TABLE OF CONTENTS

PREFACE

Geodesy begins with measurements from control points. Geometric geodesy measures heights and angles and distances on the earth. For the Global Positioning System (GPS), the control points are satellites and the accuracy is phenomenal. But even when the measurements are reliable, we make more than absolutely necessary. Mathematically, the positioning problem is overdetermined.

There are errors in the measurements. The data are nearly consistent, but not exactly. An algorithm must be chosen—very often it is *least squares*—to select the output that best satisfies all the inconsistent and overdetermined and redundant (but still accurate!) measurements. This book is about algorithms for geodesy and global positioning.

The starting point is least squares. The equations $A\boldsymbol{x} = \boldsymbol{b}$ are overdetermined. No vector $\boldsymbol{x}$ gives agreement with all measurements $\boldsymbol{b}$, so a "best solution" $\hat{\boldsymbol{x}}$ must be found. This fundamental linear problem can be understood in different ways, and all of these ways are important:

1 (Calculus) Choose $\hat{\boldsymbol{x}}$ to minimize $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$.

2 (Geometry) Project $\boldsymbol{b}$ onto the "column space" containing all vectors $A\boldsymbol{x}$.

3 (Linear algebra) Solve the normal equations $A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Chapter 4 develops these ideas. We emphasize especially how least squares is a projection: The residual error $\boldsymbol{r} = A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is orthogonal to the columns of A. That means $A^{\mathrm{T}}\boldsymbol{r} = \boldsymbol{0}$, which is the same as $A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. This is basic linear algebra, and we follow the exposition in the book by Gilbert Strang (1993). We hope that each reader will find new insights into this fundamental problem.

Another source of information affects the best answer. The measurement errors have probability distributions. When data are more reliable (with smaller variance), they should be weighted more heavily. By using statistical information on means and variances, the output is improved. Furthermore the statistics may change with time—we get new information as measurements come in.

The classical unweighted problem $A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ becomes more dynamic and realistic (and more subtle) in several steps:

– ***Weighted*** least squares (using the covariance matrix Σ to assign weights)

– ***Recursive*** least squares (for fast updating without recomputing)

– ***Dynamic*** least squares (using sequential filters).

The Kalman filter updates not only $\hat{\boldsymbol{x}}$ itself (the estimated state vector) but also its variance.

Chapter 17 develops the theory of filtering in detail, with examples of positioning problems for a GPS receiver. We describe the Kalman filter and also its variant the Bayes filter—which computes the updates in a different (and sometimes faster) order. The formulas for filtering are here based directly on matrix algebra, not on the theory of conditional probability—because more people understand matrices!

Throughout geodesy and global positioning are two other complications that cannot be ignored. This subject requires

– ***Nonlinear*** least squares (distance $\sqrt{x^2 + y^2}$ and angle $\arctan \frac{y}{x}$ are not linear!)

– ***Integer*** least squares (to count wavelengths from satellite to receiver).

Nonlinearity is handled incrementally by small linearized steps. Chapter 10 shows how to compute and use the gradient vector, containing the derivatives of measurements with respect to coordinates. This gives the (small) change in position estimates due to a (small) adjustment in the measurements.

Integer least squares resolves the "ambiguity" in counting wavelengths—because the receiver sees only the fractional part. This could be quite a difficult problem. A straightforward approach usually succeeds, and we describe (with MATLAB software) the LAMBDA method that preconditions and decorrelates harder problems.

Inevitably we must deal with numerical error, in the solution procedures as well as the data. The condition number of the least squares problem may be large—the normal equations may be nearly singular. Many applications are actually rank deficient, and we require extra constraints to produce a unique solution. The key tool from matrix analysis is the ***Singular Value Decomposition*** (SVD), which is described in Chapter 7. It is a choice of orthogonal bases in which the matrix becomes diagonal. It applies to all rectangular matrices A, by using the (orthogonal) eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}.

The authors hope very much that this book will be useful to its readers. We all have a natural desire to know where we are! Positioning is absolutely important (and relatively simple). GPS receivers are not expensive. You could control a fleet of trucks, or set out new lots, or preserve your own safety in the wild, by quick and accurate knowledge of position. From The Times of 11 July 1996, GPS enables aircraft to shave up to an hour off the time from Chicago to Hong Kong. This is one of the world's longest non-stop scheduled flights—now a little shorter.

The GPS technology is moving the old science of geodesy into new and completely unexpected applications. This is a fantastic time for everyone who deals with measurements of the Earth. We think Gauss would be pleased.

We hope that the friends who helped us will be pleased too. Our debt is gladly acknowledged, and it is a special pleasure to thank Clyde C. Goad. Discussions with him have opened up new aspects of geodesy and important techniques for GPS. He sees ideas and good algorithms, not just formulas. We must emphasize that algorithms and software are an integral part of this book.

Our algorithms are generally expressed in MATLAB. The reader can obtain all the M-files from http://www.i4.auc.dk/borre/matlab. Those M-files execute the techniques (and examples) that the book describes. The first list of available M-files is printed at the end of the book, and is continuously updated in our web homepages. Computation is now an essential part of this subject. (*The M-files will be available when the book is published*.)

This book separates naturally into three parts. The first is basic linear algebra. The second is the application to the (linear and also nonlinear) science of measurement. The third is the excitement of GPS. You will see how the theory is immediately needed and used. *Measurements are all around us, today and tomorrow. The goal is to extract the maximum information from those measurements*.

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