Chapter 3

Graphical and Numerical Methods

The world of differential equations is large (very large). This page aims to see what is already done and what remains to do.

Chapters 1 and 2 concentrated on *equations we can solve*. Compared to digging for coal or drilling for oil, this was the equivalent of picking up gold. Solutions were waiting for us. Looking back honestly, we just wrote them down (not so easy in Chapter 2).

Above all I am thinking of $\boldsymbol{e^{at}}$ in Chapter 1 and $\boldsymbol{e^{st}}$ in Chapter 2 and $\boldsymbol{e^{\lambda t}x}$ coming in Chapter 6 (with eigenvalues and eigenvectors). When the equation is linear, and its coefficients are constant, then its solutions are exponentials.

Chapter 1 First order equations (linear or separable or exact or special)

Chapter 2 Second order equations $Ay'' + By' + Cy = f(t)$

Chapter 6 First order systems $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$ with matrices A and vectors $\boldsymbol{y}$.

Chapter 3 will be different. Instead of $f(t)$ we have $f(t, y)$. Most nonlinear problems don't allow a formula for $y(t)$. "A solution exists but it has no formula." This is the hard reality of differential equations $y' = f(t, y)$. The equations are important but they don't have exponential answers. This chapter **pictures** the solution, **computes** the solution, and decides if the solution is **stable**.

Section 3.1 Pictures for nonlinear equations $y' = f(t, y)$: Stability decided by $\partial f / \partial y$.

Section 3.2 Pictures for linear second order equations and 2 by 2 systems : Stable or not.

Section 3.3 Test for stability at critical points by linearizing systems of equations.

Section 3.4 Euler methods (safe but slow) for computing approximations to y.

Section 3.5 Fast and accurate computations, by methods more efficient than Euler. Science and engineering and finance constantly use Runge-Kutta.

After this chapter, the book will move into high dimensions : **the world of linear algebra**. One particle and one resistor and one spring and one of anything : that was only a start. The reality is a network of connections : a brain, a living body, a modern machine, a web of processors. Every network leads to a matrix. *You will learn how to read a matrix*.

In my opinion, linear algebra is pure gold.

3.1 Nonlinear Equations $y' = f(t, y)$

This section aims to get a picture of $y(t)$, not a formula. The pictures will be graphs in the $t - y$ plane (t across and $y(t)$ up). The differential equation is $dy/dt = f(t, y)$ and everything depends on that function f. I can start with a linear equation $y' = 2y$.

> The solutions to $y' = 2y$ are $y(t) = Ce^{2t}$. For every number C this gives a solution curve from $t = -\infty$ to $t = \infty$. Those curves cover every point in the $t - y$ plane. This is the "solution picture" we want for nonlinear equations $y' = f(t, y)$.

That solution $y = Ce^{2t}$ has a graph. The plane is filled with those graphs. Every point t, y has one of those curves going through it (choose the right C). A different equation $y' = \sin ty$ won't have a formula. Its picture starts with just this one fact:

$\boldsymbol{dy/dt = \sin ty}$ **The solution curve through the point $\boldsymbol{t, y}$ has the slope $\boldsymbol{\sin ty}$.**

From that *point* picture we have to build a *curve* picture. This section tries to connect small arrows at points into solution curves through those points. The arrow at the point t, y has the right slope $f(t, y)$. Connecting with other arrows is the hard part.

I will separate this section into facts about $y(t)$ and pictures of $y(t)$.

Facts About $y(t)$

The facts will be answers to these questions, and the Chapter 3 Notes add more:

1. Starting from $y(0)$ at $t = 0$, **does $\boldsymbol{dy/dt = f(t, y)}$ have a solution**?
2. **Could there be two or more solutions** that start from the same $y(0)$?

Question **1** is about *existence* of $y(t)$. Is there a solution curve through $t = 0$, $y = y(0)$?
Question **2** is about *uniqueness* of $y(t)$. Could two solution curves go through one point?

When $f(t, y)$ is reasonable, we expect exactly one curve through every point t, y: *existence and also uniqueness*. Which functions are reasonable? Here are answers:

1. A solution exists if $f(t, y)$ is a continuous function for t near 0 and y near $y(0)$.
2. There can't be two solutions with the same $y(0)$ when $\partial f/\partial y$ is also continuous.

The word "continuous" has a precise technical meaning. Let me be imprecise and nontechnical. Continuity at a point rules out jumps and infinities in a small neighborhood of that point. The particular function $f = y/t$ is certainly ruled out at points where $t = 0$:

$$\frac{dy}{dt} = \frac{y}{t} \quad \text{with} \quad y(0) = 0 \quad \text{has infinitely many solutions} \quad \boldsymbol{y = Ct}.$$

The particular function $f = t/y$ is also ruled out when $y(0) = 0$ (no division by 0):

$$\frac{dy}{dt} = \frac{t}{y} \quad \text{with} \quad y(0) = 0 \quad \text{has two solutions} \quad \boldsymbol{y(t) = t} \ \textbf{and} \ \boldsymbol{y(t) = -t}.$$

In those examples, y/t and t/y are starting from $0/0$. Solutions do exist (that fact wasn't guaranteed). Solutions are not unique (no surprise). We ask more from $f(t, y)$.

There is one important point that we emphasize here, because it could easily be missed.

Continuity of f and $\dfrac{\partial f}{\partial y}$ at all points does not guarantee that solutions reach $t = \infty$.

Yes, there will be a solution starting from $y(0)$. That solution will be unique. But $y(t)$ could blow up at some finite time t. The first nonlinear equation in the book (Section 1.1) was an example of early explosion:

Blow-up at $t = 1$ The solution to $\dfrac{dy}{dt} = y^2$ with $y(0) = 1$ is $y(t) = \dfrac{\mathbf{1}}{\mathbf{1 - t}}$.

That function $f = y^2$ is certainly continuous. Its derivative $\partial f/\partial y = 2y$ is also continuous. But the derivative $2y$ grows when the solution grows. To be sure there is no explosion at a finite time t, we ask for an upper bound L on the continuous function $\partial f/\partial y$:

If $\left|\dfrac{\partial f}{\partial y}\right| \le L$ for all t and y there is a unique solution through $y(0)$ reaching all t.

For a linear differential equation $y' = a(t)y + q(t)$, the derivative $\partial f/\partial y$ of the right hand side is just $a(t)$. Then if $|a(t)| \le L$ and $q(t)$ is continuous for all time, solution curves go from $t = -\infty$ to $t = \infty$. Chapter 1 found a formula for $y(t)$ in this linear case.

I will end with one final nonlinear fact. The condition $|\partial f/\partial y| \le L$ is pushed to its limit when $\partial f/\partial y = L$ exactly. Then $y' = Ly + q(t)$. A comparison with this linear equation gives information about the nonlinear equation, when $|\partial f/\partial y| \le L$:

$$\text{If } y' = f(t, y) \text{ and } z' = f(t, z), \text{ then } |y(t) - z(t)| \le e^{Lt}|y(0) - z(0)|. \quad (1)$$

If $y(t)$ and $z(t)$ start very close, they stay close. This is the opposite of what you see on the cover of this book. The cover shows a famous example of **chaos**: solutions go wild. A slight change in $y(0)$ will send the solution on a completely different (and distant) path. We now know that Pluto's orbit is chaotic: very very unpredictable. The equations allow it, because they don't have $|\partial f/\partial y| \le L$. Pluto is not a planet.

Pictures of the Solution

Example 1 $dy/dt = 2 - y$ **Solution** $y(t) = 2 + Ce^{-t}$ $y(\infty) = 2$

The perfect picture of $y' = 2 - y$ would show a small arrow at every point t, y. **The arrow would have slope $s = 2 - y$.** Along the all-important "steady state line" $y = 2$, this slope would be *zero*. The arrows are flat ($s = 0$) along that line: a constant solution.

Above that steady line, the slope $2 - y$ is negative. The vectors have components dt across and $dy = (2 - y)dt$ down. We don't have space for an arrow at every point, but Figure 3.1 gives the idea. MATLAB calls the field of arrows a "quiver".

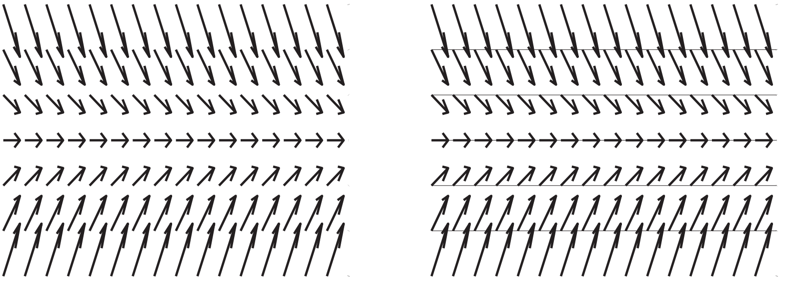

Figure 3.1: (a) Arrows with slopes $f(t, y)$ show the direction of the solution curves $y(t)$. (b) **Along an isocline $f(t, y) = s$, all arrows have the same slope s.** Here $s = 2 - y$.

Notice that all arrows point **toward** the line $y = 2$. That steady state solution is **stable**. The formula $y(t) = 2 + Ce^{-t}$ confirms that the solutions approach $y = 2$.

First key idea: **The solution curves $y(t) = 2 + Ce^{-t}$ are tangent to the arrows**. Tangent means: The curves have the same slope $s = 2 - y$ as the arrows! The curves solve the equation, the equation specifies the slopes, the arrows have correct slopes.

Second key idea: **Put your arrows along isoclines**. An isocline (meaning "same slope") is a curve $f(t, y) =$ constant. This idea makes the arrows much easier to draw. All the isoclines $2 - y = s$ are horizontal lines for this equation $y' = 2 - y$. When the differential equation is $dy/dt = f(t, y)$, **each choice of slope s produces an isocline $f(t, y) = s$.**

In our example, those isoclines $2 - y = s$ are flat because $f(t, y) = 2 - y$ does not depend on t (autonomous equation). I start the picture by drawing a few isoclines. I always draw the isocline $f(t, y) = 0$ (here $2 - y = 0$ is the steady state line $y = 2$). For this equation, that "nullcline" or "zerocline" with $s = 0$ **is also a solution curve**. The arrows have slope zero when $y = 2$, so they point along the flat line.

How to understand these pictures? **The arrows are pointing along the solution curves**. The curves cross over isoclines. But they don't cross over the zero isocline $y = 2$.

All arrows are pointing toward the line $y = 2$. Those arrows will eventually take us across every other isocline. The pictures say that the solution curves $y(t)$ are asymptotic to that line $y = 2$. For this equation $dy/dt = 2 - y$ we know the solutions $y = 2 + Ce^{-t}$.

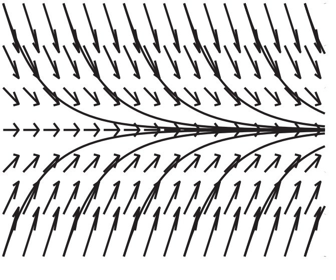

Figure 3.2: Solution curves (tangent to arrows) go through isoclines: $y' = 2 - y$.

Example 2 $\dfrac{dy}{dt} = y - y^2$ **Solutions** $y(t) = \dfrac{1}{1 + Ce^{-t}}$ $y(t) \to 1$ or $-\infty$

The slope of every small arrow is $y - y^2$. In the range $0 < y < 1$, y will be larger than y^2. The arrows have positive slope $y - y^2$ in this range (small slope near $y = 0$, small slope near $y = 1$, all up and to the right). The other two ranges are above $y = 1$ and below $y = 0$. There the slopes $y - y^2$ are negative—arrows go down and right. *The solution curves are steep when y is large*, because $y^2 >> y$.

Figure 3.3 shows the isoclines $f(t, y) = y - y^2 = s =$ constant. Again f does not depend on t ! The equation is autonomous, the isoclines are flat lines. There are **two zeroclines $y = 1$ and $y = 0$** (where $dy/dt = 0$ and y is constant). Those arrows have zero slope and the graph of $y(t)$ runs along each zerocline: a steady state.

The question is about all the other solution curves: What do they do? We happen to have a formula for $y(t)$, but the point is that *we don't need it*. Figure 3.3 shows the three possibilities for the solution curves to the *logistic equation* $y' = y - y^2$:

1. Curves above $y = 1$ go from $+\infty$ down toward the line $y = 1$ (**dropin curves**)

2. Curves between $y = 0$ and $y = 1$ go up toward that line $y = 1$ (***S*-curves**)

3. Curves below $y = 0$ go down (fast) toward $y = -\infty$ (**dropoff curves**).

The solution curves go across all isoclines except the two zeroclines where $y - y^2 = 0$.

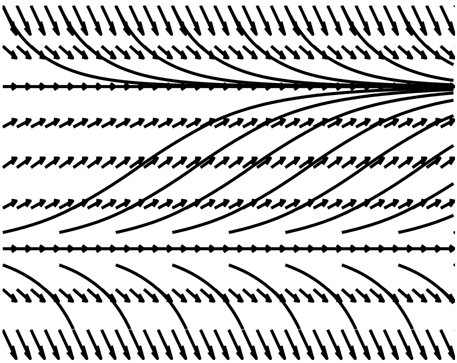

Figure 3.3: The arrows form a "direction field". Isoclines $y - y^2 = s$ attract or repel.

You see the S-curves between 0 and 1. The arrows are flat as they leave $y = 0$, steepest at $y = \frac{1}{2}$, flat again as they approach $y = 1$. The dropoff curves are below $y = 0$. Those arrows get very steep and the curves never reach $t = \infty$: $y = 1/(1 - e^{-t})$ gives $1/0 =$ *minus infinity* when $t = 0$. That dropoff curve never gets out of the third quadrant.

Important Solution curves have a special feature for autonomous equations $y' = f(y)$. Suppose the curve $y(t)$ is shifted right or left to the curve $Y(t) = y(t + C)$. Then $Y(t)$ solves the same equation $Y' = f(Y)$—both sides are just shifted in the same way.

Conclusion: The solution curves for autonomous equations $y' = f(y)$ just shift along *with no change in shape*. You can also see this by integrating $dy/f(y) = dt$ (separable equation). The right side integrates to $t + C$. We get all solutions by allowing all C.

In the logistic example, all S-curves and dropin curves and dropoff curves come from shifting *one* S-curve and *one* dropin curve and *one* dropoff curve.

Solution Curves Don't Meet

Is there a solution curve through every point (t, y)? Could two solution curves meet at that point? Could a solution curve suddenly end at a point? These "picture questions" are already answered by the facts.

At the start of this section, the functions f and $\partial f/\partial y$ were required to be continuous near $t = 0$, $y = y(0)$. Then there is a unique solution to $y' = f(t, y)$ with that start. In the picture this means: **There is exactly one solution curve going through the point.** The curve doesn't stop. By requiring f and $\partial f/\partial y$ to be continuous at and near *all* points, we guarantee one non-stopping solution curve through every point.

Example 3 will fail! The solution curves for $dy/dt = -t/y$ are half-circles and not whole circles. **They start and stop and meet on the line $y = 0$** (where $f = -t/y$ is **not continuous**). Exactly one semicircular curve passes through every point with $y \neq 0$.

Example 3 $\boldsymbol{dy/dt = -t/y}$ is separable. Then $y\,dy = -t\,dt$ leads to $\boldsymbol{y^2 + t^2 = C}$.

Start again with pictures. The isocline $f(t, y) = -t/y = s$ is the line $y = (-1/s)t$. All those isoclines go through $(0, 0)$ which is a very singular point. In this example the direction arrows with slope s are perpendicular to the isoclines with slope $dy/dt = -1/s$.

The isoclines are rays out from $(0, 0)$. The arrow directions are perpendicular to those rays and tangent to the solution curves. **The curves are half-circles $\boldsymbol{y^2 + t^2 = C}$.** (There is another half-circle on the opposite side of the axis. So two solutions start from $y = 0$ at time $-T$ and go forward to $y = 0$ at time T.) The solution curves stop at $y = 0$, where the function $f = -t/y$ loses its continuity and the solution loses its life.

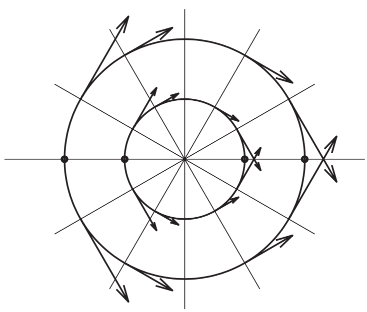

Figure 3.4: For $y' = -t/y$ the isoclines are rays. The solution curves are half-circles.

Example 4 $\boldsymbol{y' = 1 + t - y}$ is linear but not separable. The isoclines trap the solution.

Trapping between isoclines is a neat part of the picture. It is based on the arrows. **All arrows go one way across an isocline, so all solution curves go that way**. Solutions that cross the isocline can't cross back. The zero isocline $f(t, y) = 1 + t - y = 0$ in Figure 3.5 is the line $y = t + 1$. Along that isocline the arrows have slope 0. The solution curves must cross from above to below.

The central isocline $1 + t - y = 1$ in Figure 3.5 is the 45° line $y = t$. This solves the differential equation! The arrow directions are exactly along the line: slope $s = 1$. Other solution curves could never touch this one.

The picture shows solution curves in a "lobster trap" between the lines: the curves can't escape. They are trapped between the line $y = t$ and every isocline $1 + t - y = s$ above or below it. The trap gets tighter and tighter as s increases from 0 to 1, and the isocline gets closer to $y = t$. *Conclusion from the picture*: **The solution $\boldsymbol{y(t)}$ must approach $\boldsymbol{t}$.**

This is a linear equation $y' + y = 1 + t$. The null solutions to $y' + y = 0$ are Ce^{-t}. The forcing term $1 + t$ is a polynomial. A particular solution comes by substituting $y_p(t) = at + b$ into the equation and solving for those undetermined coefficients a and b:

$$(at + b)' = 1 + t - (at + b) \qquad \boldsymbol{a = 1 \text{ and } b = 0} \qquad \boldsymbol{y = y_n + y_p = Ce^{-t} + t} \tag{2}$$

The solution curves $y = Ce^{-t} + t$ do approach the line $y = t$ asymptotically as $t \to \infty$.

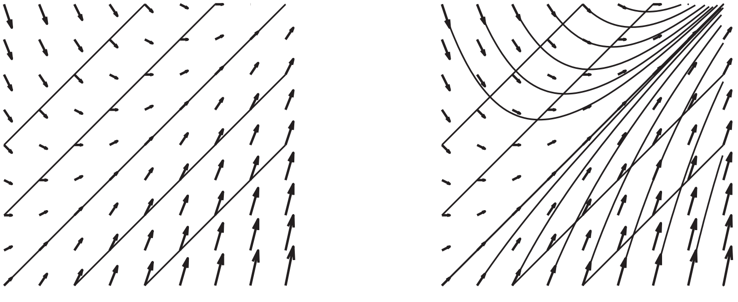

Figure 3.5: The solution curves for $y' = 1 + t - y$ get trapped between the 45° isoclines.

■ REVIEW OF THE KEY IDEAS ■

1. The direction field for $y' = f(t, y)$ has an arrow with slope f at each point t, y.
2. Along the isocline $f(t, y) = s$, all arrows have the same slope s.
3. The solution curves $y(t)$ are tangent to the arrows. One way through isoclines!
4. Fact: When f and $\partial f/\partial y$ are continuous, the curves cover the plane and don't meet.
5. The solution curves for autonomous $y' = f(y)$ shift left-right to $Y(t) = y(t - T)$.

Problem Set 3.1

1 (a) Why do two isoclines $f(t,y) = s_1$ and $f(t,y) = s_2$ never meet ?

(b) Along the isocline $f(t,y) = s$, what is the slope of all the arrows ?

(c) Then all solution curves go only one way across an ______.

2 (a) Are isoclines $f(t,y) = s_1$ and $f(t,y) = s_2$ always parallel ? Always straight ?

(b) An isocline $f(t,y) = s$ is a solution curve when its slope equals ______.

(c) The zerocline $f(t,y) = 0$ is a solution curve only when y is ______ : slope 0.

3 If $y_1(0) < y_2(0)$, what continuity of $f(t,y)$ assures that $y_1(t) < y_2(t)$ for all t ?

4 The equation $dy/dt = t/y$ is completely safe if $y(0) \neq 0$. Write the equation as $y\,dy = t\,dt$ and find its unique solution starting from $y(0) = -1$. The solution curves are hyperbolas—can you draw two on the same graph ?

5 The equation $dy/dt = y/t$ has many solutions $y = Ct$ in case $y(0) = 0$. It has no solution if $y(0) \neq 0$. When you look at all solution curves $y = Ct$, which points in the t, y plane have no curve passing through ?

6 For $y' = ty$ draw the isoclines $ty = 1$ and $ty = 2$ (those will be hyperbolas). On each isocline draw four arrows (they have slopes 1 and 2). Sketch pieces of solution curves that fit your picture between the isoclines.

7 The solutions to $y' = y$ are $y = Ce^t$. Changing C gives a higher or lower curve. But $y' = y$ is autonomous, its solution curves should be shifting right and left!

Draw $y = 2e^t$ and $y = -2e^t$ to show that they really are *right-left shifts* of $y = e^t$ and $y = -e^t$. The shifted solutions to $y' = y$ are e^{t+C} and $-e^{t+C}$.

8 For $y' = 1 - y^2$ the flat lines $y =$ constant are isoclines $1 - y^2 = s$. Draw the lines $y = 0$ and $y = 1$ and $y = -1$. On each line draw arrows with slope $1 - y^2$. The picture says that $y =$ ______ and $y =$ ______ are steady state solutions. From the arrows on $y = 0$, guess a shape for the solution curve $y = (e^t - e^{-t})/(e^t + e^{-t})$.

9 The parabola $y = t^2/4$ and the line $y = 0$ are both solution curves for $\boldsymbol{y' = \sqrt{|y|}}$. Those curves meet at the point $t = 0$, $y = 0$. What continuity requirement is failed by $f(y) = \sqrt{|y|}$, to allow more than one solution through that point ?

10 Suppose $y = 0$ up to time T is followed by the curve $y = (t - T)^2/4$. Does this solve $\boldsymbol{y' = \sqrt{|y|}}$? Draw this $y(t)$ going through flat isoclines $\sqrt{|y|} = 1$ and 2.

11 The equation $\boldsymbol{y' = y^2 - t}$ is often a favorite in MIT's course 18.03: not too easy. Why do solutions $y(t)$ rise to their maximum on $y^2 = t$ and then descend ?

12 Construct $f(t,y)$ with two isoclines so solution curves go *up* through the higher isocline and other solution curves go *down* through the lower isocline. *True or false*: Some solution curve will stay between those isoclines: **A continental divide**.