1.3 Matrices Multiplying Vectors: $A$ times $x$

An $m$ by $n$ matrix $A$ has $m$ rows and $n$ columns.

Those columns $a_1, a_2, \ldots, a_n$ are in $m$-dimensional space $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

Their combinations are $x_1 a_1 + \cdots + x_n a_n = A x = \text{matrix } A \times \text{vector } x$

There is a **row way** to multiply $Ax$ and also a **column way** to compute the vector $Ax$.

**Row way** = Dot product of vector $x$ with each row of $A$

$$Ax = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 2v_1 + 5v_2 \\ 3v_1 + 7v_2 \end{bmatrix} + v_2 \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

Find 7, then 10.

**Column way** = $Ax$ is a combination of the columns of $A$

$$Ax = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} \text{column 1} \end{bmatrix} + v_2 \begin{bmatrix} \text{column 2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

7 and 10 together.

Which way to choose? Dot products with **rows** or combination of **columns**?

For computing with numbers, I use the row way: dot products.

For understanding with vectors, I use the column way: combine columns.

Same result $Ax$ from the same multiply-adds. Just in a different order.

**$C(A) = \text{Column space of } A = \text{all combinations of the columns} = \text{all outputs } A x$**

The **identity matrix** has $Ix = x$ for every $x$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The column space of the 3 by 3 identity matrix $I$ is the whole space $\mathbb{R}^3$.

If all columns are multiples of column 1 (not zero), the column space $C(A)$ is a line.

Line containing all $cu$

Plane from all $cu + dv$

$cu = -u/2$
3.3 Independent Columns and Rows : Bases by Elimination

Remember \( A = CR \) with \( r \) independent columns in \( C \) (but how to find them?)

The good way is elimination on the \( m \) rows of \( A \) (not the columns)

In Chapter 2, elimination reduced \( A \) to the \( n \) by \( n \) identity matrix : \( A \) was invertible

Now elimination will produce an \( r \) by \( r \) identity matrix inside \( R \)

That identity matrix locates the \( r \) independent columns of \( A \)

Here is an example of elimination
\[
A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 0 & 4 & 4 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{“reduced row echelon form”}
\]

This last matrix \( R_0 \) reveals the row space and column space and nullspace of \( A \)

Basis for the row space of \( A = \text{Rows of } R = \text{Rows 1 and 2 of } R_0 \)

Basis for the column space of \( A = \text{Columns 1 and 2 of } A \). Then \( A = CR \)

Basis for the nullspace of \( A \) : Solve \( R_0 x = 0 \) to find \( x = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \) in Section 3.4

We will show how elimination works to reach this special form \( R_0 : m - r \) zero rows

---

Three types of elimination steps All of them can be reversed!

1. Subtract a multiple of one row from another row (below or above)

2. Multiply a row by a nonzero number (to produce pivot = first nonzero = 1)

3. Exchange rows (to move pivot rows in \( R \) above any zero rows in \( R_0 \))

Key point Those steps do not change the row space of a matrix

The result \( R_0 = \begin{bmatrix} R \\ 0 \end{bmatrix} \) has the same row space as \( A \) : simpler rows and \( m - r \) zero rows
3.5 Four Fundamental Subspaces $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$

The big picture

$C(A^T)$
row space all $A^T y$
dimension $r$

$R^n$
0

nullspace
$N(A)$
dimension $n - r$

$N(A^T)$
dimension $m - r$

$C(A)$
column space all $A x$
dimension $r$

$R^m$
0

left nullspace
$N(A^T)$
$A^T y = 0$

Fundamental Theorem of Linear Algebra, Part 1

The column space and row space both have dimension $r$.
The nullspaces have dimensions $n - r$ and $m - r$.

This tells us the **Counting Theorem**: How many solutions to $A x = 0$? $n - r$

$m$ equations, $n$ unknowns, rank $r \Rightarrow A x = 0$ has $n - r$ independent solutions

At least $n - m$ solutions. More solutions for dependent equations (then $r < m$)

There is always a nonzero solution $x$ to $A x = 0$ if $n > m$ Good to know

Fundamental Theorem, Part 2: **Subspaces are orthogonal**: Chapter 4

Fundamental Theorem, Part 3: **Perfect bases = singular vectors $v, u$**: Chapter 7

Row space: Basis $v_1$ to $v_r$
Nullspace: Basis $v_{r+1}$ to $v_n$
Column space: Basis $u_1$ to $u_r$
Nullspace of $A^T$: Basis $u_{r+1}$ to $u_m$
Part 7: Singular Values and Vectors:
\[ Av = \sigma u \text{ and } A = U\Sigma V^T \]

7.1 Singular Vectors in \( U \) and \( V \)—Singular Values in \( \Sigma \)

An example shows orthogonal inputs \( v \) going into orthogonal outputs \( Av \)

\[
Av_1 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad Av_2 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
\]

\( v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is orthogonal to \( v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)
\( u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) is orthogonal to \( u_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \)

Divide inputs \( v_1 \) and \( v_2 \) by \( \sqrt{2} \)

Divide outputs \( u_1 \) and \( u_2 \) by \( \sqrt{10} \)

Four unit vectors with \( Av_1 = 3\sqrt{5} u_1 \) and \( Av_2 = \sqrt{5} u_2 \)

Notice \( \sqrt{10}/\sqrt{2} = \sqrt{5} \)

\( v_1, v_2 = \) orthogonal basis for the row space of \( A = \) right singular vectors in \( V \)

\( u_1, u_2 = \) orthogonal basis for the column space of \( A = \) left singular vectors in \( U \)

\( \sigma_1 = 3\sqrt{5} \) and \( \sigma_2 = \sqrt{5} \) are the singular values of \( A \) in the diagonal matrix \( \Sigma \)

Express \( Av_1 = 3\sqrt{5} u_1 \) and \( Av_2 = \sqrt{5} u_2 \) in matrix form \( AV = U\Sigma \)

\[
V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /\sqrt{2} \quad \text{and} \quad U = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} /\sqrt{10} \quad \text{are orthogonal matrices} \quad V^TV = I \quad U^TU = I
\]

Matrix form
\[
AV = U\Sigma \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \quad \text{Multiply by} \quad V^{-1} = V^T
\]

\( A = U\Sigma V^T \) is the perfect decomposition of \( A \): orthogonal–diagonal–orthogonal