8.1 Fourier Series

This section explains three Fourier series: **sines, cosines, and exponentials e^{ikx}**. Square waves (1 or 0 or -1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

Start with $\sin x$. It has period 2π since $\sin(x+2\pi) = \sin x$. It is an odd function since $\sin(-x) = -\sin x$, and it vanishes at $x = 0$ and $x = \pi$. Every function $\sin nx$ has those three properties, and Fourier looked at *infinite combinations of the sines*:

Fourier sine series $$S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} \boldsymbol{b_n \sin nx} \quad (1)$$

If the numbers $b_1, b_2, b_3, \ldots$ drop off quickly enough (we are foreshadowing the importance of their decay rate) then the sum $S(x)$ will inherit all three properties:

Periodic $S(x+2\pi) = S(x)$ **Odd** $S(-x) = -S(x)$ $S(0) = S(\pi) = 0$

200 years ago, Fourier startled the mathematicians in France by suggesting that *any odd periodic function* $S(x)$ could be expressed as an infinite series of sines. This idea started an enormous development of Fourier series. Our first step is to **find the number b_k that multiplies $\sin kx$. The function $S(x)$ is "transformed" to a sequence of b's.**

Suppose $S(x) = \sum b_n \sin nx$. *Multiply both sides by* $\sin kx$. *Integrate from* 0 *to* π :

$$\int_0^{\pi} S(x) \sin kx \, dx = \int_0^{\pi} b_1 \sin x \sin kx \, dx + \cdots + \int_0^{\pi} \boldsymbol{b_k \sin kx \, \sin kx \, dx} + \cdots \quad (2)$$

On the right side, all integrals are zero except the highlighted one with $n = k$. This property of "**orthogonality**" will dominate the whole chapter. For sines, integral $= 0$ is a fact of calculus:

Sines are orthogonal $$\int_0^{\pi} \boldsymbol{\sin nx \, \sin kx \, dx = 0} \quad \text{if} \quad n \neq k\,. \quad (3)$$

Zero comes quickly if we integrate $\int \cos mx \, dx = \left[\frac{\sin mx}{m}\right]_0^{\pi} = 0 - 0$. So we use this:

Product of sines $$\sin nx \, \sin kx = \frac{1}{2}\cos(n-k)x - \frac{1}{2}\cos(n+k)x\,. \quad (4)$$

Integrating $\cos(n-k)x$ and $\cos(n+k)x$ gives zero, proving orthogonality of the sines.

The exception is when $n = k$. Then we are integrating $(\sin kx)^2 = \frac{1}{2} - \frac{1}{2}\cos 2kx$:

$$\int_0^{\pi} \sin kx \, \sin kx \, dx = \int_0^{\pi} \frac{1}{2}\, dx - \int_0^{\pi} \frac{1}{2} \cos 2kx \, dx = \frac{\pi}{2}\,. \quad (5)$$

The highlighted term in equation (2) is $\boldsymbol{(\pi/2) b_k}$. Multiply both sides by $2/\pi$ to find b_k.

Sine coefficients $S(-x) = -S(x)$	$$b_k = \frac{2}{\pi}\int_0^{\pi} S(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin kx\,dx.$$	(6)

Notice that $S(x)\sin kx$ is *even* (equal integrals from $-\pi$ to 0 and from 0 to π).

I will go immediately to the most important example of a Fourier sine series. $S(x)$ is an **odd square wave** with $SW(x) = 1$ for $0 < x < \pi$. It is drawn in Figure 8.1 as an odd function (with period 2π) that vanishes at $x = 0$ and $x = \pi$.

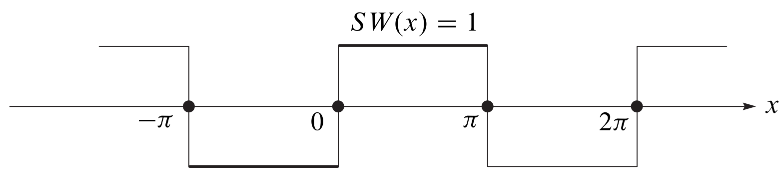

Figure 8.1: The odd square wave with $SW(x+2\pi) = SW(x) = \{1 \text{ or } 0 \text{ or } -1\}$.

Example 1 Find the Fourier sine coefficients b_k of the odd square wave $SW(x)$.

Solution For $k = 1, 2, \ldots$ use formula (6) with $S(x) = 1$ between 0 and π:

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin kx\,dx = \frac{2}{\pi}\left[\frac{-\cos kx}{k}\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{2}{1}, \frac{0}{2}, \frac{2}{3}, \frac{0}{4}, \frac{2}{5}, \frac{0}{6}, \cdots\right\} \quad (7)$$

The even-numbered coefficients b_{2k} are all zero because $\cos 2k\pi = \cos 0 = 1$. The odd-numbered coefficients $b_k = \mathbf{4/\pi k}$ decrease at the rate $1/k$. We will see that same $1/k$ decay rate for all functions formed from *smooth pieces and jumps*.

Put those coefficients $4/\pi k$ and zero into the Fourier sine series for $SW(x)$:

Square wave	$$SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]$$	(8)

Figure 8.2 graphs this sum after one term, then two terms, and then five terms. You can see the all-important **Gibbs phenomenon** appearing as these "partial sums" include more terms. Away from the jumps, we safely approach $SW(x) = 1$ or -1. At $x = \pi/2$, the series gives a beautiful alternating formula for the number π :

$$1 = \frac{4}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] \quad \text{so that} \quad \boldsymbol{\pi = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)}. \quad (9)$$

The Gibbs phenomenon is the overshoot that moves closer and closer to the jumps. Its height approaches $1.18\ldots$ and it does not decrease with more terms of the series. This overshoot is the one greatest obstacle to calculation of all discontinuous functions (like shock waves). We try hard to avoid Gibbs but sometimes we can't.

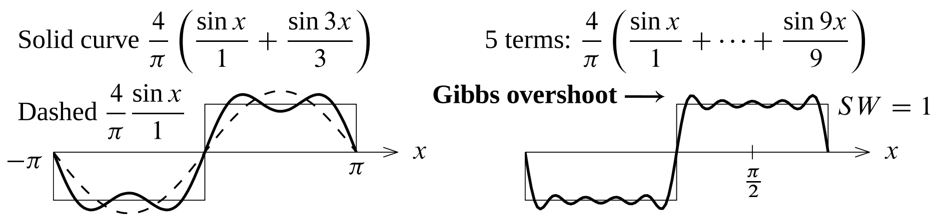

Figure 8.2: The sums $b_1 \sin x + \cdots + b_N \sin Nx$ overshoot the square wave near jumps.

Fourier Cosine Series

The cosine series applies to ***even functions*** $C(x) = C(-x)$. They are symmetric across 0 :

$$\textbf{Cosine series} \quad C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots = \boldsymbol{a_0} + \sum_{n=1}^{\infty} \boldsymbol{a_n \cos nx}. \tag{10}$$

Every cosine has period 2π. Figure 8.3 shows two even functions, the **repeating ramp** $RR(x)$ and the **up-down train** $UD(x)$ of delta functions. That sawtooth ramp RR is the integral of the square wave. The delta functions in UD give the derivative of the square wave. (For sines, the integral and derivative are cosines.) RR and UD will be valuable examples, one smoother than SW, one less smooth.

First we find formulas for the cosine coefficients a_0 and a_k. *The constant term a_0 is the average value of the function $C(x)$* :

$$\boldsymbol{a_0} = \textbf{average} \qquad \boldsymbol{a_0} = \frac{1}{\pi}\int_0^{\pi} C(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(x)\,dx. \tag{11}$$

I just integrated every term in the cosine series (10) from 0 to π. On the right side, the integral of a_0 is $a_0\pi$ (divide both sides by π). All other integrals are zero :

$$\int_0^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0 - 0 = 0. \tag{12}$$

In words, the constant function 1 is orthogonal to $\cos nx$ over the interval $[0, \pi]$.

The other cosine coefficients a_k come from the *orthogonality of cosines*. As with sines, we multiply both sides of (10) by $\cos kx$ and integrate from 0 to π:

$$\int_0^{\pi} C(x)\cos kx\,dx = \int_0^{\pi} a_0 \cos kx\,dx + \int_0^{\pi} a_1 \cos x \cos kx\,dx + \cdots + \int_0^{\pi} \boldsymbol{a_k (\cos kx)^2\,dx} + \cdots$$

You know what is coming. On the right side, only the highlighted term can be nonzero. For $k > 0$, that bold nonzero term is $\boldsymbol{a_k \pi/2}$. Multiply both sides by $2/\pi$ to find a_k :

$$\textbf{Cosine coefficients } \boldsymbol{C(-x) = C(x)} \qquad a_k = \frac{2}{\pi}\int_0^{\pi} C(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \boldsymbol{C(x)\cos kx\,dx}\,. \tag{13}$$

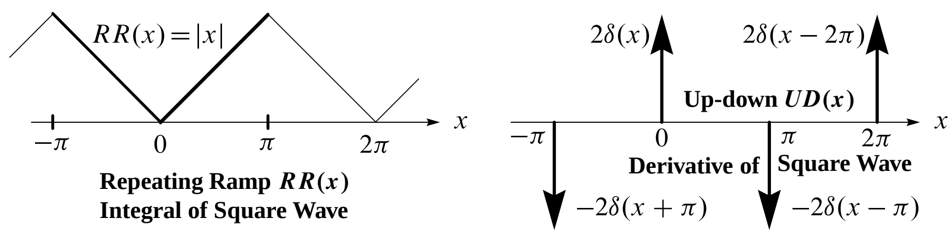

Figure 8.3: The repeating ramp RR and the up-down UD (periodic spikes) are even. The slope of RR is -1 then 1 : odd square wave SW. **The next derivative is UD** : $\pm 2\delta$.

Example 2 Find the cosine coefficients of the ramp $RR(x)$ and the up-down $UD(x)$.

Solution The simplest way is to start with the sine series for the square wave :

$$SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right] = \text{slope of } RR$$

Take the derivative of every term to produce cosines in the up-down delta function :

Up-down spikes $$\boldsymbol{UD(x)} = \frac{4}{\pi}\left[\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots\right]. \qquad (14)$$

Those coefficients don't decay at all. The terms in the series don't approach zero, so officially the series cannot converge. Nevertheless it is correct and important. At $x = 0$, the cosines are all 1 and their sum is $+\infty$. At $x = \pi$, the cosines are all -1. Then their sum is $-\infty$. (The downward spike is $-2\delta(x-\pi)$.) The true way to recognize $\delta(x)$ is by the integral test $\int \delta(x) f(x)\,dx = f(0)$ and Example 3 will do this.

For the repeating ramp, we integrate the square wave series for $SW(x)$ and add a_0. The average ramp height is $a_0 = \pi/2$, halfway from 0 to π :

Ramp series $$\boldsymbol{RR(x)} = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos x}{\mathbf{1^2}} + \frac{\cos 3x}{\mathbf{3^2}} + \frac{\cos 5x}{\mathbf{5^2}} + \frac{\cos 7x}{\mathbf{7^2}} + \cdots\right]. \qquad (15)$$

The constant of integration is a_0. *Those coefficients a_k drop off like* $1/k^2$. They could be computed directly from formula (13) using $\int x \cos kx\, dx$, and integration by parts (or an appeal to *Mathematica* or *Maple*). It was much easier to integrate every sine separately in $SW(x)$, which makes clear the crucial point**: Each "degree of smoothness" in the function brings a faster decay rate of its Fourier coefficients a_k and b_k.** Every integration divides those numbers by k.

No decay	**Delta** functions (with spikes)
$1/k$ decay	**Step** functions (with jumps)
$1/k^2$ decay	**Ramp** functions (with corners)
$1/k^4$ decay	**Spline** functions (jumps in f''')
r^k decay with $r < 1$	**Analytic** functions like $1/(2-\cos x)$

The Fourier Series for a Delta Function

Example 3 Find the (cosine) coefficients of the ***delta function*** $\boldsymbol{\delta(x)}$, made 2π-periodic.

Solution The spike in $\delta(x)$ occurs at $x = 0$. All the integrals are 1, because the cosine of 0 is 1. We divide by 2π for a_0 and by π for the other cosine coefficients a_k.

$$\textbf{Average}\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,dx = \boldsymbol{\frac{1}{2\pi}} \qquad \textbf{Cosines}\quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\cos kx\,dx = \boldsymbol{\frac{1}{\pi}}$$

Then the series for the delta function has *all cosines in equal amounts*: **No decay**.

Delta function
$$\boldsymbol{\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\cos x + \cos 2x + \cos 3x + \cdots\right].} \qquad (16)$$

This series cannot truly converge (its terms don't approach zero). But we can graph the sum after $\cos 5x$ and after $\cos 10x$. Figure 8.4 shows how these "partial sums" are doing their best to approach $\delta(x)$. They oscillate faster while going higher.

There is a neat formula for the sum δ_N that stops at $\cos Nx$. Start by writing each term $2\cos x$ as $e^{ix} + e^{-ix}$. We get a geometric progression from e^{-iNx} up to e^{iNx}.

$$\delta_N = \frac{1}{2\pi}\left[1 + e^{ix} + e^{-ix} + \cdots + e^{iNx} + e^{-iNx}\right] = \frac{1}{2\pi}\,\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}. \qquad (17)$$

This is the function graphed in Figure 8.4.

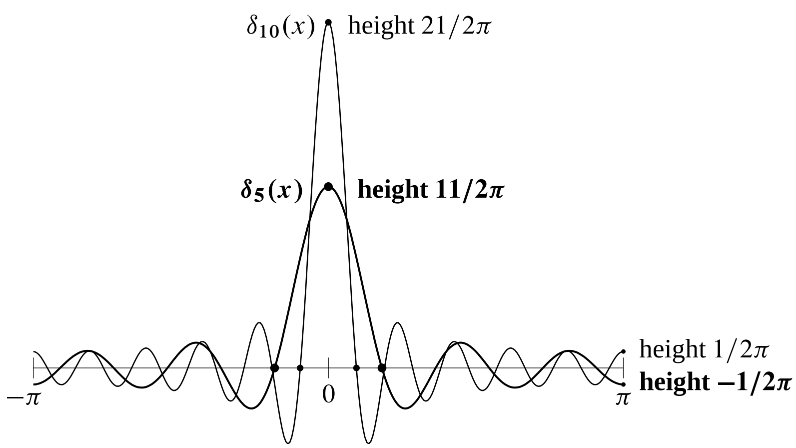

Figure 8.4: The sums $\delta_N(x) = (1 + 2\cos x + \cdots + 2\cos Nx)/2\pi$ try to approach $\delta(x)$.

Complete Series: Sines and Cosines

Over the half-period $[0, \pi]$, the sines are not orthogonal to all the cosines. In fact the integral of $\sin x$ times 1 is not zero. So for functions $F(x)$ that are not odd or even, we must move to the *complete series* (*sines plus cosines*) on the full interval. Since our functions are periodic, that "full interval" can be $[-\pi, \pi]$ or $[0, 2\pi]$. We have both a's and b's.

Complete Fourier series $$F(x) = \boldsymbol{a_0} + \sum_{n=1}^{\infty} \boldsymbol{a_n \cos nx} + \sum_{n=1}^{\infty} \boldsymbol{b_n \sin nx}\,. \qquad (18)$$

On every "2π interval" the sines and cosines are orthogonal. We find the Fourier coefficients a_k and b_k in the usual way: **Multiply** (18) **by** 1 **and** $\cos kx$ **and** $\sin kx$. **Then integrate both sides from** $-\pi$ **to** π **to get** a_0 **and** a_k **and** b_k.

$$\boldsymbol{a_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx \quad \boldsymbol{a_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx \quad \boldsymbol{b_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx$$

Orthogonality kills off infinitely many integrals and leaves only the one we want.

Another approach is to split $F(x) = C(x) + S(x)$ into an even part and an odd part. Then we can use the earlier cosine and sine formulas. The two parts are

$$C(x) = F_{\textbf{even}}(x) = \frac{F(x) + F(-x)}{2} \qquad S(x) = F_{\textbf{odd}}(x) = \frac{F(x) - F(-x)}{2}. \qquad (19)$$

The even part gives the a's and the odd part gives the b's. Test on a square pulse from $x = 0$ to $x = h$—this one-sided thin box function is not odd or even.

Example 4 Find the a's and b's if $\boldsymbol{F(x)} =$ **tall box** $= \begin{cases} \boldsymbol{1/h} & \text{for } 0 < x < h \\ \boldsymbol{0} & \text{for } h < x < 2\pi \end{cases}$

Solution The integrals for a_0 and a_k and b_k stop at $x = h$ where $F(x)$ drops to zero. The coefficients decay like $1/k$ because of the jump at $x = 0$ and the drop at $x = h$:

Coefficients of square pulse $$\boldsymbol{a_0} = \frac{1}{2\pi}\int_0^h 1/h\,dx = \frac{\mathbf{1}}{\boldsymbol{2\pi}} = \textbf{ average}$$

$$\boldsymbol{a_k} = \frac{1}{\pi h}\int_0^h \cos kx\,dx = \frac{\boldsymbol{\sin kh}}{\boldsymbol{\pi kh}} \qquad \boldsymbol{b_k} = \frac{1}{\pi h}\int_0^h \sin kx\,dx = \frac{\boldsymbol{1 - \cos kh}}{\boldsymbol{\pi kh}}.$$

Important As h approaches zero, the box gets thinner and taller. Its width is h and its height is $1/h$ and its area is 1. The box approaches a delta function! And its Fourier coefficients approach the coefficients of the delta function as $h \to 0$:

$$a_0 = \frac{\mathbf{1}}{\boldsymbol{2\pi}} \qquad a_k = \frac{\sin kh}{\pi kh} \text{ approaches } \frac{\mathbf{1}}{\boldsymbol{\pi}} \qquad b_k = \frac{1 - \cos kh}{\pi kh} \text{ approaches } \mathbf{0}. \qquad (20)$$

Energy in Function = Energy in Coefficients

There is an extremely important equation (***the energy identity***) that comes from integrating $(F(x))^2$. When we square the Fourier series of $F(x)$, and integrate from $-\pi$ to π, all the "cross terms" drop out. The only nonzero integrals come from 1^2 and $\cos^2 kx$ and $\sin^2 kx$. Those integrals give 2π and π and π, multiplied by a_0^2 and a_k^2 and b_k^2:

$$\textbf{Energy} \quad \int_{-\pi}^{\pi} (F(x))^2 dx = 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots). \qquad (21)$$

The energy in $F(x)$ equals the energy in the coefficients. The left side is like the length squared of a vector, except *the vector is a function*. The right side comes from an infinitely long vector of a's and b's. The lengths are equal, which says that the Fourier transform from function to vector is like an orthogonal matrix. Normalized by $\sqrt{2\pi}$ and $\sqrt{\pi}$, **sines and cosines are an orthonormal basis in function space**.

Complex Exponentials $c_k e^{ikx}$

This is a small step and we have to take it. In place of separate formulas for a_0 and a_k and b_k, we will have *one formula* for all the complex coefficients c_k. And the function $F(x)$ might be complex (as in quantum mechanics). The Discrete Fourier Transform will be much simpler when we use N complex exponentials for a vector.

We practice with the complex infinite series for a 2π-periodic function:

$$\textbf{Complex Fourier series} \quad F(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (22)$$

If every $c_n = c_{-n}$, we can combine e^{inx} with e^{-inx} into $2\cos nx$. Then (22) is the cosine series for an even function. If every $c_n = -c_{-n}$, we use $e^{inx} - e^{-inx} = 2i \sin nx$. Then (22) is the sine series for an odd function and the c's are pure imaginary.

To find c_k, multiply (22) by e^{-ikx} (not e^{ikx}) **and integrate from $-\pi$ to π**:

$$\int_{-\pi}^{\pi} F(x)e^{-ikx} dx = \int_{-\pi}^{\pi} c_0 e^{-ikx} dx + \int_{-\pi}^{\pi} c_1 e^{ix} e^{-ikx} dx + \cdots + \int_{-\pi}^{\pi} c_k e^{ikx} e^{-ikx} dx + \cdots$$

The complex exponentials are orthogonal. **Every integral on the right side is zero**, except for the highlighted term (when $n = k$ and $e^{ikx}e^{-ikx} = 1$). The integral of 1 is 2π. That surviving term gives the formula for c_k:

$$\textbf{Fourier coefficients} \quad \int_{-\pi}^{\pi} F(x)e^{-ikx} dx = 2\pi c_k \qquad \text{for} \quad k = 0, \pm 1, \ldots l \qquad (23)$$

Notice that $c_0 = a_0$ is still the average of $F(x)$. The orthogonality of e^{inx} and e^{ikx} is checked by integrating e^{inx} times e^{-ikx}. Remember to use that complex conjugate e^{-ikx}.

Example 5 For a delta function, all integrals are 1 and every c_k is $1/2\pi$. *Flat transform*!

Example 6 Find c_k for the 2π-periodic shifted box $F(x) = \begin{cases} 1 & \text{for } s \le x \le s+h \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$

Solution The integrals (23) have $F = 1$ from s to $s + h$:

$$c_k = \frac{1}{2\pi}\int_s^{s+h} 1 \cdot e^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_s^{s+h} = \boldsymbol{e^{-iks}}\left(\frac{1-e^{-ikh}}{2\pi ik}\right). \qquad (24)$$

Notice above all the simple effect of the shift by s. *It "modulates" each* c_k *by* e^{-iks}. The energy is unchanged, the integral of $|F|^2$ just shifts, and $|e^{-iks}| = 1$.

$$\boxed{\textbf{Shift}\quad \boldsymbol{F(x)}\quad \textbf{to}\quad \boldsymbol{F(x-s)}\quad \longleftrightarrow\quad \textbf{Multiply every}\quad \boldsymbol{c_k}\quad \textbf{by}\quad \boldsymbol{e^{-iks}}.} \qquad (25)$$

Example 7 A centered box has shift $s = -h/2$. It becomes balanced around $x = 0$. This even function equals 1 on the interval from $-h/2$ to $h/2$:

Centered by $\boldsymbol{s = -\frac{h}{2}}$ $$c_k = e^{ikh/2}\,\frac{1-e^{-ikh}}{2\pi ik} = \frac{1}{2\pi}\,\frac{\boldsymbol{\sin(kh/2)}}{\boldsymbol{k/2}}.$$

Divide by h for a tall box. The ratio of $\sin(kh/2)$ to $kh/2$ is called the "**sinc**" of $kh/2$.

Tall box $$\frac{F_{\textbf{centered}}}{h} = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \text{sinc}\left(\frac{kh}{2}\right) e^{ikx} = \begin{cases} 1/h & \text{for } -h/2 \le x \le h/2 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

That division by h produces area $= 1$. Every coefficient approaches $\frac{1}{2\pi}$ as $h \to 0$. The Fourier series for the tall thin box again approaches the Fourier series for $\delta(x)$.

The Rules for Derivatives and Integrals

The derivative of e^{ikx} *is* ike^{ikx}. This great fact puts the Fourier functions e^{ikx} in first place for applications. They are eigenfunctions for d/dx (and the eigenvalues are $\lambda = ik$). Differential equations with constant coefficients are naturally solved by Fourier series.

Multiply by $\boldsymbol{ik}$ **The derivative of** $\boldsymbol{F(x) = \sum c_k e^{ikx}}$ **is** $\boldsymbol{dF/dx = \sum ikc_k e^{ikx}}$

The second derivative has coefficients $(ik)^2 c_k = \boldsymbol{-k^2} c_k$. High frequencies are growing stronger. And in the opposite direction (when we integrate), we divide by ik and high frequencies get weaker. The solution becomes smoother. Please look at this example:

Response $\boldsymbol{1/(k^2+1)}$
to frequency $\boldsymbol{k}$ $$-\frac{d^2y}{dx^2} + y = e^{ikx} \quad \text{is solved by} \quad y(x) = \frac{e^{ikx}}{\boldsymbol{k^2+1}}$$

This was a typical problem in Chapter 2. The transfer function is $1/(k^2+1)$. There we learned: The forcing function e^{ikx} is exponential so the solution is exponential.

All we are doing now is superposition. Allow all the exponentials at once!

$$-\frac{d^2y}{dx^2} + y = \sum c_k e^{ikx} \quad \text{is solved by} \quad y(x) = \sum \frac{c_k e^{ikx}}{\boldsymbol{k^2+1}}. \tag{26}$$

1. **Derivative rule dF/dx has Fourier coefficients ikc_k** (energy moves to high k).
2. **Shift rule $F(x-s)$ has Fourier coefficients $e^{-iks}c_k$** (no change in energy).

Application: Laplace's Equation in a Circle

Our first application is to Laplace's equation $u_{xx} + u_{yy} = 0$ (Section 7.4). The idea is to construct $u(x, y)$ as an infinite series, choosing its coefficients to match $u_0(x, y)$ along the boundary. The shape of the boundary is crucial, and we take a circle of radius 1.

Begin with the solutions 1, $r\cos\theta$, $r\sin\theta$, $r^2\cos 2\theta$, $r^2\sin 2\theta$, ... to Laplace's equation. Combinations of these special solutions give all solutions in the circle:

$$\boldsymbol{u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \cdots} \tag{27}$$

It remains to choose the constants a_k and b_k to make $u = u_0$ on the boundary. For a circle, θ and $\theta + 2\pi$ give the same point. This means that $u_0(\theta)$ is periodic:

Set $r = 1$
$$u_0(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \cdots \tag{28}$$

This is exactly the Fourier series for u_0. **The constants a_k and b_k must be the Fourier coefficients of $u_0(\theta)$.** Thus Laplace's boundary value problem is completely solved, if an infinite series (27) is acceptable as the solution.

Example 8 **Point source $u_0 = \delta(\theta)$.** The boundary is held at $u_0 = 0$, except for the source at $x = 1$, $y = 0$ (where $\theta = 0$). Find the temperature $u(r, \theta)$ inside the circle.

Delta function
$$u_0(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}(\cos\theta + \cos 2\theta + \cos 3\theta + \cdots) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{in\theta}$$

Inside the circle, each $\cos n\theta$ is multiplied by r^n to solve Laplace's equation:

Inside the circle
$$u(r,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}(r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots) \tag{29}$$

Poisson managed to sum this infinite series! It involves a series of powers $(re^{i\theta})^n$. His sum gives the response at every (r, θ) to the point source at $r = 1$, $\theta = 0$:

Temperature inside circle
$$u(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta} \tag{30}$$

At the center $r = 0$, this produces the average of $u_0 = \delta(\theta)$ which is $a_0 = 1/2\pi$. On the boundary $r = 1$, this gives $u = 0$ except $u = \infty$ at the point where $\cos 0 = 1$.

Example 9 $\boldsymbol{u_0(\theta) = 1}$ **on the top half of the circle and** $\boldsymbol{u_0 = -1}$ **on the bottom half.**

Solution The boundary values u_0 are a square wave SW. We know its sine series:

$$\textbf{Square wave for } u_0(\theta) \qquad SW(\theta) = \frac{4}{\pi}\left[\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right] \tag{31}$$

Inside the circle, multiplying by $r, r^3, r^5, \ldots$ gives fast decay of high frequencies:

$$\textbf{Rapid decay inside} \qquad u(r,\theta) = \frac{4}{\pi}\left[\frac{r\sin\theta}{1} + \frac{r^3\sin 3\theta}{3} + \frac{r^5\sin 5\theta}{5} + \cdots\right] \tag{32}$$

Laplace's equation has smooth solutions inside, even when $u_0(\theta)$ is not smooth.

Problem Set 8.1

1 (a) To prove that $\cos nx$ is orthogonal to $\cos kx$ when $k \neq n$, use the formula $(\cos nx)\,(\cos kx) = \frac{1}{2}\cos(n+k)x + \frac{1}{2}\cos(n-k)x$. Integrate from $x = 0$ to $x = \pi$. What is $\int \cos^2 kx\, dx$?

(b) From 0 to π, $\cos x$ is **not** orthogonal to $\sin x$. The period has to be 2π:

$$\text{Find } \int_0^{\pi} (\sin x)\,(\cos x)\,dx \quad\text{and}\quad \int_{-\pi}^{\pi} (\sin x)\,(\cos x)\,dx \quad\text{and}\quad \int_0^{2\pi} (\sin x)\,(\cos x)\,dx.$$

2 Suppose $F(x) = x$ for $0 \le x \le \pi$. Draw graphs for $-2\pi \le x \le 2\pi$ to show three extensions of F: a 2π-periodic even function and a 2π-periodic odd function and a π-periodic function.

3 Find the Fourier series on $-\pi \le x \le \pi$ for

(a) $f_1(x) = \sin^3 x$, an odd function (sine series, only two terms)

(b) $f_2(x) = |\sin x|$, an even function (cosine series)

(c) $f_3(x) = x$ for $-\pi \le x \le \pi$ (sine series with jump at $x = \pi$)

4 Find the complex Fourier series $e^x = \sum c_k e^{ikx}$ on the interval $-\pi \le x \le \pi$. The even part of a function is $\frac{1}{2}(f(x) + f(-x))$, so that $f_{\text{even}}(x) = f_{\text{even}}(-x)$. Find the cosine series for f_{even} and the sine series for f_{odd}. *Notice the jump at* $x = \pi$.

5 From the energy formula (21), the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |SW(x)|^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Substitute the numbers b_k from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

6 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad\text{for}\quad |x| < \frac{\pi}{2}, \qquad f(x) = 0 \quad\text{for}\quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

7 Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is its second derivative?

8 Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

9 Suppose $G(x)$ has period $2L$ instead of 2π. Then $G(x + 2L) = G(x)$. Integrals go from $-L$ to L or from 0 to $2L$. The Fourier formulas change by a factor π/L:

The coefficients in $G(x) = \sum_{-\infty}^{\infty} \boldsymbol{C}_k e^{ik\pi x/L}$ are $\boldsymbol{C}_k = \frac{1}{2L}\int_{-L}^{L} G(x)e^{-ik\pi x/L}dx.$

Derive this formula for C_k: Multiply the first equation for $G(x)$ by ____ and integrate both sides. Why is the integral on the right side equal to $2LC_k$?

10 For G_{even}, use Problem 9 to find the cosine coefficient A_k from $(C_k + C_{-k})/2$:

$$G_{\text{even}}(x) = \sum_{0}^{\infty} A_k \cos\frac{k\pi x}{L} \quad \text{has} \quad A_k = \frac{1}{L}\int_{0}^{L} G_{\text{even}}(x) \cos\frac{k\pi x}{L}dx.$$

G_{even} is $\frac{1}{2}(G(x) + G(-x))$. Exception for $A_0 = C_0$: Divide by $2L$ instead of L.

11 Problem 10 tells us that $\boldsymbol{a_k = \frac{1}{2}(c_k + c_{-k})}$ on the usual interval from 0 to π. Find a similar formula for b_k from c_k and c_{-k}. In the reverse direction, find the complex coefficient c_k in $F(x) = \sum c_k e^{ikx}$ from the real coefficients a_k and b_k.

12 Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3 \cdots) = 2\log(1 + z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2\log(1 + z)$ agrees with θ.

13 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin $r = 0$?

14 With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum this geometric series.

15 (a) Verify that the fraction in Poisson's formula (30) satisfies Laplace's equation.

(b) Find the response $u(r, \theta)$ to an impulse at $x = 0, y = 1$ (where $\theta = \frac{\pi}{2}$).

16 With complex exponentials in $F(x) = \sum c_k e^{ikx}$, the energy identity (21) changes to $\int_{-\pi}^{\pi} |F(x)|^2\, dx = 2\pi \sum |c_k|^2$. Derive this by integrating $(\sum c_k e^{ikx})(\sum \overline{c}_k e^{-ikx})$.

17 A centered square wave has $F(x) = 1$ for $|x| \le \pi/2$.

(a) Find its energy $\int |F(x)|^2\, dx$ by direct integration

(b) Compute its Fourier coefficients c_k as specific numbers

(c) Find the sum in the energy identity (Problem 16).

18 $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is analytic: infinitely smooth.

(a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

(b) Does that series still converge quickly? Compare n^{10} with 2^n for $n = 2^{10}$.

19 If $f(x) = 1$ for $|x| \le \pi/2$ and $f(x) = 0$ for $\pi/2 < |x| < \pi$, find its cosine coefficients. Can you graph and compute the Gibbs overshoot at the jumps?

20 Find all the coefficients a_k and b_k for F, I, and D on the interval $-\pi \le x \le \pi$:

$$F(x) = \delta\left(x - \frac{\pi}{2}\right) \qquad I(x) = \int_0^x \delta\left(x - \frac{\pi}{2}\right) dx \qquad D(x) = \frac{d}{dx}\delta\left(x - \frac{\pi}{2}\right).$$

21 For the one-sided tall box function in Example 4, with $F = 1/h$ for $0 \le x \le h$, what is its odd part $\frac{1}{2}(F(x) - F(-x))$? I am surprised that the Fourier coefficients of this odd part disappear as h approaches zero and $F(x)$ approaches $\delta(x)$.

22 Find the series $F(x) = \sum c_k e^{ikx}$ for $F(x) = e^x$ on $-\pi \le x \le \pi$. That function e^x looks smooth, but there must be a hidden jump to get coefficients c_k proportional to $1/k$. Where is the jump?

23 (a) (Old particular solution) Solve $Ay'' + By' + Cy = e^{ikx}$.

(b) (New particular solution) Solve $Ay'' + By' + Cy = \sum c_k e^{ikx}$.