Preface

This is a basic textbook for linear algebra, combining the theory with the applications. The central equations in this subject are the linear system $Ax = b$ and the eigenvalue problem $Ax = \lambda x$. It is simply amazing how much there is to say (and to learn) about those two equations. This book comes from years of teaching and organizing and thinking about the linear algebra course — and still this subject feels new and very alive.

I am really happy that the need for linear algebra is widely recognized. It is *absolutely* as important as calculus. I don't concede anything, when I look at how mathematics is actually used. So many applications are discrete rather than continuous, digital rather than analog, linearizable rather than erratic and chaotic. Then vectors and matrices are the right language to know.

The equation $Ax = b$ uses that language right away. The left side has a known matrix A times an unknown vector x. There is a good way to see their product: **Ax is a combination of the columns of A**. The equation is asking us to find the combination that produces b. Our solution can come at three levels and they are all important:

1. **Direct solution** by forward elimination and back substitution.

2. **Matrix solution** by $x = A^{-1}b$ using the inverse matrix A^{-1}.

3. **Vector space solution** by finding all combinations of the columns of A and all solutions to $Ax = 0$.

And there is another possibility: *$Ax = b$ may have no solution.* The direct approach by elimination may lead to $0 = 1$. The matrix approach may fail to find A^{-1}. The vector space approach can look at all combinations of the columns, but b might not be in that column space. Part of the mathematics is understanding when an equation is solvable and when it's not.

Another part is to visualize vectors. A single vector v with two components is not hard. The components v_1 and v_2 tell us how far to go across and up — we can draw an arrow. A second vector w may be perpendicular to v (and Chapter 1 tells exactly when). If each vector has six components, the artist is lost but our imagination keeps trying. We can think of a right angle in six-dimensional space. We can see $2v$ (twice as far) and $-w$ (opposite direction to w). We can *almost* see a combination like $2v - w$.

Most important is the effort to imagine **all the combinations of v and w**. They fill out some kind of "two-dimensional plane" inside the six-dimensional space. As I write those words, I am not at all sure that I see that subspace. But linear algebra gives a simple way to work with vectors and matrices of any size. And if we have six currents in a network or six forces on a structure or six prices for our products, we are certainly in six-dimensional space.

Already in this preface, you can see the style of the book and its goal. The style is informal but the goal is serious. Linear algebra is great mathematics, and I try to explain it as clearly as I can. I certainly hope that the professor in this course learns new things. The author always does. The student will see the ideas and the applications reinforcing each other. That is the whole point for all of us — to learn how to think. I hope you will see how this book moves forward, *gradually but steadily.*

Mathematics is continually asking you to look beyond the most specific case (those six prices at a particular moment). The goal is to see the broader pattern. Whether we have pixel intensities on a TV screen or forces on an airplane or schedules for every pilot, those are all vectors and they all get multiplied by matrices.

Structure of the Course

I want to note four points about the organization of the book:

1. Chapter 1 provides a brief introduction to the basic ideas of vectors and dot products. If the class has met them before, there is no problem to start the course with Chapter 2.

2. I now use the *row reduced echelon form* more heavily than before. In MATLAB this is $R = \mathrm{rref}(A)$. Reducing A to R produces bases for the row space and column space and nullspace of A. Better than that, reducing the combined matrix $[A\ I]$ produces total information about all four of the fundamental subspaces.

3. I am still convinced that these four subspaces (two in $\mathbf{R}^m$ and two in $\mathbf{R}^n$) are the right way to learn about linear independence and bases and dimension. The examples are so natural, and they are genuinely the key to applications. I hate just making up artificial examples of vector spaces when there are so many good ones. If the class sees plenty of examples of independence and dependence, then the definition is virtually understood in advance.

4. Section 6.1 introduces eigenvalues for 2 *by* 2 *matrices*. Many courses want to meet eigenvalues early (to apply them in another course or to avoid missing them completely). It is absolutely possible to go directly from Chapter 3 to Section 6.1. The determinant is easy for a 2 by 2 matrix.

A one-semester course that moves steadily can reach eigenvalues. The idea of *diagonalizing a matrix* is the key. For most matrices that is $S^{-1}AS$. Later for symmetric matrices it is Q^TAQ, and for all matrices it is U^TAV. I do my best to reach that Singular Value Decomposition because it has become extremely useful. If I had two semesters I could do so much more. But I feel very good about this course and the response to it.

Options for the Course

Chapters 1–6 are the heart of a basic course in linear algebra — *theory plus applications*. The beauty of this subject is in the way those two parts come together. The exams and homeworks available on the web page `http://web.mit.edu/18.06/www` concentrate on these six chapters.

Chapter 7 connects linear transformations with matrices. The matrix depends on the choice of basis! We show how vectors and matrices change when the basis changes. And we show the linear transformation behind the matrix. I don't start the course with that deeper idea, it is better to understand subspaces first.

Chapter 8 gives important applications — often I choose Markov matrices for a lecture without an exam. Chapter 9 comes back to numerical linear algebra, to explain how $Ax = b$ and $Ax = \lambda x$ are actually solved. Chapter 10 moves from real to complex numbers, as entries in the vectors and matrices. The ten chapters are very appropriate for a two-semester course and I wish we had that at MIT.

Computing in Linear Algebra

The text gives first place to MATLAB, a beautiful system that was developed specifically for linear algebra. This is the first language of our Teaching Codes, written by Cleve Moler for the first edition and much extended by Steve Lee for this edition. They are on the web:

`http://www-math.mit.edu/~linalg`

This page has a link to the home page for the 18.06 linear algebra course, with exams and MATLAB homeworks. Please use the page freely. I hope to offer a prize to the $25{,}000^{\text{th}}$ user and it is close.

We will provide a similar library of Teaching Codes for Maple and Mathematica. The codes execute the steps of linear algebra in the way we teach them. Then the reader can see matrix theory both ways — the algebra and also the algorithms. Fortunately those work together perfectly. This textbook supports a course that includes computing and also a course that doesn't. Either way, there is so much good mathematics to learn and to do.