this is zero). For orthogonality of **all** sines and cosines, the period has to be 2π.

Solution (a)

$$\int_0^\pi (\cos nx)(\cos kx)dx = \frac{1}{2}\int_0^\pi \cos(n+k)x\,dx + \frac{1}{2}\int_0^\pi \cos(n-k)x\,dx$$

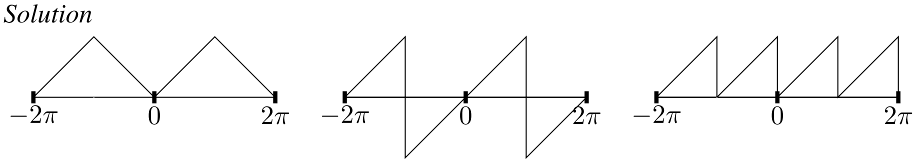
$$= \left[\frac{\sin(n+k)x}{2(n+k)} + \frac{\sin(n-k)x}{2(n-k)}\right]_0^\pi = 0+0$$

Solution (b) $$\int_0^\pi (\cos x)(\sin 2x)\,dx = \int_0^\pi (\cos x)(2\sin x\cos x)\,dx = \left[-\frac{2}{3}\cos^3 x\right]_0^\pi = \frac{\mathbf{4}}{\mathbf{3}} \neq 0.$$

Non-orthogonality comes from $\int_0^\pi \cos mx\ \sin nx\,dx$ when $m-n$ is an odd number.

2 Suppose $F(x) = x$ for $0 \leq x \leq \pi$. Draw graphs for $-2\pi \leq x \leq 2\pi$ to show three extensions of F: a 2π-periodic even function and a 2π-periodic odd function and a π-periodic function.

Solution



3 Find the Fourier series on $-\pi \leq x \leq \pi$ for

(a) $f_1(x) = \sin^3 x$, an odd function (sine series, only two terms)

Solution (a) The fast way is to know the identity $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$. This must be the Fourier sine series! It has only two terms.

More slowly, use Euler's great formula to produce complex exponentials :

$$(\sin x)^3 = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 = \frac{e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}}{8i^3} = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x.$$

Or slowly compute the usual formulas $\int \sin^3 x\sin x\,dx$ and $\int \sin^3 x\sin 3x\,dx$.

(b) $f_2(x) = |\sin x|$, an even function (cosine series)

Solution (b)

$$a_0 = \frac{1}{\pi}\int_0^\pi |\sin x|\,dx = \frac{2}{\pi}$$

$$a_k = \frac{1}{2\pi}\int_0^\pi |\sin x|\,\cos\,kx\,dx = -\frac{1}{4\pi}\left[\frac{\cos(k-1)x}{k-1} + \frac{\cos(k+1)x}{k+1}\right]_{x=0}^{x=\pi}$$

$$= \mathbf{0}\ (\text{odd } k) \text{ or } -\frac{1}{4\pi}\left[\frac{-2}{k-1} + \frac{-2}{k+1}\right] = \boldsymbol{\frac{k}{\pi(k^2-1)}}\ (\text{even } k)$$

(c) $f_3(x) = x$ for $-\pi \le x \le \pi$ (sine series with jump at $x = \pi$)

Solution (c) $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\sin\,kx\,dx = \left[\frac{1}{\pi\,k^2}\sin\,kx - \frac{x}{\pi\,k}\cos\,kx\right]_{-\pi}^{\pi}$

$$= -\frac{1}{k}(\cos\,k\pi + \cos(-k\pi)) = -\frac{2}{k}(-1)^k.$$

4 Find the complex Fourier series $e^x = \sum c_k e^{ikx}$ on the interval $-\pi \le x \le \pi$. The even part of a function is $\frac{1}{2}(f(x) + f(-x))$, so that $f_{\text{even}}(x) = f_{\text{even}}(-x)$. Find the cosine series for f_{even} and the sine series for f_{odd}. *Notice the jump at* $x = \pi$.

Solution

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-ik)}\,dx$$

$$= \left[\frac{1}{2\pi(1-ik)}e^{x(1-ik)}\right]_{-\pi}^{\pi} = \frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2\pi(1-ik)}$$

The even part of the function is : $\frac{1}{2}(e^x + e^{-x})$. The cosine coefficients are

$$a_0 = \frac{1}{4\pi}\int_{-\pi}^{\pi}(e^x + e^{-x})\,dx = \frac{1}{2\pi}(e^\pi - e^{-\pi})$$

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^x + e^{-x})\cos kx\,dx = \frac{2k\cosh[\pi]\sin[k\pi] + 2\cos[k\pi]\sinh[\pi]}{\pi + k^2\pi}$$

The odd part of the function is: $\frac{1}{2}(e^x - e^{-x})$. The sine series is:

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^x - e^{-x})\sin kx\,dx = \frac{2\cosh[\pi]\sin[k\pi] - 2k\cos[k\pi]\sinh[\pi]}{\pi + k^2\pi}$$

5 From the energy formula (21), the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi}|SW(x)|^2\,dx = \int_{-\pi}^{\pi}1\,dx = 2\pi.$$

Substitute the numbers b_k from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

Solution The sine coefficients for the odd square wave are

$$b_k = \frac{4}{\pi}\left(\frac{1-(-1)^k}{2k}\right) = \frac{4}{\pi}\left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right)$$

Energy identity gives $\boldsymbol{\pi^2} = 8\sum_{k=1}^{\infty}\left(\frac{1-(-1)^k}{2k}\right)^2 = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right)$

6 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \quad f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos kx\, dx = \frac{2}{k\pi}\sin\frac{k\pi}{2} = \sin\, c\left(\frac{k\pi}{2}\right)$$

$$b_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin kx\, dx = 0$$

7 Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is its second derivative?

Solution The parabola $y = x(\pi - x) = x\pi - x^2$ starts at $y(0) = 0$ with slope $y'(0) = \pi$ and second derivative $y''(0) = -2$. Its sine series makes it an odd function $x\pi + x^2$ from $-\pi$ to 0. This odd extension has **second derivative** $= \boldsymbol{\pm 2}$. That jump in y'' means that the Fourier coefficients b_k will decay like $1/k^3$. (Remember $1/k$ for jumps in $y(x)$ and $1/k^2$ for jumps in $y'(x)$—no jumps in y, y' for this example.)

8 Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

Solution The function is not odd or even, so integrals must go from $-\pi$ to π. The function is zero from $-\pi$ to 0 leaving only these integrals for a_0, a_k, b_k :

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{2\pi}\left[-\cos x\right]_0^{\pi} = \boldsymbol{\frac{1}{\pi}}$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sin x \cos kx\,dx = -\frac{1}{2\pi}\left[\frac{\cos(1-k)x}{1-k} + \frac{\cos(1+k)x}{1+k}\right]_0^{\pi} =$$

$$[k \text{ even}]\ \frac{1}{\pi}\left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \boldsymbol{\frac{2}{\pi(1-k^2)}} \quad [\text{and } \mathbf{0} \text{ for k odd}]$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \sin x \sin kx\,dx \text{ gives } b_1 = \frac{1}{2} \text{ and other } b_k = 0.$$

9 Suppose $G(x)$ has period $2L$ instead of 2π. Then $G(x + 2L) = G(x)$. Integrals go from $-L$ to L or from 0 to $2L$. The Fourier formulas change by a factor π/L:

The coefficients in $G(x) = \sum_{-\infty}^{\infty} \boldsymbol{C}_k e^{ik\pi x/L}$ are $\boldsymbol{C}_k = \frac{1}{2L}\int_{-L}^{L} G(x)e^{-ik\pi x/L}dx.$

Derive this formula for C_k: Multiply the first equation for $G(x)$ by _____ and integrate both sides. Why is the integral on the right side equal to $2LC_k$?

Solution Multiply $G(x) = \sum_{-\infty}^{\infty} C_k e^{ik\pi x/L}$ by $e^{-ik\pi x/L}$. Integrate.

$$\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx = \int_{-L}^{L} e^{-ik\pi x/L}\sum_{-\infty}^{\infty} C_k e^{ik\pi x/L}\,dx$$

$$\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx = C_k \int_{-L}^{L} dx = 2LC_k \text{ (orthogonality)}$$

$$C_k = \frac{1}{2L}\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx$$

10 For G_{even}, use Problem 9 to find the cosine coefficient A_k from $(C_k + C_{-k})/2$:

$G_{\text{even}}(x) = \sum_{0}^{\infty} A_k \cos\frac{k\pi x}{L}$ has $A_k = \frac{1}{L}\int_0^{L} G_{\text{even}}(x) \cos\frac{k\pi x}{L}dx.$

G_{even} is $\frac{1}{2}(G(x) + G(-x))$. Exception for $A_0 = C_0$: Divide by $2L$ instead of L.

Solution The result comes directly from $\frac{1}{2}(C_k + C_{-k})$.

11 Problem 10 tells us that $\boldsymbol{a_k = \frac{1}{2}(c_k + c_{-k})}$ on the usual interval from 0 to π. Find a similar formula for b_k from c_k and c_{-k}. In the reverse direction, find the complex coefficient c_k in $F(x) = \sum c_k e^{ikx}$ from the real coefficients a_k and b_k.

Solution **Solution and correction** We are comparing two ways to write a Fourier series :

$$\sum_{-\infty}^{\infty} c_k e^{ikx} = a_0 + \sum_{1}^{\infty} a_k \cos kx + \sum_{1}^{\infty} b_k \sin kx$$

Pick out the terms for k and $-k$:

$$c_k e^{ikx} + c_{-k} e^{-ikx} = a_k \cos kx + b_k \sin kx$$

Use Euler's formula to reach cosines/sines on both sides :

$$(c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx = a_k \cos kx + b_k \sin kx$$

This shows that $\boldsymbol{a_k = c_k + c_{-k}}$ (**correction from text**) and $\boldsymbol{b_k = i(c_k - c_{-k})}$.

Reverse Euler's formula to reach complex exponentials on both sides :

$$c_k e^{ikx} + c_{-k} e^{-ikx} = \frac{1}{2} a_k (e^{ikx} + e^{-ikx}) + \frac{1}{2i} b_k (e^{ikx} - e^{-ikx})$$

This shows that $\boldsymbol{c_k = \frac{1}{2} a_k + \frac{1}{2i} b_k}$ and $\boldsymbol{c_{-k} = \frac{1}{2} a_k - \frac{1}{2i} b_k}$.

Real functions with real a's and b's lead to $c_{-k} = \overline{c_k}$ (complex conjugates)

12 Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3 \cdots) = 2 \log(1 + z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2 \log(1 + z)$ agrees with θ.

Solution The sine series of the odd function $f(\theta) = \theta$ has coefficients $b_n =$

$$\frac{2}{\pi} \int_0^{\pi} \theta \sin n\theta \, d\theta = \frac{2}{\pi} \left[\frac{1}{n^2} \sin n\theta - \frac{\theta}{n} \cos n\theta \right]_0^{\pi} = -\frac{2 \cos n\pi}{n} = \mathbf{2} \left[\frac{\mathbf{1}}{\mathbf{1}}, -\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{3}}, -\frac{\mathbf{1}}{\mathbf{4}}, \cdots \right]$$

The solution to Laplace's equation inside the circle has factors r^n :

$$u(r, \theta) = \sum \boldsymbol{b_n r^n \sin n\theta} = 2r \sin \theta - \frac{2}{2} r^2 \sin 2\theta + \frac{2}{3} r^3 \sin 3\theta \ldots$$

$$= \text{Im} \left[2z - \frac{2}{2} z^2 + \frac{2}{3} z^3 \ldots \right] = \text{Im}[2 \log(1 + z)].$$

13 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin $r = 0$?

Solution This 0-1 step function $u_0(\theta)$ equals $\frac{1}{2} + \frac{1}{2}$ (square wave). Equation (8) of the text gives the Fourier sine series for the square wave :

$$\text{0-1 Step Function} \quad u_0(\theta) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{\sin \theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots \right]$$

Then the solution to Laplace's equation includes factors r^n :

$$u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{r \sin \theta}{1} + \frac{r^3 \sin 3\theta}{3} + \frac{r^5 \sin 5\theta}{5} + \cdots \right] = \frac{\mathbf{1}}{\mathbf{2}} \quad \textbf{at} \quad \boldsymbol{r = 0}.$$

14 With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum this geometric series.

Solution Inside the circle we see factors r^n (and $1 + x + x^2 + \cdots = 1/(1-x)$) :

$$u(r,\theta) = 1 + \frac{1}{2}re^{i\theta} + \frac{1}{4}r^2e^{2i\theta} + \cdots = \mathbf{1}/\left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{re^{i\theta}}\right).$$

15 (a) Verify that the fraction in Poisson's formula (30) satisfies Laplace's equation.

Solution (a) We could verify Laplace's equation in r, θ coordinates or recognize that every term in the sum (29) solves that equation :

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(b) Find the response $u(r,\theta)$ to an impulse at $x = 0, y = 1$ (where $\theta = \frac{\pi}{2}$).

Solution (b) When the source is at the point $\theta = \pi$, this replaces $r\cos\theta$ by $-r\cos\theta$ in equation (30). Then the response to a point source is infinite at $r = 1, \theta = \pi$:

$$u(r,\theta) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}} \quad \frac{\mathbf{1} - \boldsymbol{r^2}}{\mathbf{1} + \boldsymbol{r^2} + \mathbf{2}\boldsymbol{r}\cos\boldsymbol{\theta}}$$

16 With complex exponentials in $F(x) = \sum c_k e^{ikx}$, the energy identity (21) changes to $\int_{-\pi}^{\pi} |F(x)|^2\,dx = 2\pi \sum |c_k|^2$. Derive this by integrating $(\sum c_k e^{ikx})(\sum \overline{c}_k e^{-ikx})$.

Solution All products $e^{ikx}e^{-ikx}$ integrate to zero except when $n = k$:

$$\int_{-\pi}^{\pi} (c_k e^{ikx})(\overline{c_k} e^{-ikx})\,dx = 2\pi c_k \overline{c_k} = 2\pi |c_k|^2.$$

The total energy is the sum over all k.

17 A centered square wave has $F(x) = 1$ for $|x| \le \pi/2$.

(a) Find its energy $\int |F(x)|^2\,dx$ by direct integration

Solution (a) $\displaystyle\int |F(x)|^2\,dx = \int_{-\pi/2}^{\pi/2} dx = \boldsymbol{\pi}.$

(b) Compute its Fourier coefficients c_k as specific numbers

Solution (b)
$$c_k = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\,dx = \left[\frac{1}{2\pi}\,\frac{e^{-ikx}}{-ik}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi\, ik}\left(e^{ik\pi/2} - e^{-ik\pi/2}\right) = \frac{\mathbf{1}}{\boldsymbol{\pi\, k}}\,\mathbf{sin}\left(\frac{\boldsymbol{k\pi}}{\mathbf{2}}\right)$$

(c) Find the sum in the energy identity (Problem 8).

Solution (c) $\sin\dfrac{k\,\pi}{2} = 1, 0, -1, 0$ (repeated) so $2\pi \sum |c_k|^2 = \dfrac{2}{\pi}\left(\dfrac{1}{1} + \dfrac{1}{9} + \dfrac{1}{25} + \cdots\right) = 1.$

18 $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is analytic : infinitely smooth.

(a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

(b) Does that series still converge quickly ? Compare n^{10} with 2^n for $n = 2^{10}$.

Solution (a) 10 derivatives of $\cos\, nx$ gives $-n^{10}\cos\, nx$:

$$\frac{d^{10}F}{dx^{10}} = -\frac{1}{2}\cos x - \frac{2^{10}}{2^2}\cos\, 2x - \frac{3^{10}}{2^3}\cos\, 3x \cdots - \frac{n^{10}}{2^n}\cos\, nx - \cdots$$

Solution (b) Yes, 2^n gets large much faster than n^{10} so the series easily converges.

At $n = 2^{10} = 1024$ we have $2^n = 2^{1024}$, much larger than $n^{10} = 2^{100}$.

19 If $f(x) = 1$ for $|x| \le \pi/2$ and $f(x) = 0$ for $\pi/2 < |x| < \pi$, find its cosine coefficients. Can you graph and compute the Gibbs overshoot at the jumps ?

Solution $a_0 =$ **average value** $= \mathbf{\frac{1}{2}}$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\, kx\, dx = \left[\frac{1}{\pi\, k}\sin\, kx\right]_{-\pi/2}^{\pi/2} = \boldsymbol{\frac{2}{\pi\, k}\sin\frac{k\pi}{2}}$$

20 Find all the coefficients a_k and b_k for F, I, and D on the interval $-\pi \le x \le \pi$:

$$F(x) = \delta\left(x - \frac{\pi}{2}\right) \qquad I(x) = \int_0^x \delta\left(x - \frac{\pi}{2}\right) dx \qquad D(x) = \frac{d}{dx}\delta\left(x - \frac{\pi}{2}\right).$$

Solution (a) Integrate $\cos\, kx$ and $\sin\, kx$ against $\delta(x - \frac{\pi}{2})$ to get

$$a_0 = \boldsymbol{\frac{1}{2\pi}} \qquad a_k = \boldsymbol{\frac{1}{\pi}\cos\frac{k\pi}{2}} \text{ and } b_k = \boldsymbol{\frac{1}{\pi}\sin\frac{k\pi}{2}}$$

Solution (b) The integral $I(x)$ is the unit step function $H(x - \frac{\pi}{2})$ with jump at $x = \frac{\pi}{2}$:

$$a_0 = \frac{1}{2\pi}\int_{\pi/2}^{\pi} 1\, dx = \frac{1}{4}$$

$$a_k = \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos\, kx\, dx = \frac{1}{\pi\, k}\left(\sin\, k\pi - \sin\frac{k\pi}{2}\right) = -\frac{1}{\pi\, k}\sin\frac{k\pi}{2}$$

$$b_k = \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin\, kx\, dx = -\frac{1}{\pi\, k}\left(\cos\, k\pi - \cos\frac{k\pi}{2}\right)$$

Solution (c) $D(x)$ is the "doublet" = derivative of the delta function $\delta\left(x - \frac{\pi}{2}\right)$. You must integrate by parts (and $D(-\pi) = D(\pi) = 0$ fortunately).

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D(x)\cos\, kx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta\left(x - \frac{\pi}{2}\right)(k\sin\, kx)\, dx$$

So a_k for $D(x)$ is $\boldsymbol{kb_k}$ **in part (b)**, and b_k for $D(x)$ is $\boldsymbol{-ka_k}$ **in part (b)**.

21 For the one-sided tall box function in Example 4, with $F = 1/h$ for $0 \le x \le h$, what is its odd part $\frac{1}{2}(F(x) - F(-x))$? I am surprised that the Fourier coefficients of this odd part disappear as h approaches zero and $F(x)$ approaches $\delta(x)$.

Solution Every function has an even part and an odd part :

$$F_{\text{even}}(x) = \frac{1}{2}(F(x) + F(-x)) \quad F_{\text{odd}}(x) = \frac{1}{2}(F(x) - F(-x)) \quad F = F_{\text{even}} + F_{\text{odd}}$$

For the one-sided box function, those even and odd parts are

$$F_{\text{even}}(x) = \frac{1}{2h} \text{ for } |x| \le h \quad F_{\text{odd}}(x) = -\frac{1}{h} \text{ for } -h \le x \le 0, +\frac{1}{h} \text{ for } 0 < x \le h.$$

The Fourier coefficients of F_{odd} don't really "disappear" as $h \to 0$, because the energy $\int |F_{\text{odd}}|^2\,dx$ is growing. But it is growing in the high frequencies and any particular coefficient c_k (at a fixed frequency k) approaches zero as $h \to 0$.

22 Find the series $F(x) = \sum c_k e^{ikx}$ for $F(x) = e^x$ on $-\pi \le x \le \pi$. That function e^x looks smooth, but there must be a hidden jump to get coefficients c_k proportional to $1/k$. Where is the jump ?

Solution When e^x is made into a periodic function there is a jump (or a drop) at $x = \pi$. The drop from e^π to $e^{-\pi}$ starts the next 2π-interval. That drop shows up as a factor multiplying the $1/k$ decay that all jump functions show in their Fourier expansion :

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\,dx = \left[\frac{1}{2\pi}\,\frac{e^{(1-ik)x}}{1-ik}\right]_{x=-\pi}^{\pi}$$
$$= \frac{1}{2\pi}\,\frac{\boldsymbol{e^{\pi} - e^{-\pi}}}{1-ik}.$$

23 (a) (Old particular solution) Solve $Ay'' + By' + Cy = e^{ikx}$.

(b) (New particular solution) Solve $Ay'' + By' + Cy = \sum c_k e^{ikx}$.

Solution This problem shows directly the power of **linearity** to deal with complicated forcing functions as combinations of simple forcing functions e^{ikx} :

$$Ay'' + By' + Cy = e^{ikx} \quad \text{has} \quad y_p = \frac{1}{(ik)^2A + ikB + C}\,e^{ikx} = Y_k e^{ikx}$$
$$Ay'' + By' + Cy = \sum c_k e^{ikx} \quad \text{has} \quad y_p = \sum c_k Y_k e^{ikx}.$$

Problem Set 8.2, page 453

1 Multiply the three matrices in equation (11) and compare with F. In which six entries do you need to know that $i^2 = -1$? This is $(w_4)^2 = w_2$. If $M = N/2$, why is $(w_N)^M = -1$?

Solution

2 *Why is row i of $\overline{F}$ the same as row $N - i$ of F* (numbered from 0 to $N - 1$)?

Solution

3 From Problem 8, find the 4 by 4 permutation matrix P so that $F = P\overline{F}$. Check that $P^2 = I$ so that $P = P^{-1}$. Then from $\overline{F}F = 4I$ show that $F^2 = 4P$.

It is amazing that $F^4 = 16P^2 = 16I$. *Four transforms of any* $\boldsymbol{c}$ *bring back* $\mathbf{16}\,\boldsymbol{c}$. For all N, F^2/N is a permutation matrix P and $\boldsymbol{F^4 = N^2 I}$.

Solution

4 Invert the three factors in equation (11) to find a fast factorization of F^{-1}.

5 F is symmetric. Transpose equation (11) to find a new Fast Fourier Transform.

Solution

6 All entries in the factorization of F_6 involve powers of $w =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these factors with $1, w, w^2$ in D and powers of w^2 in F_3. Multiply!

Solution

7 Put the vector $\boldsymbol{c} = (1, 0, 1, 0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0, 1, 0, 1)$.

Solution

8 Compute $\boldsymbol{y} = F_8\boldsymbol{c}$ by the three FFT steps for $\boldsymbol{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $\boldsymbol{c} = (0, 1, 0, 1, 0, 1, 0, 1)$.

Solution

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

Solution

10 F is a symmetric matrix. Its eigenvalues aren't real. How is this possible?

Solution

The three great symmetric tridiagonal matrices of applied mathematics are K**,** B**,** C. The eigenvectors of K, B, and C are discrete **sines**, **cosines**, and **exponentials**. The eigenvector matrices give the **DST**, **DCT**, and **DFT** — discrete transforms for signal processing. Notice that diagonals of the circulant matrix C loop around to the far corners.

$$\boldsymbol{K} = \begin{bmatrix} \mathbf{2} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{2} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 2 & -1 & \cdot & \mathbf{-1} \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ \mathbf{-1} & \cdot & -1 & 2 \end{bmatrix} \qquad \begin{aligned} K_{11} &= K_{NN} = \mathbf{2} \\ B_{11} &= B_{NN} = \mathbf{1} \\ C_{1N} &= C_{N1} = \mathbf{-1} \end{aligned}$$

11 The eigenvectors of K_N and B_N are the discrete sines $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N$ and the discrete cosines $\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{N-1}$. Notice the eigenvector $\boldsymbol{c}_0 = (1, 1, \ldots, 1)$. Here are $\boldsymbol{s}_k$ and $\boldsymbol{c}_k$—these vectors are samples of $\sin kx$ and $\cos kx$ from 0 to π.

$$\left(\sin\frac{\pi k}{N+1}, \sin\frac{2\pi k}{N+1}, \ldots, \sin\frac{N\pi k}{N+1}\right) \text{ and } \left(\cos\frac{\pi k}{2N}, \cos\frac{3\pi k}{2N}, \ldots, \cos\frac{(2N-1)\pi k}{2N}\right)$$

For 2 by 2 matrices K_2 and B_2, verify that $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ are eigenvectors.

Solution

12 Show that C_3 has eigenvalues $\lambda = 0, 3, 3$ with eigenvectors $\boldsymbol{e}_0 = (1, 1, 1)$, $\boldsymbol{e}_1 = (1, w, w^2)$, $\boldsymbol{e}_2 = (1, w^2, w^4)$. You may prefer the real eigenvectors $(1, 1, 1)$ and $(1, 0, -1)$ and $(1, -2, 1)$.

Solution

13 Multiply to see the eigenvectors $\boldsymbol{e}_k$ and eigenvalues λ_k of C_N. Simplify to $\lambda_k = 2 - 2\cos(2\pi k/N)$. Explain why C_N is only semidefinite. It is not positive definite.

$$C\boldsymbol{e}_k = \begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ -1 & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix} = (2 - w^k - w^{-k}) \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix}.$$

Solution

14 The eigenvectors $\boldsymbol{e}_k$ of C are automatically perpendicular because C is a _____ matrix. (To tell the truth, C has repeated eigenvalues as in Problem 12. There was a plane of eigenvectors for $\lambda = 3$ and we chose orthogonal $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in that plane.)

Solution

15 Write the 2 eigenvalues for K_2 and the 3 eigenvalues for B_3. Always K_N and B_{N+1} have the same N eigenvalues, with the extra eigenvalue _____ for B_{N+1}. (This is because $K = A^{\mathrm{T}}A$ and $B = AA^{\mathrm{T}}$.)

Solution

Problem Set 8.5, page 477

1 When the driving function is $f(t) = \delta(t)$, the solution starting from rest is the **impulse response**. The impulse is $\delta(t)$, the response is $y(t)$. Transform this equation to find the **transfer function** $Y(s)$. Invert to find the impulse response $y(t)$.

$$y'' + y = \delta(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution Take the Laplace Transform of $y'' + y = \delta(t)$ with $y(0) = y'(0) = 0$:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 1$$

$$Y(s)(s^2 + 1) = 1$$

$$Y(s) = \frac{1}{s^2 + 1} \text{ is the transform of } y(t) = \boldsymbol{\sin t}.$$

2 (Important) Find the first derivative and second derivative of $f(t) = \sin t$ for $t \geq 0$. Watch for a jump at $t = 0$ which produces a spike (delta function) in the derivative.

Solution The first derivative of $\sin(t)$ is $\cos(t)$, and the second derivative is $-\sin(t) + \delta(t)$.

3 Find the Laplace transform of the unit box function $b(t) = \{1 \text{ for } 0 \leq t < 1\} = H(t) - H(t-1)$. The unit step function is $H(t)$ in honor of Oliver Heaviside.

Solution The unit box function is $f(t) = H(t) - H(t-1)$

The transform is $F(s) = \dfrac{1}{s} - \dfrac{e^{-s}}{s} = \boldsymbol{\dfrac{1}{s}(1 - e^{-s})}$

The same result comes from $F(s) = \displaystyle\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^1 e^{-st}\, dt.$

4 If the Fourier transform of $f(t)$ is defined by $\widehat{f}(k) = \int f(t) e^{-ikt} dt$ and $f(t) = 0$ for $t < 0$, what is the connection between $\widehat{f}(k)$ and the Laplace transform $F(s)$?

Solution The Fourier Transform is the Laplace Transform with $s = ik$: $\boldsymbol{\widehat{f}(k) = F(ik)}$.

5 What is the Laplace transform $R(s)$ of the standard **ramp function** $r(t) = t$? For $t < 0$ all functions are zero. The derivative of $r(t)$ is the unit step $H(t)$. Then multiplying $R(s)$ by s gives _____.

Solution The Laplace Transform $R(s)$ of the Ramp Function $r(t) = t$ is

$$R(s) = \int_0^\infty te^{-st}\, dt = -\frac{te^{-st}}{s}\bigg|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\, dt = 0 - \frac{e^{-st}}{s^2}\bigg|_0^\infty = \frac{1}{s^2}$$

Multiplying $R(s)$ by s gives the Laplace transform $1/s$ of the step function.

6 Find the Laplace transform $F(s)$ of each $f(t)$, and the poles of $F(s)$:

(a) $f = 1 + t$ (b) $f = t \cos \omega t$ (c) $f = \cos(\omega t - \theta)$
(d) $f = \cos^2 t$ (e) $f = e^{-2t} \cos t$ (f) $f = te^{-t} \sin \omega t$

Solution (a) The transform of $f(t) = 1 + t$ has a **double pole** at $s = 0$:

$$F(s) = \int_0^\infty (1+t) e^{-st}\, dt = \int_0^\infty e^{-st}\, dt + \int_0^\infty te^{-st}\, dt = \frac{1}{s} + \frac{1}{s^2} = \boldsymbol{\frac{1+s}{s^2}}$$

Solution (b)

$$f(t) = t\cos(\omega t) = t\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) = \frac{te^{i\omega t}}{2} + \frac{te^{-i\omega t}}{2} \quad \text{transforms to}$$

$$\begin{aligned} F(s) &= \int_0^\infty \frac{te^{(i\omega - s)t}}{2}\, dt + \int_0^\infty \frac{te^{-(i\omega - s)t}}{2}\, dt \\ &= \frac{-e^{-t(s - i\omega)}(st - it\omega + 1)}{2(s - i\omega)^2}\bigg|_0^\infty + \frac{-e^{-t(s+i\omega)}(st + it\omega + 1)}{2(s + i\omega)^2}\bigg|_0^\infty \\ &= \frac{1}{2(s - i\omega)^2} + \frac{1}{2(s + i\omega)^2} = \frac{(s - i\omega)^2 + (s + i\omega)^2}{2(s - i\omega)^2(s + i\omega)^2} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} \end{aligned}$$

Poles occur at $s = i\omega$ and $s = -i\omega$, the two exponents of $f(t)$.

Solution (c) $f(t) = \cos(\omega t - \theta) = \cos\omega t\cos\theta + \sin\omega t \sin\theta$ transforms to

$$F(s) = \frac{s}{s^2+\omega^2}\cos\theta + \frac{\omega}{s^2+\omega^2}\sin\theta$$

Poles occur at $s = \pm i\omega$.

Solution (d)

$$f(t) = \cos^2(t) = \frac{1}{4}(e^{it}+e^{-it})^2 = \frac{1}{4}(e^{2it}+2+e^{-2it})$$

$$F(s) = \int_0^\infty \frac{1}{4}(e^{2it}+e^{-2it}+2)e^{-st}\,dt$$

$$= -\frac{1}{4(s-2i)} + \frac{1}{4(s+2i)} + \frac{1}{2s} = \frac{2s}{4(s^2+4)} + \frac{1}{2s} = \frac{s^2+2}{s(s^2+4)}$$

Poles occur at $s=0$ and $s=\pm 2i$. Another way is to write $\cos^2 t = \dfrac{1+\cos 2t}{2}$

Solution (e)

$$f(t) = e^{-2t}\cos t = \frac{1}{2}e^{(i-2)t} + \frac{1}{2}e^{-(i+2)t}$$

$$F(s) = \int_0^\infty \frac{1}{2}e^{(i-2)t}e^{-st}\,dt + \int_0^\infty \frac{1}{2}e^{-(i+2)t}e^{-st}\,dt$$

$$= \frac{1}{2(-i+2+s)} + \frac{1}{2(i+2+s)} = \frac{s+2}{(s+2)^2+1}$$

Poles occur at the exponents $s=-2\pm i$ in $f(t)$.

Solution (f)

$$f(t) = te^{-t}\sin\omega t = \frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t}$$

$$F(s) = \int_0^\infty \left(\frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t}\right)e^{-st}\,dt$$

$$= \int_0^\infty \frac{t}{2i}e^{(i\omega-1-s)t}\,dt - \int_0^\infty \frac{t}{2i}e^{-(i\omega+1+s)t}\,dt$$

$$= \left.\frac{ie^{-t(s-i\omega+1)}(1+t(s-i\omega+1))}{2(s-i\omega+1)^2} - \frac{ie^{-t(s+i\omega+1)}(1+t(s+i\omega+1))}{2(s+i\omega+1)^2}\right|_0^\infty$$

Poles of $F(s)$ occur at $s = \boldsymbol{-1 \pm i\omega}$, the exponents of $f(t)$.

7 Find the Laplace transform s of $f(t) =$ next integer above t and $f(t) = t\,\delta(t)$.

A staircase $f(t) = [t] = H(t) + H(t-1) + H(t-2) + \cdots =$ next integer above t is a sum of step functions. The transform is

$$\frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots = \frac{1}{s}(1+e^{-s}+e^{-2s}+\cdots) = \frac{1}{s}\left(\frac{1}{1-e^{-s}}\right).$$

The differentiation rule $\mathcal{L}(tf(t)) = -F'(s)$ with $f(t)=\delta(t)$ and $F(s)=1$ gives

$$\mathcal{L}(t\delta(t)) = -\frac{d}{ds}(1) = \mathbf{0} \text{ (this is correct because } t\delta(t) \text{ is the zero function).}$$

8 *Inverse Laplace Transform*: Find the function $f(t)$ from its transform $F(s)$:

(a) $\dfrac{1}{s-2\pi i}$ (b) $\dfrac{s+1}{s^2+1}$ (c) $\dfrac{1}{(s-1)(s-2)}$

(d) $1/(s^2+2s+10)$ (e) $e^{-s}/(s-a)$ (f) $2s$

Solution (a) $F(s) = \dfrac{1}{s-2\pi i}$ is the transform of $f(t) = \boldsymbol{e^{2\pi i t}}$.

Solution (b) $F(s) = \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$ is the transform of $f(t) = \mathbf{cos}\,\boldsymbol{t} + \mathbf{sin}\,\boldsymbol{t}$.

Solution (c) $F(s) = \dfrac{1}{(s-1)(s-2)} = \dfrac{1}{s-2} - \dfrac{1}{s-1}$ is the transform of $f(t) = \boldsymbol{e^{2t} - e^{t}}$.

Solution (d)

$$\begin{aligned} F(s) &= \frac{1}{s^2+2s+10} = \frac{1}{(s+1+3i)(s+1-3i)} \\ &= \frac{i}{6(s+(1+3i))} - \frac{i}{6(s+(1-3i))} \\ f(t) &= \frac{i}{6}e^{-(1+3i)t} - \frac{i}{6}e^{-(1-3i)t} \\ &= -\boldsymbol{\frac{e^{-t}\sin(3t)}{3}} \end{aligned}$$

Solution (e)

$$\begin{aligned} F(s) &= \frac{e^{-s}}{s-a} \\ f(t) &= e^{a(t-1)}H(t-1) = \textbf{ shift of } \boldsymbol{e^{at}} \end{aligned}$$

Solution (f)

$$\begin{aligned} F(s) &= 2s \\ f(t) &= \boldsymbol{2\, d\delta/dt} \end{aligned}$$

9 Solve $y''+y=0$ from $y(0)$ and $y'(0)$ by expressing $Y(s)$ as a combination of $s/(s^2+1)$ and $1/(s^2+1)$. Find the inverse transform $y(t)$ from the table.

Solution

$$\begin{aligned} y'' + y &= 0 \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= 0 \\ Y(s)(s^2+1) &= sy(0) + y'(0) \\ Y(s) &= y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1} \end{aligned}$$

The inverse transform is $y(t) = \boldsymbol{y(0)\cos(t) + y'(0)\sin(t)}$.

10 Solve $y'' + 3y' + 2y = \delta$ starting from $y(0) = 0$ and $y'(0) = 1$ by Laplace transform. Find the poles and partial fractions for $Y(s)$ and invert to find $y(t)$.

Solution The transform of $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = \delta(t)$ with $y(0) = 0$ and $y'(0) = 1$ is

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) &= 1 \\ Y(s)(s^2+3s+2) - 1 &= 1 \\ Y(s) &= \frac{2}{(s+1)(s+2)} \\ Y(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\ y(t) &= \boldsymbol{2e^{-t} - 2e^{-2t}} \end{aligned}$$

11 Solve these initial-value problems by Laplace transform :

(a) $y' + y = e^{i\omega t}, y(0) = 8$ (b) $y'' - y = e^t,\ y(0) = 0,\ y'(0) = 0$
(c) $y' + y = e^{-t}, y(0) = 2$ (d) $y'' + y = 6t,\ y(0) = 0,\ y'(0) = 0$
(e) $y' - i\omega y = \delta(t), y(0) = 0$ (f) $my'' + cy' + ky = 0,\ y(0) = 1, y'(0) = 0$

Solution (a)

$$\begin{aligned} y' + y &= e^{i\omega t} \text{ with } y(0) = 8 \\ sY(s) - 8 + Y(s) &= \frac{1}{s - i\omega} \\ Y(s)(s+1) &= \frac{1}{s-i\omega} + 8 \\ Y(s) &= \frac{1}{(s+1)(s-i\omega)} + \frac{8}{s+1} \\ Y(s) &= \frac{1}{1+i\omega}\left(\frac{1}{s-i\omega} - \frac{1}{s+1}\right) + \frac{8}{s+1} \\ \text{Particular} \ + \ \text{null} \ y(t) &= \boldsymbol{\frac{1}{1+i\omega}\left(e^{i\omega t} - e^{-t}\right) + 8e^{-t}} \end{aligned}$$

Solution (b)

$$\begin{aligned} y'' - y &= e^t \text{ with } y(0) = 0 \text{ and } y'(0) = 0 \\ s^2Y(s) - Y(s) &= \frac{1}{s-1} \\ Y(s) &= \frac{1}{(s-1)(s+1)(s-1)} \\ &= \frac{1}{4(s+1)} - \frac{1}{4(s-1)} + \frac{1}{2(s-1)^2} \\ y(t) &= \boldsymbol{\frac{e^{-t}}{4} - \frac{e^t}{4} + \frac{te^t}{2}} \end{aligned}$$

Solution (c)

$$\begin{aligned} y' + y &= e^{-t} \text{ with } y(0) = 2 \\ sY(s) - 2 + Y(s) &= \frac{1}{s+1} \\ Y(s) &= \frac{1}{(s+1)^2} + \frac{2}{s+1} \\ y(t) &= \boldsymbol{te^{-t} + 2e^{-t}} \end{aligned}$$

Solution (d)

$$\begin{aligned}
y'' + y &= 6t \text{ with } y(0) = y'(0) = 0\\
s^2Y(s) + Y(s) &= \frac{6}{s^2}\\
Y(s)(s^2+1) &= \frac{6}{s^2}\\
Y(s) &= \frac{6}{s^2} - \frac{3i}{s+i} + \frac{3i}{s-i}\\
y(t) &= 6t - 3ie^{-it} + 3ie^{it} = \boldsymbol{6t - 6\sin t}
\end{aligned}$$

Solution (e)

$$\begin{aligned}
y' - i\omega y &= \delta(t) \text{ with } y(0) = 0\\
sY(s) - i\omega Y(s) &= 1\\
Y(s) &= \frac{1}{s - i\omega}\\
y(t) &= \boldsymbol{e^{i\omega t}}
\end{aligned}$$

Solution (f) $my'' + cy' + ky = 0$ with $y(0) = 1$ and $y'(0) = 0$

$ms^2Y(s) - msy(0) + csY(s) - cy(0) + kY(s) = 0$

$Y(s)(ms^2 + cs + k) = ms + c$

$Y(s) = \dfrac{ms + c}{ms^2 + cs + k}$ has the form $\dfrac{a}{s - s_1} + \dfrac{b}{s - s_2}$

We used this *Mathematica* command to find $y(t)$

Simplify[InverseLaplaceTransform$[(m * s\ +\ c)/(m * s\hat{\ }2\ +\ c * s\ +\ k),\ s,\ t]]$

$$y(t) = \frac{e^{-\frac{\left(c+\sqrt{c^2-4km}\right)t}{2m}}\left(c\left(-1+e^{\frac{\sqrt{c^2-4kmt}}{m}}\right)+\left(1+e^{\frac{\sqrt{c^2-4kmt}}{m}}\right)\sqrt{c^2-4km}\right)}{2\sqrt{c^2-4km}}$$

12 The transform of e^{At} is $(sI - A)^{-1}$. Compute that matrix (the transfer function) when $A = [1\ \ 1;\ 1\ \ 1]$. Compare the poles of the transform to the eigenvalues of A.

Solution When $A = [1\ \ 1; 1\ \ 1]$ we have:

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}^{-1} = \frac{\mathbf{1}}{\boldsymbol{s^2 - 2s}}\begin{bmatrix} \boldsymbol{s-1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{s-1} \end{bmatrix}.$$

The poles of the system are $s = 2$ and $s = 0$, the eigenvalues of A.

13 If dy/dt decays exponentially, show that $sY(s) \to y(0)$ as $s \to \infty$.

Solution

$$\begin{aligned}
sY(s) &= \int_0^\infty se^{-st}y(t)\,dt \quad \textit{(integrate by parts)}\\
&= \int_0^\infty e^{-st}\frac{dy}{dt}\,dt - \left[e^{-st}y(t)\right]_0^\infty\\
&= \int_0^\infty e^{-st}\frac{dy}{dt}\,dt + y(0) \to y(0) \text{ as } s \to \infty
\end{aligned}$$

$$\text{Example: } \frac{dy}{dt} = e^{-at} \text{ has } sY(s) - y(0) = \frac{1}{s+a} \to 0 \text{ as } s \to \infty$$

14 Transform Bessel's time-varying equation $ty''+y'+ty=0$ using $\mathscr{L}[ty]=-dY/ds$ to find a first-order equation for Y. By separating variables or by substituting $Y(s)=C/\sqrt{1+s^2}$, find the Laplace transform of the Bessel function $y=J_0$.

Solution The transform of ty'' applies the $\mathscr{L}(t,y)$ rule to y'' instead of y:

$$\mathscr{L}(t,y'')=-\frac{d}{ds}(\text{transform of} y'') \;=-\frac{d}{ds}(s^2Y(s)-sy(0)-y'(0)).$$

Apply this to the transform of $t\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+ty=0$

$$-2sY(s)-s^2\frac{dY}{ds}+y(0)+sY(s)-y(0)-\frac{dY}{ds}=0$$

$$-sY(s)-s^2\frac{dY}{ds}-\frac{dY}{ds}=0$$

$$sY(s)=-(s^2+1)\frac{dY}{ds}$$

$$\frac{dY}{Y(s)}=-\frac{s\,ds}{s^2+1}$$

$$\log Y(s)=\log\left(\frac{1}{\sqrt{s^2+1}}\right)$$

The transform of the Bessel solution $y=J_0$ is Y(s)=$\boldsymbol{\dfrac{1}{\sqrt{s^2+1}}}$

15 Find the Laplace transform of a single arch of $f(t)=\sin\pi t$.

Solution A single arch of $\sin\pi t$ extends from $t=0$ to $t=1$:

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^1\sin(\pi t)e^{-st}dt=\int_0^1\frac{e^{i\pi t-st}}{2i}dt-\int_0^1\frac{e^{-i\pi t-st}}{2i}dt$$

$$=\left[\frac{e^{i\pi t-st}}{2i(i\pi-s)}+\frac{e^{-i\pi t-st}}{2i(i\pi+s)}\right]_{t=0}^{t=1}$$

$$=\frac{e^{i\pi-s}-1}{2i(i\pi-s)}+\frac{e^{-i\pi-s}-1}{2i(i\pi+s)}$$

$$=\left(\frac{-e^{-s}-1}{2i}\right)\left(\frac{1}{i\pi-s}-\frac{1}{i\pi+s}\right)=\left(\frac{e^{-s}+1}{i}\right)\left(\frac{s}{\pi^2+s^2}\right)$$

A faster and more direct approach: One arch of the sine curve agrees with $\sin\pi t+$ unit shift of $\sin\pi t$, because those cancel after one arch.

$$\sin\pi t+\sin\pi(t-1)=\sin\pi t+\sin\pi t\cos\pi=\sin\pi t-\sin\pi t=0.$$

16 Your acceleration $v'=c(v^*-v)$ depends on the velocity v^* of the car ahead:

(a) Find the ratio of Laplace transforms $V^*(s)/V(s)$.

(b) If that car has $v^*=t$ find your velocity $v(t)$ starting from $v(0)=0$.

Solution (a) Take the Laplace Transform of $\dfrac{dv}{dt}=c(v^*-v)$ assuming $v(0)=0$;

$$\begin{aligned} sV(s) - v(0) &= cV^*(s) - cV(s) \\ V(s)(s+c) &= cV^*(s) \\ \frac{V^*(s)}{V(s)} &= \boldsymbol{\frac{s+c}{c}} \end{aligned}$$

Solution (b) If $v^*(t) = t$ then $V^*(s) = \dfrac{1}{s^2}$. Therefore

$$\begin{aligned} V(s)(s+c) &= \frac{c}{s^2} \\ V(s) &= \frac{c}{s^3 + cs^2} \\ &= \frac{1}{c(s+c)} - \frac{1}{cs} + \frac{1}{s^2} \\ v(t) &= \boldsymbol{\frac{e^{-ct}}{c} - \frac{1}{c} + t} \end{aligned}$$

17 A line of cars has $v_n' = c[v_{n-1}(t-T) - v_n(t-T)]$ with $v_0(t) = \cos \omega t$ in front.

(a) Find the growth factor $A = 1/(1 + i\omega e^{i\omega T}/c)$ in oscillation $v_n = A^n e^{i\omega t}$.

(b) Show that $|A| < 1$ and the amplitudes are safely decreasing if $cT < \frac{1}{2}$.

(c) If $cT > \frac{1}{2}$ show that $|A| > 1$ (dangerous) for small ω. (Use $\sin\theta < \theta$.)

Human reaction time is $T \geq 1\,\text{sec}$ and human aggressiveness is $c = 0.4/\text{sec}$.

Danger is pretty close. Probably drivers adjust to be barely safe.

Solution (a) $\dfrac{dv_n}{dt} = c(v_{n-1}(t-T) - v_n(t-T))$ with $v_n = A^n e^{i\omega t}$

$$\begin{aligned} i\omega A^n e^{i\omega t} &= cA^{n-1} e^{i\omega(t-T)} - cA^n e^{i\omega(t-T)} \\ A\frac{i\omega e^{i\omega T}}{c} &= 1 - A \\ A\left(1 + \frac{i\omega e^{i\omega T}}{c}\right) &= 1 \end{aligned}$$

Solution (b)

$$\text{For } |A| < 1 \text{ we need } \left|1 + \frac{i\omega}{c} e^{i\omega T}\right| > 1$$

$$\left|1 - \frac{\omega}{c}\sin(\omega T) + \frac{\omega}{c}\cos(\omega T)\right| > 1$$

$$\left(1 - \frac{\omega}{c}\sin(\omega T)\right)^2 + \frac{\omega^2}{c^2}\cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c}\sin(\omega T) + \frac{\omega^2}{c^2}\sin^2(\omega T) + \frac{\omega^2}{c^2}\cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c}\sin(\omega T) + \frac{\omega^2}{c^2} > 1$$

$$\frac{\omega^2}{c^2} > \frac{2\omega}{c}\sin(\omega T)$$

$$\text{Since } \sin\omega T < \omega T, \text{ we are safe if } \frac{\omega^2}{c^2} > \frac{2\omega}{c}\omega T \text{ which is } cT < \frac{1}{2}$$

Solution (c) $\sin\omega T \approx \omega T$ when this number is small. Then the same steps show $|A| > 1$ when $cT > \frac{1}{2}$.

18 For $f(t) = \delta(t)$, the transform $F(s) = 1$ is the limit of transforms of tall thin box functions $b(t)$. The boxes have width $\epsilon \to 0$ and height $1/\epsilon$ and area 1.

$$\text{Inside integrals, } b(t) = \left\{\begin{array}{ll} 1/\epsilon & \text{for } 0 \le t < \epsilon \\ 0 & \text{otherwise} \end{array}\right\} \text{approaches } \delta(t).$$

Find the transform $B(s)$, depending on ϵ. Compute the limit of $B(s)$ as $\epsilon \to 0$.

Solution We begin by finding the transform of the box:

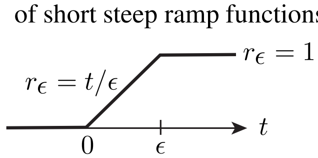
$$B(s) = \int_0^\epsilon \frac{1}{\epsilon} e^{-st}\,dt = \left.\frac{-1}{s\epsilon} e^{-st}\right|_0^\epsilon = \frac{1 - e^{-s\epsilon}}{s\epsilon}$$

We take the limit as $\epsilon \to 0$—the box approaches a delta function!

$$B_\epsilon(s) = \lim_{\epsilon\to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon}$$

$$= \lim \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{s\epsilon} = \mathbf{1}.$$

19 The transform $1/s$ of the unit step function $H(t)$ comes from the limit of the transforms of short steep ramp functions $r_\epsilon(t)$. These ramps have slope $1/\epsilon$:



Compute $R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st}dt + \int_\epsilon^\infty e^{-st}dt$. Let $\epsilon \to 0$.

Solution
$$R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st}\,dt + \int_\epsilon^\infty e^{-st}\,dt = \left[\frac{e^{-st}(-st-1)}{\epsilon s^2}\right]_{t=0}^{t=\epsilon} + \left[\frac{e^{-st}}{-s}\right]_{t=\epsilon}^{t=\infty}$$

$$= \frac{e^{-s\epsilon}(-s\epsilon - 1) + 1}{\epsilon s^2} + \frac{e^{-s\epsilon}}{s} = \boldsymbol{\frac{1 - e^{-s\epsilon}}{\epsilon s^2}}$$

$$\lim R_\epsilon(s) = \lim \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{\epsilon s^2} = \boldsymbol{\frac{1}{s}}.$$

20 In Problems 18 and 19, show that the derivative of the ramp function $r_\epsilon(t)$ is the box function $b(t)$. The "generalized derivative" of a step is the _____ function.

Solution The generalized derivative of the short ramp $r_\epsilon(t)$ is the thin box $b(t)/\epsilon$. We say "generalized" because this is not a true derivative at $t = 0$: the ramp has zero slope left of $t = 0$ and nonzero slope right of $t = 0$. But the transforms of r_ϵ and b_ϵ follow the rule for derivatives.

The generalized derivative of a step function is a **delta** function.

21 What is the Laplace transform of $y'''(t)$ when you are given $Y(s)$ and $y(0), y'(0), y''(0)$?

Solution The Laplace Transform of $y'''(t)$ is $s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

22 The *Pontryagin maximum principle* says that the optimal control is "bang-bang"— it only takes on the extreme values permitted by the constraints. To go from rest at $x = 0$ to rest at $x = 1$ in minimum time, use maximum acceleration A and deceleration $-B$. At what time t do you change from the accelerator to the brake ? (This is the fastest driving between two red lights.)

Solution The maximum principle requires full acceleration A to an unknown time t_0 and then full deceleration $-B$ to reach $x = 1$ with zero velocity. The velocities are

$$v = At \quad \text{for} \quad t \le t_0$$
$$v = At_0 - B(t - t_0) \quad \text{for} \quad t > t_0$$

Integrating the velocity $v = dx/dt$ gives the distance $x(t)$:

$$x = \tfrac{1}{2}At^2 \quad \text{for} \quad t < t_0$$
$$x = \tfrac{1}{2}At_0^2 \quad \text{at} \quad t = t_0$$
$$x = \tfrac{1}{2}At_0^2 + At_0(t - t_0) - \tfrac{1}{2}B(t - t_0)^2 \quad \text{for} \quad t > t_0$$

At the final time T we reach $x = 1$ with velocity $v = 0$. This gives two equations for t_0 and T :

$$v = At_0 - B(T - t_0) = 0$$
$$x = At_0T - \tfrac{1}{2}At_0^2 - \tfrac{1}{2}B(T - t_0)^2 = 1$$

Substitute $T = \frac{1}{B}t_0(A + B)$ from the first equation into the second equation. This leaves an ordinary quadratic equation to solve for t_0.

Problem Set 8.6, page 453

1 Find the convolution $\boldsymbol{v} * \boldsymbol{w}$ and also the cyclic convolution $\boldsymbol{v} \circledast \boldsymbol{w}$:

(a) $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (2, 1)$

Solution (a)

Convolution : $(1, 2) * (2, 1)$ $$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

Cyclic Convolution : $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

(b) $\boldsymbol{v} = (1,2,3)$ and $\boldsymbol{w} = (4,5,6)$.

Solution (b)

$$(1,2,3) * (4,5,6) \qquad \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}$$

$$\text{Cyclic Convolution:} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}$$

2 Compute the convolution $(1,3,1) * (2,2,3) = (a,b,c,d,e)$. To check your answer, add $a+b+c+d+e$. That total should be 35 since $1+3+1=5$ and $2+2+3=7$ and $5 \times 7 = 35$.

Solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 11 \\ 11 \\ 3 \end{bmatrix}$$

$1+3+1$ times $2+2+3$ is $2+8+11+11+3$: $(5)(7) = (35)$.

3 Multiply $1+3x+x^2$ times $2+2x+3x^2$ to find $a+bx+cx^2+dx^3+ex^4$. Your multiplication was the same as the convolution $(1,3,1)*(2,2,3)$ in Problem 8. When $x=1$, your multiplication shows why $1+3+1=5$ times $2+2+3=7$ agrees with $a+b+c+d+e=35$.

Solution

$$(1+3x+x^2) \times (2+2x+3x^2) = 2+2x+3x^2+6x+6x^2+9x^3+2x^2+2x^3+3x^4$$
$$= \mathbf{2+8x+11x^2+11x^3+3x^4}$$

At $x=1$ this is again $(5) \times (7) = (35)$.

4 (Deconvolution) Which vector $\boldsymbol{v}$ would you convolve with $\boldsymbol{w} = (1,2,3)$ to get $\boldsymbol{v} * \boldsymbol{w} = (0,1,2,3,0)$? Which $\boldsymbol{v}$ gives $\boldsymbol{v} \circledast \boldsymbol{w} = (3,1,2)$?

Solution

$$\begin{bmatrix} v_0 & 0 & 0 \\ v_1 & v_0 & 0 \\ v_2 & v_1 & v_0 \\ 0 & v_2 & v_1 \\ 0 & 0 & v_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

The first and last equation give $v_0 = v_2 = 0$. Substituting into the second, third, fourth equation gives $v_1 = 1$. Therefore $v = (\mathbf{0,1,0})$.

$$\text{For cyclic convolution} \quad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_0 & v_2 & v_1 \\ v_1 & v_0 & v_2 \\ v_2 & v_1 & v_0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{gives} \quad \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

5 (a) For the periodic functions $f(x) = 4$ and $g(x) = 2\cos x$, show that $f * g$ is **zero** (the zero function) !

Solution (a) From equation (4) we have

$$(f * g)(x) = \int_0^{2\pi} g(y) f(x-y)\, dy = 4 \int_0^{2\pi} 2\cos y\, dy = 4 \cdot 0 = 0 \text{ for all } x.$$

(b) In frequency space (k-space) you are multiplying the Fourier coefficients of 4 and $2\cos x$. Those coefficients are $c_0 = 4$ and $d_1 = d_{-1} = 1$. Therefore every product $c_k d_k$ is _____ .

Solution (b) In frequency space (k-space) you are multiplying the Fourier coefficients of 4 and $2\cos x$. Those coefficients are $c_0 = 4$ and $d_1 = d_{-1} = 1$. **Therefore every product $\boldsymbol{c_k d_k}$ is zero**. These are the coefficients of the zero function.

6 For periodic functions $f = \sum c_k e^{ikx}$ and $g = \sum d_k e^{ikx}$, the Fourier coefficients of $f * g$ are $\boldsymbol{2\pi c_k d_k}$. Test this factor 2π when $f(x) = 1$ and $g(x) = 1$ by computing $f * g$ from its definition (6.4).

Solution From equation (4) :

$$(f * g)(x) = \int_0^{2\pi} f(y) g(x-y)\, dy = \int_0^{2\pi} 1 \cdot 1\, dy = 2\pi.$$

The same convolution in k-space has $c_0 = 1$ and $d_0 = 1$ (all other $c_k = d_k = 0$). Then $2\pi c_k d_k$ gives the correct coefficients (2π and 0) of the convolution $f * g$ (which equals 2π).

7 Show by integration that the periodic convolution $\int_0^{2\pi} \cos x \cos(t-x) dx$ is $\pi \cos t$. In k-space you are squaring Fourier coefficients $c_1 = c_{-1} = \frac{1}{2}$ to get $\frac{1}{4}$ and $\frac{1}{4}$; these are the coefficients of $\frac{1}{2}\cos t$. The 2π in Problem 8 makes $\pi \cos t$ correct.

Solution

$$\int_0^{2\pi} \cos x \cos(t-x)\, dx = \int_0^{2\pi} \cos x(\cos t \cos x + \sin t \sin x)\, dx = \pi \cos t + 0.$$

8 Explain why $f * g$ is the same as $g * f$ (periodic or infinite convolution).

Solution In Fourier space convolution $f * g$ or $f \circledast g$ leads to multiplication $c_k d_k$, which is certainly the same as $d_k c_k$. So $f \circledast g = g \circledast f$ in x-space.

9 What 3 by 3 circulant matrix C produces cyclic convolution with the vector $\boldsymbol{c} = (1, 2, 3)$? Then $C\boldsymbol{d}$ equals $\boldsymbol{c} \circledast \boldsymbol{d}$ for every vector $\boldsymbol{d}$. Compute $\boldsymbol{c} \circledast \boldsymbol{d}$ for $\boldsymbol{d} = (0, 1, 0)$.

Solution The circulant matrix $C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ gives cyclic convolution with $(1, 2, 3)$.

When $d = (0, 1, 0)$ we have $\boldsymbol{c} \circledast \boldsymbol{d} = C\boldsymbol{d} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

10 What 2 by 2 circulant matrix C produces cyclic convolution with $\boldsymbol{c} = (1,1)$? Show in four ways that this C is not invertible. Deconvolution is impossible.

(1) Find the determinant of C. (2) Find the eigenvalues of C.

(3) Find $\boldsymbol{d}$ so that $C\boldsymbol{d} = \boldsymbol{c} \circledast \boldsymbol{d}$ is zero. (4) $F\boldsymbol{c}$ has a zero component.

Solution The 2 by 2 circulant matrix $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ gives $(1,1) \circledast \boldsymbol{d} = C\boldsymbol{d}$.

(1) The determinant of this matrix is zero.

(2) The eigenvalues of C come from $\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = 0$. Then $(1-\lambda)^2 = 1$ and $\lambda = 0, 2$. That zero eigenvalue means that the matrix C is singular.

(3) $C\boldsymbol{d} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so C is not invertible: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in nullspace.

(4) The Fourier matrix F gives $F\boldsymbol{c} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This again shows $\lambda = 2$ and 0.

11 (a) Change $b(x) * \delta(x-1)$ to a multiplication $\widehat{b}(k)\, \widehat{d}(k)$:

The box $b(x) = \{1 \text{ for } 0 \le x \le 1\}$ transforms to $\widehat{b}(k) = \int_0^1 e^{-ikx} dx$.

The shifted delta transforms to $\widehat{d}(k) = \int \delta(x-1) e^{-ikx} dx$.

(b) Show that your result $\widehat{b}\,\widehat{d}$ is the transform of a shifted box function. This shows how convolution with $\delta(x-1)$ shifts the box.

Solution This question shows that continuous convolution with $\delta(x-1)$ produces a shift in the box function $b(x)$, just like discrete convolution with the shifted delta vector $(\ldots, 0, 0, 1, \ldots)$ produces a one-step shift.

We compute $\delta(x-1) * b(x)$ in x-space to find $b(x-1)$, or in k-space to see the effect on the coefficients:

$$\widehat{b}(k) = \int_0^1 e^{-ikx} dx = \left[\frac{e^{-ikx}}{-ik}\right]_{x=0}^{x=1} = \frac{1-e^{-ik}}{ik}$$

$$\textbf{Shifted box} \quad e^{-ik}\left(\frac{1-e^{-ik}}{ik}\right) \quad \text{agrees with} \quad \int_1^2 e^{-ikx} dx = \left[\frac{e^{-ikx}}{-ik}\right]_{x=1}^{x=2}.$$

12 Take the Laplace transform of these equations to find the transfer function $G(s)$:

(a) $Ay'' + By' + Cy = \delta(t)$ (b) $y' - 5y = \delta(t)$ (c) $2y(t) - y(t-1) = \delta(t)$

Solution (a) $As^2Y(s) + BsY(s) + CY(s) = 1$ gives the transfer function $\dfrac{1}{As^2 + Bs + C}$

Solution (b) $sY(s) - 5Y(s) = 1$ gives the transfer function $Y(s) = \dfrac{1}{s-5}$

Solution (c) $2Y(s) - Y(s)e^{-s} = 1$ gives the transfer function $Y(s) = \dfrac{1}{2 - e^{-s}}$

13 Take the Laplace transform of $y'''' = \delta(t)$ to find $Y(s)$. From the Transform Table in Section 8.5 find $y(t)$. You will see $y''' = 1$ and $y'''' = 0$. But $y(t) = 0$ for negative t, so your y''' is actually a unit step function and your y'''' is actually $\delta(t)$.

Solution $y'''' = \delta$ transforms to $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = 1$

Assume zero initial values to get $s^4Y(s) = 1$ and $Y(s) = \dfrac{1}{s^4}$ and $y^3 = \dfrac{t^3}{6}$.

This is also the solution to $y'''' = 0$ with initial values $y, y', y'', y''' = \mathbf{0, 0, 0, 1}$.

14 Solve these equations by Laplace transform to find $Y(s)$. Invert that transform with the Table in Section 8.5 to recognize $y(t)$.

(a) $y' - 6y = e^{-t}$, $y(0) = 2$ (b) $y'' + 9y = 1$, $y(0) = y'(0) = 0$.

Solution (a) The transform of $y' - 6y = e^{-t}$ with $y(0) = 2$ is

$$\begin{aligned} sY(s) - 2 - 6Y(s) &= \frac{1}{s+1} \\ Y(s) &= \frac{2}{s-6} + \frac{1}{(s+1)(s-6)} \\ &= \frac{2}{s-6} + \frac{1}{7(s-6)} - \frac{1}{7(s+1)} \\ &= \frac{15}{7(s-6)} - \frac{1}{7(s+1)} \end{aligned}$$

The inverse transform is $y(t) = \boldsymbol{\dfrac{15}{7}e^{6t} - \dfrac{1}{7}e^{-t}}$

Solution (b) The transform of $y'' + 9y = 1$ with $y(0) = y'(0) = 0$ is

$$\begin{aligned} s^2Y(s) + 9Y(s) &= \frac{1}{s} \\ Y(s) &= \frac{1}{s(s^2+9)} \\ &= \frac{1}{9s} - \frac{1}{18(-3i+s)} - \frac{1}{18(3i+s)} \end{aligned}$$

The inverse transform is $y(t) = \dfrac{1}{9} - \dfrac{1}{18}e^{3it} - \dfrac{1}{18}e^{-3it} = \boldsymbol{y_p + y_n}$.

15 Find the Laplace transform of the shifted step $H(t-3)$ that jumps from 0 to 1 at $t = 3$. Solve $y' - ay = H(t-3)$ with $y(0) = 0$ by finding the Laplace transform $Y(s)$ and then its inverse transform $y(t)$: one part for $t < 3$, second part for $t \geq 3$.

Solution The transform of $H(t-3)$ multiplies e^{-3s} by the transform $\frac{1}{s}$ of $H(t)$.

$$y' - ay = H(t-3) \qquad y(0) = 0$$

$$\begin{aligned} sY(s) - aY(s) &= \frac{e^{-3s}}{s} \\ Y(s) &= \frac{e^{-3s}}{s(s-3)} = \frac{e^{-3x}}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right). \end{aligned}$$

The inverse transform $y(t)$ is the shift of $\frac{1}{3}\left(e^{-3t} - 1\right)$: zero until $t = 3$.

16 Solve $y' = 1$ with $y(0) = 4$—a trivial question. Then solve this problem the slow way by finding $Y(s)$ and inverting that transform.

Solution The trivial solution is : $y = t + 4$. The transform method gives

$$sY(s) - 4 = \frac{1}{s}$$
$$Y(s) = \frac{1}{s^2} + \frac{4}{s}$$
$$y(t) = t + 4$$

17 The solution $y(t)$ is the convolution of the input $f(t)$ with what function $g(t)$?

(a) $y' - ay = f(t)$ with $y(0) = 3$

Solution (a)

$$y' - ay = f(t) \qquad \text{with } \; y(0) = 3$$
$$sY(s) - 3 - aY(s) = F(s)$$
$$Y(s) = \frac{3 + F(s)}{s - a}$$
$$y(t) = \boldsymbol{3e^{-t} + f(t) * e^{-at}}$$

(b) $y' -$ (integral of y) $= f(t)$.

Solution (b) The transform of $y' -$ (integral of y) $= f(t)$ is $sY(s) - \dfrac{Y(s)}{s} = F(s)$, if $y(0) = 0$.

The inverse transform of $\dfrac{1}{s - \frac{1}{s}} = \dfrac{s}{s^2 - 1}$ is $\cos(it)$.

Then $Y(s) = \dfrac{F(s)}{s - \frac{1}{s}}$ is the transform of the convolution $f(t) * \cos(it)$.

18 For $y' - ay = f(t)$ with $y(0) = 3$, we could replace that initial value by adding $3\delta(t)$ to the forcing function $f(t)$. Explain that sentence.

Solution For a first order equation, an initial condition $y(0)$ is equivalent to adding $y(0)\delta(t)$ to the equation and starting that new equation at zero.

19 What is $\delta(t) * \delta(t)$? What is $\delta(t-1) * \delta(t-2)$? What is $\delta(t-1)$ *times* $\delta(t-2)$?

Solution $\delta(t) * \delta(t) = \delta(t)$

$\delta(t-1) * \delta(t-2) = \delta(t-3)$

$\delta(t-1)$ *times* $\delta(t-2)$ equals the zero function.

20 By Laplace transform, solve $y' = y$ with $y(0) = 1$ to find a very familiar $y(t)$.

Solution

$$y' = y \qquad y(0) = 1$$
$$sY(s) - 1 = Y(s)$$
$$Y(s) = \frac{1}{s - 1} \;\text{ gives }\; y(t) = \boldsymbol{e^t}.$$

21 By Fourier transform as in (9), solve $-y'' + y =$ box function $b(x)$ on $0 \le x \le 1$.

Solution The Fourier transform of $-y'' + y = b(x)$ is

$$(k^2+1)\,\widehat{y}\,(k) = \widehat{b}(k) = \int_0^1 e^{-ikx}\,dx = \frac{1-e^{-ik}}{ik}.$$

$$\widehat{y}(k) = \frac{1-e^{-ik}}{(k^2+1)(ik)}$$

This transform must be inverted to find $\boldsymbol{y(x)}$. In reality I would solve separately on $x \le 0$ and $0 \le x \le 1$ and $x \ge 1$. Then matching at the breakpoints $x = 0$ and $x = 1$ determines the free constants in the separate solutions.

22 There is a big difference in the solutions to $y'' + By' + Cy = f(x)$, between the cases $B^2 < 4C$ and $B^2 > 4C$. Solve $y'' + y = \delta$ and $y'' - y = \delta$ with $y(\pm\infty) = 0$.

Solution (a) The delta function produces a unit jump in y' at $x = 0$:

$y'' + y = 0$ has $y = c_1 \cos x + c_2 \sin x$ for $x < 0$, $y = C_1 \sin x$ for $x > 0$.

The jump in y' gives $C_2 - c_2 = 1$. The condition on $y(\pm\infty)$ does not apply to this first equation.

$y'' - y = 0$ has $y = ce^x$ for $x < 0$ and $y = Ce^{-x}$ for $x > 0$; then $y(\pm\infty) = 0$.

Matching y at $x = 0$ gives $c = C$.

Jump in y' at $x = 0$ gives $-C - c = 1$ so $c = C = -\frac{1}{2}$

Solution $y(x) = -\frac{1}{2}e^x$ for $x \le 0$ and $y(x) = -\frac{1}{2}e^{-x}$ for $x \ge 0$

23 (*Review*) Why do the constant $f(t) = 1$ and the unit step $H(t)$ have the same Laplace transform $1/s$? Answer: Because the transform does not notice _____.

Solution The Laplace Transform **does not notice any values of $\boldsymbol{f(t)}$ for $\boldsymbol{t < 0}$.**