Chapter 1

First Order Equations

1.1 Four Examples : Linear versus Nonlinear

A first order differential equation connects a function $y(t)$ to its derivative dy/dt. That rate of change in y is decided by y itself (and possibly also by the time t).

Here are four examples. Example **1** is the most important differential equation of all.

1) $\dfrac{dy}{dt} = y$	2) $\dfrac{dy}{dt} = -y$	3) $\dfrac{dy}{dt} = 2ty$	4) $\dfrac{dy}{dt} = y^2$

Those examples illustrate three **linear** differential equations (**1**, **2**, and **3**) and a **nonlinear** differential equation. The unknown function $y(t)$ is squared in Example **4**. The derivative y or $-y$ or $2ty$ is proportional to the function y in Examples **1**, **2**, **3**. The graph of dy/dt versus y becomes a parabola in Example **4**, because of y^2.

It is true that t multiplies y in Example **3**. That equation is still linear in y and dy/dt. It has a *variable coefficient* $2t$, changing with time. Examples **1** and **2** have *constant coefficient* (the coefficients of y are 1 and -1).

Solutions to the Four Examples

We can write down a solution to each example. This will be one solution but it is not the *complete* solution, because each equation has a family of solutions. Eventually there will be a constant C in the complete solution. This number C is decided by the starting value of y at $t = 0$, exactly as in ordinary integration. The integral of $f(t)$ solves the simplest differential equation of all, with $y(0) = C$:

5) $\dfrac{dy}{dt} = f(t)$ The complete solution is $$y(t) = \int_0^t f(s)\, ds + C \quad .$$

For now we just write one solution to Examples **1 – 4**. They all start at $y(0) = 1$.

1 $\dfrac{dy}{dt} = y$ is solved by $y(t) = \boldsymbol{e^t}$

2 $\dfrac{dy}{dt} = -y$ is solved by $y(t) = \boldsymbol{e^{-t}}$

3 $\dfrac{dy}{dt} = 2ty$ is solved by $y(t) = \boldsymbol{e^{t^2}}$

4 $\dfrac{dy}{dt} = y^2$ is solved by $y(t) = \boldsymbol{\dfrac{1}{1-t}}$.

Notice: The three linear equations are solved by exponential functions (***powers of e***). The nonlinear equation **4** is solved by a different type of function; here it is $1/(1-t)$. Its derivative is $dy/dt = 1/(1-t)^2$, which agrees with y^2.

Our special interest now is in linear equations with *constant coefficients*, like **1** and **2**. In fact $dy/dt = y$ is the most important property of the great function $y = e^t$. Calculus had to create e^t, because a function from algebra (like $y = t^n$) cannot equal its derivative (the derivative of t^n is nt^{n-1}). But a combination of all the powers t^n can do it. That good combination is e^t in Section 1.3.

The final example extends **1** and **2**, to allow **any constant coefficient** $\boldsymbol{a}$:

6) $\boldsymbol{\dfrac{dy}{dt} = ay}$ is solved by $\boldsymbol{y = e^{at}}$ (and also $\boldsymbol{y = Ce^{at}}$).

If the constant growth rate a is positive, the solution increases. If a is negative, as in $dy/dt = -y$ with $a = -1$, the slope is negative and the solution e^{-t} decays toward zero. Figure 1.1 shows three exponentials, with dy/dt equal to y and $2y$ and $-y$.

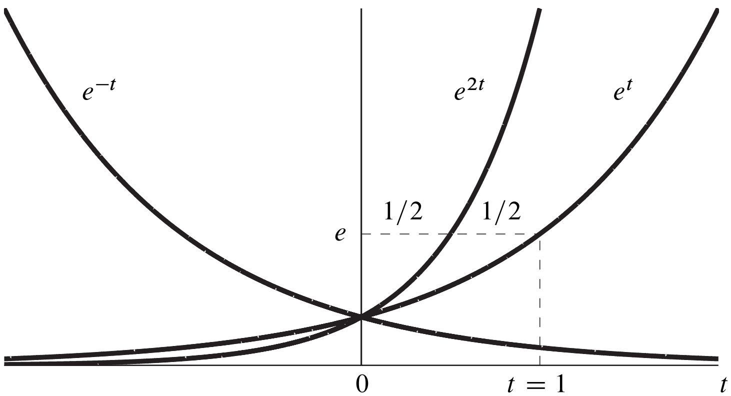

Figure 1.1: Growth, faster growth, and decay. The solutions are e^t and e^{2t} and e^{-t}.

When a is larger than 1, the solution grows faster than e^t. That is natural. The neat thing is that we still follow the exponential curve—but e^{at} climbs that curve faster. You could see the same result by *rescaling the time axis.* In Figure 1.1, the steepest curve (for $a = 2$) is the same as the first curve—but the time axis is compressed by 2.

Calculus sees this factor of 2 from the chain rule for e^{2t}. It sees the factor $2t$ from the chain rule for e^{t^2}. This exponent is t^2, the factor $2t$ is its derivative:

$$\frac{d}{dt}\left(e^u\right) = e^u \frac{du}{dt} \qquad \frac{d}{dt}\left(e^{2t}\right) = \left(e^{2t}\right) \text{ times } 2 \qquad \frac{d}{dt}\left(e^{t^2}\right) = \left(e^{t^2}\right) \text{ times } 2t$$

Problem Set 1.1

1 Draw the graph of $y = e^t$ by hand, for $-1 \le t \le 1$. What is its slope dy/dt at $t = 0$? Add the straight line graph of $y = et$. Where do those two graphs cross?

2 Draw the graph of $y_1 = e^{2t}$ on top of $y_2 = 2e^t$. Which function is larger at $t = 0$? Which function is larger at $t = 1$?

3 What is the slope of $y = e^{-t}$ at $t = 0$? Find the slope dy/dt at $t = 1$.

4 What "logarithm" do we use for the number t (the exponent) when $e^t = 4$?

5 State the chain rule for the derivative dy/dt if $y(t) = f(u(t))$ (chain of f and u).

6 The *second* derivative of e^t is again e^t. So $y = e^t$ solves $d^2y/dt^2 = y$. A second order differential equation should have another solution, different from $y = Ce^t$. What is that second solution?

7 Show that the nonlinear example $dy/dt = y^2$ is solved by $y = C/(1 - Ct)$ for every constant C. The choice $C = 1$ gave $y = 1/(1-t)$, starting from $y(0) = 1$.

8 Why will the solution to $dy/dt = y^2$ grow faster than the solution to $dy/dt = y$ (if we start them both from $y = 1$ at $t = 0$)? The first solution blows up at $t = 1$. The second solution e^t grows exponentially fast but it never blows up.

9 Find a solution to $dy/dt = -y^2$ starting from $y(0) = 1$. Integrate dy/y^2 and $-dt$. (Or work with $z = 1/y$. Then $\boldsymbol{dz/dt} = (dz/dy)\,(dy/dt) = (-1/y^2)(-y^2) = \mathbf{1}$. From $dz/dt = 1$ you will know $z(t)$ and $y = 1/z$.)

10 Which of these differential equations are linear (in y)?

(a) $y' + \sin y = t$ (b) $y' = t^2(y - t)$ (c) $y' + e^t y = t^{10}$.

11 The product rule gives what derivative for $e^t e^{-t}$? This function is constant. At $t = 0$ this constant is 1. Then $e^t e^{-t} = 1$ for all t.

12 $dy/dt = y + 1$ is not solved by $y = e^t + t$. Substitute that y to show it fails. We can't just add the solutions to $y' = y$ and $y' = 1$. What number c makes $y = e^t + c$ into a correct solution?