1.3 Matrices Multiplying Vectors : A times x

An $\boldsymbol{m}$ **by** $\boldsymbol{n}$ **matrix** $\boldsymbol{A}$ has m rows and n columns

Those columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are in m-dimensional space

Their combinations are $x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = A\boldsymbol{x} =$ **matrix $\boldsymbol{A}$ times vector $\boldsymbol{x}$**

$$\boldsymbol{A} = \begin{bmatrix} & & & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ & & & \end{bmatrix}$$

There is a **row way** to multiply $A\boldsymbol{x}$ and also a **column way** to compute the vector $A\boldsymbol{x}$

Row way $=$ **Dot product** of vector $\boldsymbol{x}$ with each row of A

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 5v_2 \\ 3v_1 + 7v_2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} \quad \begin{matrix}\text{Find 7} \\ \text{Then 10}\end{matrix}$$

Column way $=$ $A\boldsymbol{x}$ is a **combination of the columns of $\boldsymbol{A}$**

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} \text{column} \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} \text{column} \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \mathbf{7} \\ \mathbf{10} \end{bmatrix} \quad \begin{matrix}\text{7 and 10} \\ \text{together}\end{matrix}$$

Which way to choose ? Dot products with **rows** or combination of **columns** ?

For computing with numbers, I use the row way : dot products

For understanding with vectors, I use the column way : combine columns

Same result $A\boldsymbol{x}$ from the same multiply-adds. Just in a different order

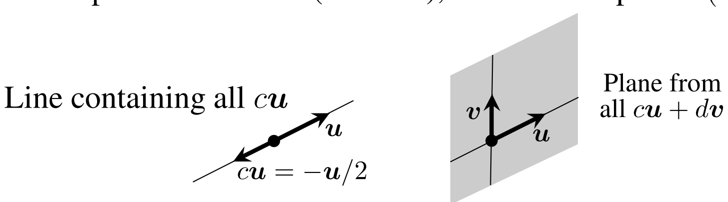
$\mathbf{C}(A) =$ **Column space of $\boldsymbol{A}$ $=$ all combinations of the columns $=$ all outputs $\boldsymbol{A}\boldsymbol{x}$**

The **identity matrix** has $I\boldsymbol{x} = \boldsymbol{x}$ for every $\boldsymbol{x}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The column space of the 3 by 3 identity matrix I is the whole space $\mathbf{R}^3$.

If all columns are multiples of column 1 (not zero), the column space $\mathbf{C}(A)$ is a line.



3.3 Independent Columns and Rows : Bases by Elimination

Remember $\boldsymbol{A} = \boldsymbol{CR}$ with r independent columns in $\boldsymbol{C}$ (but how to find them **?**)

The good way is **elimination on the $\boldsymbol{m}$ rows of $\boldsymbol{A}$** (not the columns)

In Chapter 2, elimination reduced A to the n by n identity matrix : A was invertible

Now elimination will produce an **$\boldsymbol{r}$ by $\boldsymbol{r}$ identity matrix inside $\boldsymbol{R}$**

That identity matrix locates the **$\boldsymbol{r}$ independent columns of $\boldsymbol{A}$**

Here is an example of elimination

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\text{steps}]{\text{two}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{steps}]{\text{two}} \begin{bmatrix} \mathbf{1} & \mathbf{0} & 2 \\ \mathbf{0} & \mathbf{1} & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{“reduced row echelon form”} \ \boldsymbol{R_0}$$

This last matrix $\boldsymbol{R_0}$ reveals the row space and column space and nullspace of A

Basis for the row space of $A =$ **Rows of $\boldsymbol{R}$ = Rows 1 and 2 of $\boldsymbol{R_0}$**

Basis for the column space of $A =$ **Columns 1 and 2 of $\boldsymbol{A}$. Then $\boldsymbol{A} = \boldsymbol{CR}$**

Basis for the nullspace of A : Solve $R_0\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ in Section 3.4

We will show how elimination works to reach this special form R_0 : $m - r$ zero rows

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Three types of elimination steps All of them can be reversed !

1 Subtract a multiple of one row from another row (below or above)

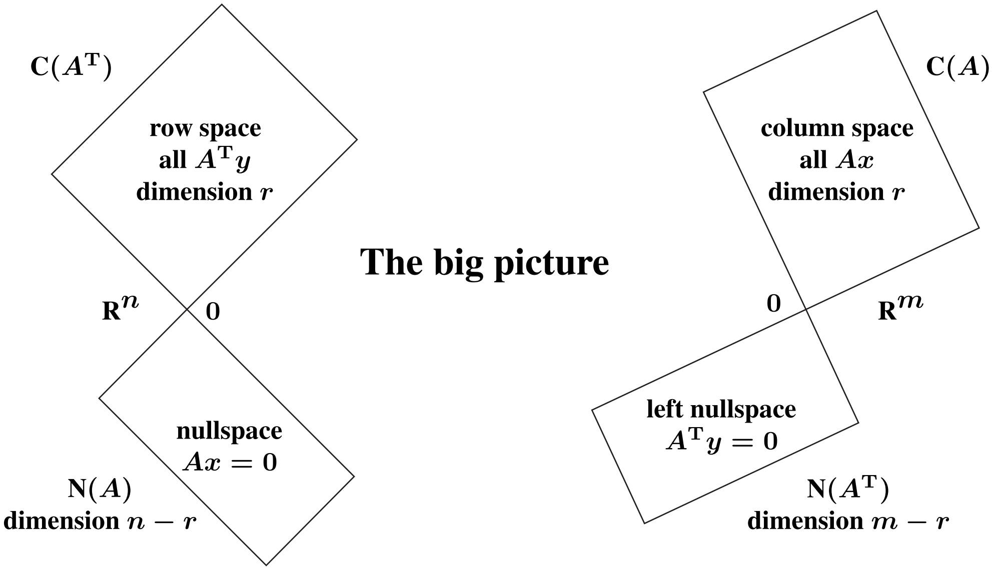
2 Multiply a row by a nonzero number (to produce pivot = first nonzero = $\mathbf{1}$)

3 Exchange rows (to move pivot rows in R above any zero rows in R_0)

Key point **Those steps do not change the row space of a matrix**

The result $\boldsymbol{R_0} = \begin{bmatrix} \boldsymbol{R} \\ \mathbf{0} \end{bmatrix}$ has the same row space as A : simpler rows and $m - r$ zero rows

3.5 Four Fundamental Subspaces $\mathbf{C}(A), \mathbf{C}(A^{\mathrm{T}}), \mathbf{N}(A), \mathbf{N}(A^{\mathrm{T}})$



***Fundamental Theorem of Linear Algebra*, Part 1**

The column space and row space both have dimension r.
The nullspaces have dimensions $n - r$ and $m - r$.

This tells us the **Counting Theorem** : How many solutions to $A\boldsymbol{x} = \mathbf{0}$? $\boldsymbol{n - r}$

$\boldsymbol{m}$ equations, $\boldsymbol{n}$ unknowns, rank $\boldsymbol{r} \Rightarrow A\boldsymbol{x} = \mathbf{0}$ has $\boldsymbol{n - r}$ independent solutions

At least $n - m$ solutions. More solutions for dependent equations (then $r < m$)

There is always a nonzero solution x to $Ax = 0$ if $n > m$ Good to know

Fundamental Theorem, Part 2 : **Subspaces are orthogonal : Chapter** 4

Fundamental Theorem, Part 3 : **Perfect bases = singular vectors $\boldsymbol{v}, \boldsymbol{u}$: Chapter 7**

Row space :	Basis $\boldsymbol{v}_1$ to $\boldsymbol{v}_r$	**Column space** :	Basis $\boldsymbol{u}_1$ to $\boldsymbol{u}_r$
Nullspace :	Basis $\boldsymbol{v}_{r+1}$ to $\boldsymbol{v}_n$	**Nullspace of A^{T}** :	Basis $\boldsymbol{u}_{r+1}$ to $\boldsymbol{u}_m$

Part 7 : Singular Values and Vectors : $Av = \sigma u$ and $A = U\Sigma V^{\mathrm{T}}$

7.1 Singular Vectors in U and V—Singular Values in Σ

An example shows **orthogonal inputs** $\boldsymbol{v}$ going into **orthogonal outputs** $A\boldsymbol{v}$

$$A\boldsymbol{v}_1 = \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{4} & \mathbf{5} \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{9} \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_2 = \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{4} & \mathbf{5} \end{bmatrix}\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is orthogonal to } \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ is orthogonal to } \boldsymbol{u}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Divide inputs $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by $\sqrt{2}$ Divide outputs $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ by $\sqrt{10}$

Four unit vectors with $A\boldsymbol{v}_1 = 3\sqrt{5}\,\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \sqrt{5}\,\boldsymbol{u}_2$ Notice $\sqrt{10}/\sqrt{2} = \sqrt{5}$

$\boldsymbol{v}_1, \boldsymbol{v}_2$ = orthogonal basis for the **row space** of A = right singular vectors in V

$\boldsymbol{u}_1, \boldsymbol{u}_2$ = orthogonal basis for the **column space** of A = left singular vectors in U

$\boldsymbol{\sigma}_1 = 3\sqrt{5}$ and $\boldsymbol{\sigma}_2 = \sqrt{5}$ are the **singular values** of A in the diagonal matrix $\boldsymbol{\Sigma}$

Express $A\boldsymbol{v}_1 = 3\sqrt{5}\,\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \sqrt{5}\,\boldsymbol{u}_2$ in matrix form $\boldsymbol{AV} = \boldsymbol{U\Sigma}$

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/\sqrt{2} \text{ and } \boldsymbol{U} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}/\sqrt{10} \text{ are orthogonal matrices} \quad \begin{matrix} \boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} = \boldsymbol{I} \\ \boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} = \boldsymbol{I} \end{matrix}$$

Matrix form $\boldsymbol{AV} = \boldsymbol{U\Sigma}$

$$\begin{bmatrix} \mathbf{3} & 0 \\ 4 & \mathbf{5} \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \mathbf{3\sqrt{5}} & 0 \\ 0 & \mathbf{\sqrt{5}} \end{bmatrix}$$

Multiply by $\boldsymbol{V}^{-1} = \boldsymbol{V}^{\mathrm{T}}$

$\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathrm{T}}$ is the perfect decomposition of A: **orthogonal–diagonal–orthogonal**