LINEAR ALGEBRA, GEODESY, AND GPS

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Preface

Geodesy begins with measurements from control points. Classical geometric geodesy measures heights and angles and distances on the earth. For the Global Positioning System (GPS), the control points are satellites and the accuracy is phenomenal. But even when the measurements are reliable, we make more than absolutely necessary. Mathematically, the positioning problem is overdetermined.

There are errors in the measurements. The data are nearly consistent, but not exactly. An algorithm must be chosen—very often it is *least squares*—to select the output that best satisfies all the inconsistent and overdetermined and redundant (but still accurate!) measurements. This book is about algorithms for geodesy and global positioning.

The starting point is least squares. The equations $A\mathbf{x} = \mathbf{b}$ are overdetermined. They have no solution, but a "best solution" must be found. This fundamental linear problem can be understood in different ways, and all of these ways are important:

1. (Calculus) Choose $\hat{\mathbf{x}}$ to minimize $\|A\mathbf{x} - \mathbf{b}\|^2$.

2. (Geometry) Project $\mathbf{b}$ onto the "column space" containing all vectors $A\mathbf{x}$.

3. (Linear algebra) Solve the normal equations $A^{\mathrm{T}}A\hat{\mathbf{x}} = A^{\mathrm{T}}\mathbf{b}$.

Chapter 4 develops these ideas. We emphasize especially how least squares is a projection: The residual error $\mathbf{r} = A\hat{\mathbf{x}} - \mathbf{b}$ is orthogonal to the columns of A. This is basic linear algebra, and we follow the exposition in the book by Gilbert Strang (1993). We hope that each reader will find new insights into this fundamental problem.

Another source of information affects the best answer. The measurement errors have probability distributions. When data are more reliable (with smaller variance), they should be weighted more heavily. By using statistical information on means and variances, the output is improved. Furthermore the statistics may change with time—we get new information as measurements come in.

The classical unweighted problem $A^{\mathrm{T}}A\hat{\mathbf{x}} = A^{\mathrm{T}}\mathbf{b}$ becomes more dynamic and realistic (and more subtle) in several steps:

- ***Weighted*** least squares (using the covariance matrix to assign weights)
- ***Recursive*** least squares (for fast updating without recomputing)
- ***Dynamic*** least squares (using the Kalman filter).

The Kalman filter (Chapter 17) updates not only $\hat{\mathbf{x}}$ itself but also its variance—which decreases as more information comes in. Throughout geodesy and global positioning is another complication that cannot be ignored. This subject requires

- ***Nonlinear*** least squares (distance $\sqrt{x^2 + y^2}$ and angle $\arctan \frac{y}{x}$ are not linear!)

Nonlinearity is handled incrementally by small linearized steps. Chapter 10 shows how to compute and use the gradient vector, containing the derivatives of measurements with respect to coordinates. This gives the (small) change in positioning due to a (small) adjustment in the measurements.

Inevitably we must deal with numerical error, in the solution procedure as well as the data. The condition number of the least squares problem may be large—the normal equations may be nearly singular. Many applications are actually rank deficient, and we require extra constraints to produce a unique solution. The key tool has become the ***Singular Value Decomposition*** (SVD), which is described in Chapter 7. It is a choice of orthogonal bases in which the matrix becomes diagonal. It applies to all rectangular matrices A, by using the (orthogonal) eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}.

The authors hope very much that this book—a combination of *linear algebra* and *geodesy*—will be useful to its readers. We all have a natural desire to know where we are! Positioning is absolutely important (and relatively simple). GPS receivers are not expensive. You could control a fleet of trucks, or set out new lots, or preserve your own safety in the wild, by quick and accurate knowledge of position. From the Times of 11 July 1996, GPS enables aircraft to fly an exact track and shave up to an hour off the time to Hong Kong. This is on the "world's longest non-stop scheduled flight" from Chicago.

The new GPS technology is moving the old science of geodesy into completely unexpected applications. We think Gauss would be pleased, to see old problems solved and new ones appear.

We hope that the friends who helped us will be pleased too. Our greatest debt is to Clyde C. Goad, for discussions that opened up new aspects of geodesy. He sees *ideas* and not just formulas. And we remember a letter long ago from Helmut Moritz, who connected geodesy to the most basic framework of applied mathematics. Then Stephen Hilla and Peter Teunissen and Tom Herring gave valuable help as the book developed and its structure unfolded. For the chapter on filtering, Søren Lundbye-Christensen has given excellent advice. We thank you all.

Our algorithms are generally expressed in MATLAB. The reader will be able (not yet!) to obtain all the M-files by ftp, in order to execute techniques (and examples) from the book. The first list of available M-files is printed at the end of the book, and is continuously updated in our web homepages. Computation is now an essential part of this subject.

This book separates naturally into three parts. The first is basic linear algebra. The second is the application to the (linear and also nonlinear) science of measurement. The third is the excitement of GPS. You will see how the theory is immediately needed and used. *Measurements are all around us, today and tomorrow. The goal is to minimize the error.*

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