3.2 Sources, Sinks, Saddles, and Spirals

The pictures in this section show solutions to $Ay'' + By' + Cy = 0$. These are linear equations with constant coefficients A, B, and C. The graphs show solutions y on the horizontal axis and their slopes $y' = dy/dt$ on the vertical axis. These pairs $(y(t), y'(t))$ depend on time, *but time is not in the pictures*. The paths show *where* the solution goes, but they don't show when.

Each specific solution starts at a particular point $(y(0), y'(0))$ given by the initial conditions. The point moves along its path as the time t moves forward from $t = 0$. We know that the solutions to $Ay'' + By' + Cy = 0$ depend on the two solutions to $As^2 + Bs + C = 0$ (an ordinary quadratic equation for s). When we find the roots s_1 and s_2, we have found all possible solutions:

$$y = c_1 e^{s_1 t} + c_2 e^{s_2 t} \qquad y' = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t} \tag{1}$$

The numbers s_1 and s_2 tell us which picture we are in. Then the numbers c_1 and c_2 tell us which path we are on.

Since s_1 and s_2 determine the picture for each equation, it is essential to see the six possibilities. We write all six here in one place, to compare them. Later they will appear in six different places, one with each figure. The first three have real solutions s_1 and s_2. The last three have complex pairs $\boldsymbol{s = a \pm i\omega}$.

Sources	**Sinks**	**Saddles**	**Spiral out**	**Spiral in**	**Center**
$s_1 > s_2 > 0$	$s_1 < s_2 < 0$	$s_2 < 0 < s_1$	$a = \text{Re } s > 0$	$a = \text{Re } s < 0$	$a = \text{Re } s = 0$

In addition to those six, there will be limiting cases $s = 0$ and $s_1 = s_2$ (as in resonance).

Stability This word is important for differential equations. *Do solutions decay to zero*? The solutions are controlled by $e^{s_1 t}$ and $e^{s_2 t}$ (and in Chapter 6 by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$). We can identify the two pictures (out of six) that are displaying full stability: the sinks. **A center $\boldsymbol{s = \pm i\omega}$ is at the edge of stability** ($e^{i\omega t}$ is neither decaying or growing).

2. Sinks are stable	$\boldsymbol{s_1 < s_2 < 0}$	**Then** $\boldsymbol{y(t) \to 0}$
5. Spiral sinks are stable	$\text{Re } s_1 = \text{Re } s_2 < 0$	**Then** $\boldsymbol{y(t) \to 0}$

Special note. May I mention here that the same six pictures also apply to a system of *two first order equations*. Instead of y and y', the equations have unknowns y_1 and y_2. Instead of the constant coefficients A, B, C, the equations will have a 2 by 2 matrix. Instead of the roots s_1 and s_2, that matrix will have eigenvalues λ_1 and λ_2. **Those eigenvalues are the roots of an equation $\boldsymbol{A\lambda^2 + B\lambda + C = 0}$**, just like s_1 and s_2.

We will see the same six possibilities for the λ's, and the same six pictures. The eigenvalues of the 2 by 2 matrix give the growth rates or decay rates, in place of s_1 and s_2.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ has solutions } \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}.$$

The eigenvalue is λ and the eigenvector is $\boldsymbol{v} = (v_1, v_2)$. The solution is $\boldsymbol{y(t) = v e^{\lambda t}}$.

The First Three Pictures

We are starting with the case of *real roots* s_1 and s_2. In the equation $Ay'' + By' + Cy = 0$, this means that $\boldsymbol{B^2 \geq 4AC}$. Then B is relatively large. The square root in the quadratic formula produces a real number $\sqrt{B^2 - 4AC}$. If A, B, C have the same sign, we have overdamping and **negative roots** and stability. The solutions decay to $(0, 0)$: a **sink**.

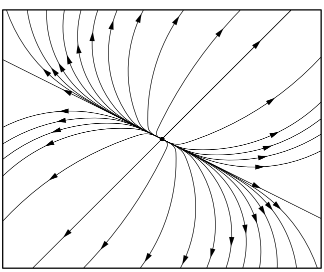
If A and C have opposite sign to B as in $y'' - 3y' + 2y = 0$, we have negative damping and **positive roots** s_1, s_2. The solutions grow (this is instability : a **source** at $(0, 0)$).

Suppose A and C have different signs, as in $y'' - 3y' - 2y = 0$. Then s_1 and s_2 also have **different signs** and the picture shows a **saddle**. The moving point $(y(t), y'(t))$ can start in toward $(0, 0)$ before it turns out to infinity. The positive s gives $e^{st} \to \infty$. *Second example for a saddle* : $y'' - 4y = 0$ leads to $s^2 - 4 = (s-2)(s+2) = 0$. The roots $s_1 = \mathbf{2}$ and $s_2 = \mathbf{-2}$ have opposite signs. Solutions $c_1 e^{2t} + c_2 e^{-2t}$ grow unless $c_1 = 0$. Only that one line with $c_1 = 0$ has arrows inward.

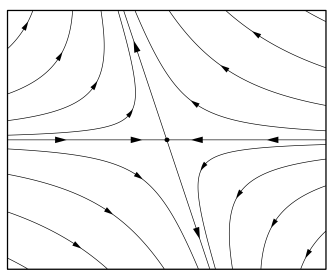
In every case with $B^2 \geq 4AC$, the roots are real. The solutions $y(t)$ have growing exponentials or decaying exponentials. We don't see sines and cosines and oscillation.

The first figure shows growth : $0 < s_2 < s_1$. Since $e^{s_1 t}$ grows faster than $e^{s_2 t}$, the larger number s_1 will dominate. The solution path for (y, y') will approach the straight line of slope s_1. That is because the ratio of $y' = c_1 s_1 e^{s_1 t}$ to $y = c_1 e^{s_1 t}$ is exactly s_1.

If the initial condition is on the "s_1 line" then the solution (y, y') stays on that line: $c_2 = 0$. If the initial condition is exactly on the "s_2 line" then the solution stays on that secondary line : $c_1 = 0$. You can see that if $c_1 \neq 0$, the $c_1 e^{s_1 t}$ part takes over as $t \to \infty$.



Reverse all the arrows in the left figure. Paths go in toward $(0, 0)$



$\boldsymbol{0 < s_2 < s_1}$ **Source : Unstable**

$\boldsymbol{s_1 < s_2 < 0}$ **Sink : Stable**

$\boldsymbol{s_2 < 0 < s_1}$ **Saddle : Unstable**

Figure 3.6: **Real roots $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$.** The paths of the point $(y(t), y'(t))$ lead out when roots are positive and lead in when roots are negative. With $s_2 < 0 < s_1$, the s_2-line leads in but all other paths eventually go out near the s_1-line : *The picture shows a saddle point.*

Example for a source: $y'' - 3y' + 2y = 0$ leads to $s^2 - 3s + 2 = (s-2)(s-1) = 0$. The roots **1** and **2** are positive. The solutions grow and e^{2t} dominates.

Example for a sink: $y'' + 3y' + 2y = 0$ leads to $s^2 + 3s + 2 = (s+2)(s+1) = 0$. The roots **−2** and **−1** are negative. The solutions decay and e^{-t} dominates.

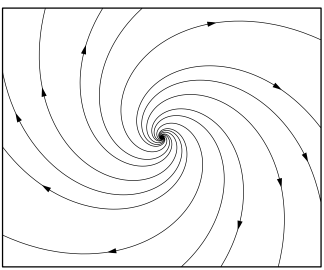
The Second Three Pictures

We move to the case of **complex roots** s_1 and s_2. In the equation $Ay'' + By' + Cy = 0$, this means that $B^2 < 4AC$. Then A and C have the same signs and B is relatively small (underdamping). The square root in the quadratic formula (2) is an imaginary number. *The exponents s_1 and s_2 are now a complex pair $a \pm i\omega$*:

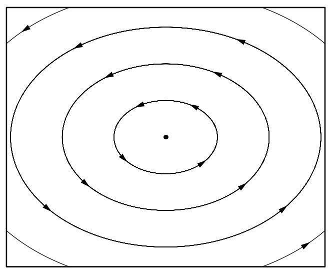
Complex roots of $\boldsymbol{As^2 + Bs + C = 0}$
$$s_1, s_2 = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} = a \pm i\omega. \tag{2}$$

The path of (y, y') **spirals around the center**. Because of e^{at}, the spiral goes out if $a > 0$: **spiral source**. Solutions spiral in if $a < 0$: **spiral sink**. The frequency ω controls how fast the solutions oscillate and how quickly the spirals go around $(0, 0)$.

In case $a = -B/2A$ is zero (no damping), we have a **center** at (0, 0). The only terms left in y are $e^{i\omega t}$ and $e^{-i\omega t}$, in other words $\cos\omega t$ and $\sin\omega t$. Those paths are ellipses in the last part of Figure 3.7. The solutions $y(t)$ are **periodic**, because increasing t by $2\pi/\omega$ will not change $\cos\omega t$ and $\sin\omega t$. That circling time $2\pi/\omega$ is the **period**.



Reverse all the arrows in the left figure. Paths go in toward (0, 0).



$a = \mathrm{Re}\, s > 0$ **Spiral source : Unstable**

$a = \mathrm{Re}\, s < 0$ **Spiral sink : Stable**

$a = \mathrm{Re}\, s = 0$ **Center : Neutrally stable**

Figure 3.7: **Complex roots s_1 and s_2.** The paths go once around $(0, 0)$ when t increases by $2\pi/\omega$. The paths spiral in when A and B have the same signs and $a = -B/2A$ is negative. They spiral out when a is positive. If $B = 0$ (no damping) and $4AC > 0$, we have a center. The simplest center is $y = \sin t$, $y' = \cos t$ (circle) from $y'' + y = 0$.

First Order Equations for y_1 and y_2

On the first page of this section, a "Special Note" mentioned another application of the same pictures. Instead of graphing the path of $(y(t), y'(t))$ for one second order equation, we could follow the path of $(y_1(t), y_2(t))$ for **two first order equations**. The two equations look like this:

$$\textbf{First order system } \boldsymbol{y}' = A\boldsymbol{y} \qquad \begin{aligned} dy_1/dt &= ay_1 + by_2 \\ dy_2/dt &= cy_1 + dy_2 \end{aligned} \tag{3}$$

The starting values $y_1(0)$ and $y_2(0)$ are given. The point (y_1, y_2) will move along a path in one of the six figures, depending on the numbers a, b, c, d.

Looking ahead, those four numbers will go into a 2 by 2 matrix A. Equation (3) will become $d\boldsymbol{y}/dt = A\boldsymbol{y}$. The symbol $\boldsymbol{y}$ in boldface stands for the vector $\boldsymbol{y} = (y_1, y_2)$. And most important for the six figures, *the exponents s_1 and s_2 in the solution $\boldsymbol{y}(t)$ will be the* **eigenvalues λ_1 and λ_2** of the matrix A.

Companion Matrices

Here is the connection between a second order equation and two first order equations. All equations on this page are linear and all coefficients are constant. I just want you to see the special "*companion matrix*" that appears in the first order equations $\boldsymbol{y}' = A\boldsymbol{y}$.

Notice that $\boldsymbol{y}$ is printed in **boldface type** because it is a **vector**. It has two components y_1 and y_2 (those are in lightface type). The first y_1 is the same as the unknown y in the second order equation. The second component y_2 is the velocity dy/dt:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned} \qquad \boldsymbol{y'' + 4y' + 3y} = 0 \quad \text{becomes} \quad \boldsymbol{y_2' + 4y_2 + 3y_1} = 0. \tag{4}$$

On the right you see one of the first order equations connecting y_1 and y_2. We need a second equation (two equations for two unknowns). *It is hiding at the far left!* There you see that $\boldsymbol{y_1' = y_2}$. In the original second order problem this is the trivial statement $y' = y'$. In the vector form $\boldsymbol{y}' = A\boldsymbol{y}$ it gives the first equation in our system. The first row of our matrix is **0 1**. When y and y' become y_1 and y_2,

$$y'' + 4y' + 3y = 0 \quad \text{becomes} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -3y_1 - 4y_2 \end{aligned} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-3} & \mathbf{-4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{5}$$

That first row **0 1** makes this a 2 by 2 **companion matrix**. It is the companion to the second order equation. The key point is that the first order and second order problems have the same "characteristic equation" because they are the same problem.

The equation $s^2 + 4s + 3 = 0$ gives the exponents $s_1 = \mathbf{-3}$ and $s_2 = \mathbf{-1}$

The equation $\lambda^2 + 4\lambda + 3 = 0$ gives the eigenvalues $\lambda_1 = \mathbf{-3}$ and $\lambda_2 = \mathbf{-1}$

The problems are the same, the exponents -3 and -1 are the same, the figures will be the same. Those figures show a **sink** because -3 and -1 are real and both negative. Solutions approach $(0,0)$. These equations are **stable**.

$$\textbf{The companion matrix for } \boldsymbol{y'' + By' + Cy = 0} \textbf{ is } \boldsymbol{A = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}}.$$

Row 1 of $y' = Ay$ is $y_1' = y_2$. Row 2 is $y_2' = -Cy_1 - By_2$. When you replace y_2 by y_1', this means that $y_1'' + By_1' + Cy_1 = 0$: *correct*.

Stability for 2 by 2 Matrices

I can explain when a 2 by 2 system $\boldsymbol{y}' = A\boldsymbol{y}$ is stable. This requires that all solutions $\boldsymbol{y}(t) = (y_1(t), y_2(t))$ approach zero as $t \to \infty$. When the matrix A is a companion matrix, this 2 by 2 system comes from one second order equation $y'' + By' + Cy = 0$. In that case we know that stability depends on the roots of $s^2 + Bs + C = 0$. **Companion matrices are stable when $\boldsymbol{B > 0}$ and $\boldsymbol{C > 0}$.**

From the quadratic formula, the roots have $s_1 + s_2 = -B$ and $s_1 s_2 = C$.

If s_1 and s_2 are negative, this means that $B > 0$ and $C > 0$.

If $s_1 = a + i\omega$ and $s_2 = a - i\omega$ and $a < 0$, this again means $B > 0$ and $C > 0$

Those complex roots add to $s_1 + s_2 = 2a$. Negative a (stability) means positive B, since $s_1 + s_2 = -B$. Those roots multiply to $s_1 s_2 = a^2 + \omega^2$. This means that C is positive, since $s_1 s_2 = C$.

For companion matrices, stability is decided by $B > 0$ and $C > 0$. **What is the stability test for any 2 by 2 matrix**? This is the key question, and Chapter 6 will answer it properly. We will find the equation for the eigenvalues of any matrix (Section 6.1). We will test those eigenvalues for stability (Section 6.4). Eigenvalues and eigenvectors are a major topic, the most important link between differential equations and linear algebra. Fortunately, the eigenvalues of 2 by 2 matrices are especially simple.

The eigenvalues of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ have $\lambda^2 - T\lambda + D = 0$.

The number T is $a + d$. The number D is $ad - bc$.

Companion matrices have $a = 0$ and $b = 1$ and $c = -C$ and $d = -B$. Then the characteristic equation $\lambda^2 - T\lambda + D = 0$ is exactly $s^2 + Bs + C = 0$.

$$\text{Companion matrices have} \quad \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \qquad T = a+d = -B \;\text{ and }\; D = ad-bc = C.$$

The stability test $\boldsymbol{B > 0}$ and $\boldsymbol{C > 0}$ is turning into the stability test $\boldsymbol{T < 0}$ and $\boldsymbol{D > 0}$.

This is the test for any 2 by 2 matrix. Stability requires $T < 0$ and $D > 0$. Let me give four examples and then collect together the main facts about stability.

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ is } \textit{unstable} \text{ because } T = 0 + 3 \text{ is positive}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \text{ is } \textit{unstable} \text{ because } D = -(1)(2) \text{ is negative}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ is } \textit{stable} \text{ because } T = -3 \text{ and } D = +2$$

$$A_4 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \text{ is } \textit{stable} \text{ because } T = -1 - 1 \text{ is negative and } D = 1 + 1 \text{ is positive}$$

The eigenvalues always come from $\lambda^2 - T\lambda + D = 0$. For that last matrix A_4, this eigenvalue equation is $\lambda^2 + 2\lambda + 2 = 0$. The eigenvalues are $\lambda_1 = \mathbf{-1+i}$ and $\lambda_2 = \mathbf{-1-i}$. They add to $T = -2$ and they multiply to $D = +2$. **This is a spiral sink and it is stable**.

Stability for 2 by 2 matrices	$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	is stable if	$T = a + d < 0$ $D = ad - bc > 0$

The six pictures for (y, y') become six pictures for (y_1, y_2). The first three pictures have real eigenvalues from $T^2 \geq 4D$. The second three pictures have complex eigenvalues from $T^2 < 4D$. This corresponds perfectly to the tests for $y'' + By' + Cy = 0$ and its companion matrix:

Real eigenvalues	$T^2 \geq 4D$	$B^2 \geq 4C$	Overdamping
Complex eigenvalues	$T^2 < 4D$	$B^2 < 4C$	Underdamping

That gives one picture of eigenvalues λ: *Real or complex*. The second picture is different: *Stable or unstable*. Both of those splittings are decided by T and D (or $-B$ and C).

1.	**Source**	$T > 0,\ D > 0,\ T^2 \geq 4D$	*Ustable*
2.	**Sink**	$T < 0,\ D > 0,\ T^2 \geq 4D$	*Stable*
3.	**Saddle**	$D < 0$ and $T^2 \geq 4D$	*Unstable*
4.	**Spiral source**	$T > 0,\ D > 0,\ T^2 < 4D$	*Unstable*
5.	**Spiral Sink**	$T < 0,\ D > 0,\ T^2 < 4D$	*Stable*
6.	**Center**	$T = 0,\ D > 0,\ T^2 < 4D$	*Neutral*

That neutrally stable center has eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ and undamped oscillation.

Section 3.3 will use this information to decide the stability of *nonlinear* equations.

Eigenvectors of Companion Matrices

Eigenvalues of A come with eigenvectors. If we stay a little longer with a companion matrix, we can see its eigenvectors. Chapter 6 will develop these ideas for any matrix, and we need more linear algebra to understand them properly. But our vectors (y_1, y_2) come from (y, y') in a differential equation, and that connection makes the eigenvectors of a companion matrix especially simple.

The fundamental idea for constant coefficient linear equations is always the same: **Look for exponential solutions**. For a second order equation those solutions are $y = e^{st}$. For a system of two first order equations those solutions are $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$. **The vector $\boldsymbol{v} = (v_1, v_2)$ is the eigenvector that goes with the eigenvalue $\boldsymbol{\lambda}$.**

$$\text{Substitute} \quad \begin{aligned} y_1 &= v_1 e^{\lambda t} \\ y_2 &= v_2 e^{\lambda t} \end{aligned} \quad \text{into the equations} \quad \begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned} \quad \text{and factor out } e^{\lambda t}.$$

Because $e^{\lambda t}$ is the same for both y_1 and y_2, it will appear in every term. When all factors $e^{\lambda t}$ are removed, we will see the equations for v_1 and v_2. That vector $\boldsymbol{v} = (v_1, v_2)$ will satisfy the eigenvector equation $A\boldsymbol{v} = \lambda \boldsymbol{v}$. This is the key to Chapter 6.

Here I only look at eigenvectors for companion matrices, because v has a specially nice form. The equations are $y_1' = y_2$ and $y_2' = -Cy_1 - By_2$.

$$\text{Substitute} \quad \begin{aligned} y_1 &= v_1 e^{\lambda t} \\ y_2 &= v_2 e^{\lambda t} \end{aligned} \quad \text{Then} \quad \begin{aligned} \lambda v_1 e^{\lambda t} &= v_2 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} &= -C v_1 e^{\lambda t} - B v_2 e^{\lambda t}. \end{aligned}$$

Cancel every $e^{\lambda t}$. The first equation becomes $\lambda v_1 = v_2$. This is our answer:

$$\text{Eigenvectors of companion matrices are multiples of the vector } \boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

■ REVIEW OF THE KEY IDEAS ■

1. If $B^2 \neq 4AC \neq 0$, six pictures show the paths of (y, y') for $Ay'' + By' + Cy = 0$.

2. Real solutions to $As^2 + Bs + C = 0$ lead to sources and sinks and saddles at $(0, 0)$.

3. Complex roots $s = a \pm i\omega$ give spirals around $(0, 0)$ (or closed loops if $a = 0$).

4. Roots s become eigenvalues λ for $\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$. Same six pictures.

Problem Set 3.2

1 Draw Figure 3.6 for a sink (the missing middle figure) with $y = c_1 e^{-2t} + c_2 e^{-t}$. Which term dominates as $t \to \infty$? The paths approach the dominating line as they go in toward zero. **The slopes of the lines are –2 and –1** (the numbers s_1 and s_2).

2 Draw Figure 3.7 for a spiral sink (the missing middle figure) with roots $s = -1 \pm i$. The solutions are $y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$. They approach zero because of the factor e^{-t}. They spiral around the origin because of $\cos t$ and $\sin t$.

3 Which path does the solution take in Figure 3.6 if $y = e^t + e^{t/2}$? Draw the curve $(y(t), y'(t))$ more carefully starting at $t = 0$ where $(y, y') = (2, 1.5)$.

4 Which path does the solution take around the saddle in Figure 3.6 if $y = e^{t/2} + e^{-t}$? Draw the curve more carefully starting at $t = 0$ where $(y, y') = (2, -\frac{1}{2})$.

5 Redraw the first part of Figure 3.6 when the roots are equal: $s_1 = s_2 = 1$ and $y = c_1 e^t + c_2 t e^t$. *There is no s_2-line.* Sketch the path for $y = e^t + t e^t$.

6 The solution $y = e^{2t} - 4e^t$ gives a source (Figure 3.6), with $y' = 2e^{2t} - 4e^t$. Starting at $t = 0$ with $(y, y') = (-3, -2)$, where is (y, y') when $e^t = 1.1$ and $e^t = .25$ and $e^t = 2$?

7 The solution $y = e^t(\cos t + \sin t)$ has $y' = 2e^t \cos t$. This spirals out because of e^t. Plot the points (y, y') at $t = 0$ and $t = \pi/2$ and $t = \pi$, and try to connect them with a spiral. Note that $e^{\pi/2} \approx 4.8$ and $e^{\pi} \approx 23$.

8 The roots s_1 and s_2 are $\pm 2i$ when the differential equation is ______. Starting from $y(0) = 1$ and $y'(0) = 0$, draw the path of $(y(t), y'(t))$ around the center. Mark the points when $t = \pi/2, \pi, 3\pi/2, 2\pi$. Does the path go clockwise?

9 The equation $y'' + By' + y = 0$ leads to $s^2 + Bs + 1 = 0$. For $B = -3, -2, -1, 0, 1, 2, 3$ decide which of the six figures is involved. For $B = -2$ and 2, why do we not have a perfect match with the source and sink figures?

10 For $y'' + y' + Cy = 0$ with damping $B = 1$, the characteristic equation will be $s^2 + s + C = 0$. Which C gives the changeover from a *sink* (overdamping) to a spiral *sink* (underdamping)? Which figure has $C < 0$?

Problems 11–18 are about $dy/dt = Ay$ with companion matrices $\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$.

11 The eigenvalue equation is $\boldsymbol{\lambda^2 + B\lambda + C = 0}$. Which values of B and C give complex eigenvalues? Which values of B and C give $\lambda_1 = \lambda_2$?

12 Find λ_1 and λ_2 if $B = 8$ and $C = 7$. Which eigenvalue is more important as $t \to \infty$? Is this a sink or a saddle?

13 Why do the eigenvalues have $\lambda_1 + \lambda_2 = -B$? Why is $\lambda_1 \lambda_2 = C$?

14 Which second order equations did these matrices come from?

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (saddle)} \qquad\qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ (center)}$$

15 The equation $y'' = 4y$ produces a saddle point at $(0,0)$. Find $s_1 > 0$ and $s_2 < 0$ in the solution $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. If $c_1 c_2 \neq 0$, this solution will be (large) (small) as $t \to \infty$ and also as $t \to -\infty$.

The only way to go toward the saddle $(y, y') = (0,0)$ as $t \to \infty$ is $c_1 = 0$.

16 If $B = 5$ and $C = 6$ the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. The vectors $\boldsymbol{v} = (1,3)$ and $\boldsymbol{v} = (1,2)$ are *eigenvectors* of the matrix A: Multiply $A\boldsymbol{v}$ to get $3\boldsymbol{v}$ and $2\boldsymbol{v}$.

17 In Problem 16, write the two solutions $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$ to the equations $\boldsymbol{y}' = A\boldsymbol{y}$. Write the complete solution as a combination of those two solutions.

18 The eigenvectors of a companion matrix have the form $v = (1, \lambda)$. Multiply by A to show that $Av = \lambda v$ gives one trivial equation and the characteristic equation $\lambda^2 + B\lambda + C = 0$.

$$\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad \text{is} \quad \begin{aligned} \lambda &= \lambda \\ -C - B\lambda &= \lambda^2 \end{aligned}$$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

19 An equation is stable and all its solutions $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ go to $y(\infty) = 0$ exactly when

$(s_1 < 0$ **or** $s_2 < 0)$ $\qquad$ $(s_1 < 0$ **and** $s_2 < 0)$ $\qquad$ $(\text{Re } s_1 < 0$ **and** $\text{Re } s_2 < 0)$?

20 If $Ay'' + By' + Cy = D$ is stable, what is $y(\infty)$?