# On the Maximum Number of Triangles in Wheel-Free Graphs


Zoltán Füredi*
Michel X. Goemans†
Daniel J. Kleitman‡

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139


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**Abstract**

Gallai [1] raised the question of determining $t(n)$, the maximum number of triangles in graphs of $n$ vertices with acyclic neighborhoods. Here we disprove his conjecture ($t(n) \sim n^2/8$) by exhibiting graphs having $n^2/7.5$ triangles. We improve the upper bound [11] of $(n^2 - n)/6$ to $t(n) \leq n^2/7.02 + O(n)$. For regular graphs, we further decrease this bound to $n^2/7.75 + O(n)$.


## 1 Introduction

Let $\mathrm{WFG}_n$ be the class of graphs on $n$ vertices with the property that the neighborhood of any vertex is acyclic. A graph $G$ is given by its vertex set, $V(G)$, and edge set, $E(G)$. The subgraph *induced* by $X \subset V(G)$ is denoted by $G[X]$. The neighborhood, $N(v)$, of vertex $v$ is the set of vertices adjacent to $v$. Note that $v \notin N(v)$. The *degree* of $v \in V(G)$, denoted by $d_v$ or $d_v(G)$, is the size of the neighborhood, $d_v = |N(v)|$. The maximum (minimum) degree is denoted by $\Delta$ ($\delta$), or $\Delta(G)$ ($\delta(G)$, respectively) to avoid misunderstandings. A


*Research supported in part by the Hungarian National Science Foundation under grant No. 1812.


New address: Dept. Math., Univ. Illinois, Urbana, IL 61801-2917. E-mail: zoltan@math.uiuc.edu


†Research supported by Air Force contract F49620-92-J-0125 and by DARPA contract N00014-89-J-1988.


E-mail: goemans@math.mit.edu


‡Research supported by Air Force contract F49620-92-J-0125 and by NSF contract 8606225.


E-mail: djk@math.mit.edu

*matching* $M \subset E(G)$ is a set of pairwise disjoint edges. A *wheel*, $W_i$, is obtained from a cycle, $C_i$, by adding a new vertex and edges joining it to all the vertices of the cycle; the new edges are called the *spokes* of the wheel, ($i \geq 3$, $W_3 = K_4$). Therefore, $\mathrm{WFG}_n$ consists of all graphs on $n$ vertices containing no wheel. Let $t(G)$ denote the number of triangles in $G$ and let $t(n)$ be the maximum of $t(G)$ over $\mathrm{WFG}_n$. Gallai (see Erdős [1]) raised the question of determining $t(n)$.

Take a complete bipartite graph $K_{2a,n-2a}$, where $a$ is a closest integer to $n/4$, and add a maximum matching on the side of size $2a$. We obtain a wheel-free graph, $G_n^1$, having $\lfloor n^2/8 \rfloor$ triangles [1]. Gallai and, independently, Zelinka [10] in 1983 conjectured that this is the maximum possible. However, Zhou [11] recently constructed wheel-free graphs having $(n^2+n)/8$ triangles whenever $n$ is of the form $8q+7$. Zhou [11] also found an upper bound, $t(n) \leq (n^2-n)/6$. In this paper we improve both bounds.

**Theorem 1** *There exists a wheel-free graph on $n$ vertices with $n^2/7.5 + n/15$ triangles whenever $n$ is a multiple of* 15*, i.e., $t(n) \geq n^2/7.5 + n/15$ .*

This theorem is proved by giving a construction, $G_n^2$, in Section 2. As $t(n)$ is monotone, we get $t(n) \geq n^2/7.5 - O(n)$ for all $n$. P. Haxell observed that $G_n^2$ has the additional property that it is *locally tree-like*, i.e., every neighborhood induces a tree. (More exactly, she has improved it so.) Zelinka [10] proved that any locally tree-like graph with $n$ vertices has at least $2n-3$ edges and posed the question which is the maximum number of edges of these graphs. As $G_n^2$ has $n^2/5 + O(n)$ edges, we got a counterexample for a conjecture of Fronček [5], who believed that a locally tree-like graph on $n$ vertices contains at most $\lfloor n(3n+8)/16 \rfloor$ edges (for $n \geq 8$).

**Theorem 2** *Every wheel-free graph on $n$ vertices contains at most $n^2/7.02+O(n)$ triangles, we have $t(n) \leq n^2/7.02 + 5n$ for all $n$.*

The main tool of the proof is Proposition 11 (proved in Section 4), which gives $t(G) \leq n^2/8 + o(n^2)$ for several types of graphs. To mention one:

**Theorem 3** *Let $G \in \mathrm{WFG}_n$ be a wheel-free graph on $n$ vertices, $n \geq 100$. If $\delta > \frac{2}{5}n + \frac{16}{5}$, then $t(G) \leq n^2/8$.*

Considering regular wheel-free graphs one can notice that, if $n$ is of the form $4a-1$, then the previously mentioned construction, $G_n^1$, is regular. Hence, there exist regular graphs in $\mathrm{WFG}_n$ having $\lfloor n^2/8 \rfloor$ triangles. But the graph constructed in Section 2 is *not* regular. In fact, the upper bound we prove for regular graphs in Section 7 is lower than the lower bound for general graphs.

**Theorem 4** *If $G$ is a regular wheel-free graph, then $t(G) \leq n^2/7.75 + O(n)$.*

We conjecture that Gallai's conjecture holds for regular graphs.

**Conjecture 5** *If $G$ is a regular wheel-free graph, then $t(G) \leq n^2/8$.*

## 2 Wheel-free graphs having more than $n^2/7.5$ triangles

Define the graph $G_n^2$ on $n$ vertices where $n$ is a multiple of 15 (see Figure 1) as follows. Its vertex set $V(G_n^2)$ consists of $a_i, b_i, c_i, d_i$ for $i = 1, \ldots, n/5$ and $e_k, f_k, g_k$ for $k = 1, \ldots, n/15$. Its edge set $E(G_n^2)$ consists of

- two matchings of size $n/5$: $(a_i, b_i)$ and $(c_i, d_i)$,
- three matchings of size $n/15$: $(e_k, f_k)$, $(f_k, g_k)$ and $(e_k, g_k)$, and
- all the edges of types: $(a_i, c_j)$, $(a_i, d_j)$, $(a_i, g_k)$, $(b_i, d_j)$, $(b_i, f_k)$, $(b_i, g_k)$, $(c_j, e_k)$, $(c_j, f_k)$ and $(d_j, e_k)$. (Here, again, $1 \leq i, j \leq n/5$ and $1 \leq k \leq n/15$).

It is easy to verify that this graph belongs to WFG$_n$. For example, the neighborhood of the vertex $a_i$ consists of the matching $\{(c_j, d_j) : 1 \leq j \leq n/5\}$ as well as a star rooted at $b_i$ with edges $\{(b_i, g_k) : 1 \leq k \leq n/15\}$ and $\{(b_i, d_j) : 1 \leq j \leq n/5\}$.

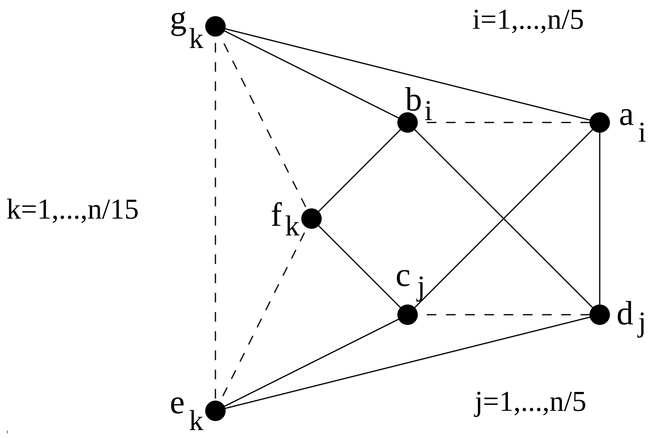


Figure 1: A wheel-free graph having $n^2/7.5 + n/15$ triangles.

Each triangle in $G_n^2$ contains an edge from the matchings and its vertices are in three different classes. An easy calculation shows that $t(G_n^2) = n^2/7.5 + n/15$.

## 3 Improved upper bounds for $t(n)$

In this section we prove a series of Lemmas that lead to Theorem 6, an upper bound for all $n$. This theorem was independently proved by P. Haxell [8].

**Theorem 6** *Every wheel-free graph on $n$ vertices contains at most $\frac{1}{7}n^2 + \frac{9}{7}n$ triangles, i.e.,* $t(n) \leq \dfrac{1}{7}n^2 + \dfrac{9}{7}n$.

Let $G \in \mathrm{WFG}_n$ be an arbitrary wheel-free graph. By definition, $G[N(v)]$, the neighborhood of any vertex $v$, is acyclic. Hence, the number of edges in $N(v)$ is less than or equal to $d_v - 1$, where $d_v = |N(v)|$. By summing $d_v - 1$ over all vertices of $G$, we obtain an upper bound for $3t(G)$. On the other hand, the summation of the degrees over all vertices is precisely twice the number of edges of $G$. Hence,

$$3t(G) \leq 2|E(G)| - n. \tag{1}$$

Since $2|E(G)| \leq n\Delta$, where $\Delta$ stands for the maximum degree, (1) gives

$$t(G) \leq \frac{n(\Delta - 1)}{3}. \tag{2}$$

Our next aim is to obtain the following upper bound on the number of edges of $G$.

$$|E(G)| \leq \frac{n^2}{4} + \frac{n}{4}. \tag{3}$$

Together with (1), this gives Zhou's upper bound, $t(n) \leq (n^2 - n)/6$. The upper bound (3) follows from the following theorem of Erdős and Simonovits [3]: If $G$ does not contain a $W_3$, neither $W_4$, then $|E(G)| \leq \lfloor (n^2 + n)/4 \rfloor$, whenever $n > n_0$. They wrote: "... are easy to prove by induction and can be left for the reader." For completeness we reconstruct their argument in a somewhat simplified form.

A graph is called $(k, \ell)$-free if $|E(G[K])| < \ell$ holds for every $k$-element subset $K \subset V(G)$. Let $f(n; k, \ell)$ be the maximum number of edges of a $(k, \ell)$-free graph with $n$ vertices. Turán's classical theorem determines $f(n; k, \binom{k}{2})$, for example, $f(n; 3, 3) = \lfloor n^2/4 \rfloor$. He proposed the problem of determining $f(n; k, \ell)$, but it is still unsolved in a number of cases. Erdős [2] investigated first all the cases $k \leq 5$, he also proved that excluding only $W_4$ implies $|E(G)| \leq \lfloor n^2/4 \rfloor + \lfloor (n+1)/2 \rfloor$ (for $n > n_0$). For recent accounts see Golberg and Gurvich [6] or Griggs, Simonovits and Thomas [7]. A wheel-free graph has neither $W_3$ nor $W_4$, so it is $(5, 8)$-free. Thus (3) follows from the following.

**Claim 7** *If $G$ is a graph with $n$ vertices such that any 5 vertices span at most 7 edges then $|E(G)| \leq (n^2 + n)/4$ holds for $n \geq 5$.*

**Proof:** The case $n = 5$ is trivial. Let $G$ be a $(5, 8)$-free graph with $n$ vertices, $n \geq 6$. Considering all the $n$ subgraphs $G \setminus x$, we have $(n-2)|E(G)| = \sum_{x \in V} |E(G \setminus x)|$ implying

$$f(n; 5, 8) \leq \lfloor \frac{n}{n-2} f(n-1; 5, 8) \rfloor. \tag{4}$$

Let $s_n = \lfloor (n^2 + n)/4 \rfloor$, $n \geq 1$. It is easy to see (by induction, distinguishing four subcases according to the residue of $n$ modulo 4) that $s_n = \lfloor \frac{n}{n-2} s_{n-1} \rfloor$ holds for all $n \geq 2$. As $f(5; 5, 8) = s_5 = 7$, (4) implies the desired upper bound. □

A *vertex cover* $C$ of the graph $H$ is a subset of vertices with the property that all edges of $H$ are incident to at least one vertex in $C$.

**Lemma 8** *If $C$ is a vertex cover of $G[N(w)]$, where $w$ is a vertex of maximum degree $\Delta$, then*

$$t(G) \leq \frac{1}{2}(\Delta - 1)(n - \Delta + |C|).$$

**Proof:** Since $C$ is a vertex cover of $G[N(w)]$, the set $S = (V - N(w)) \cup C$ is a vertex cover of $G$. Hence, all triangles of $G$ contain at least two vertices belonging to $S$. Summing $(d_v - 1)$ over all vertices $v$ in $S$, we obtain an upper bound on twice the number of triangles in $G$:

$$2t(G) \leq \sum_{v \in S} (d_v - 1) \leq (\Delta - 1)|S| = (\Delta - 1)(n - \Delta + |C|).$$

□

Observe that if $|C| = o(n)$ then the function $\frac{1}{2}(p-1)(n - p + |C|)$ is maximized at $p = \frac{1}{2}n + o(n)$. In this case, we obtain that

$$t(G) \leq \frac{n^2}{8} + o(n^2).$$

On the other hand, if the acyclic graph $G[N(w)]$ has no vertex cover of "small" cardinality then we can use the following two lemmas to isolate two vertices of $G$ which are contained in a small number of triangles. The *total degree* of the subset $S$ (in the graph $H$) means the sum of degrees, $\sum_{x \in S} d_x(H)$.

**Lemma 9** *Let $H$ be an acyclic graph on $p$ vertices with at least one edge, and let $b$ be any positive integer. Then either there exist two adjacent vertices whose total degree is at most $b + 1$, or there exists a vertex cover of $H$ of cardinality at most $(p-2)/b$.*

**Proof:** Suppose that the size of each cover exceeds $(p-2)/b$. As $H$ is a bipartite graph, König's theorem implies that one can find a matching, $M$, of size $m > (p-2)/b$. If we sum up all the degrees of the vertices of $\cup M$, then we count all edges of $H$ at most once except the edges inside $\cup M$. Hence $\sum \{d_u : u \in \cup M\}$ is at most $(p-1) + (2m-1)$. This guarantees a pair $(a, b) \in M$ such that $d_a + d_b \leq (p - 2 + 2m)/m$ which is less than $b + 2$. □

**Lemma 10** *Suppose that $G \in \mathrm{WFG}_n$, $w \in V(G)$, and $H = G[N(w)]$. Suppose also that the $d_u(H)+d_v(H) \leq s$, where $u$, $v$ are adjacent vertices in $H$. Then $G$ has at most $n-d_w+s-2$ triangles containing at least one of $\{u, v\}$.*

**Proof:** We claim that, given any two adjacent vertices $u$ and $v$, the total number of triangles containing $u$ or $v$ is at most $|N(u) \cup N(v)| - 2$. Indeed, $u$ is contained in at most $|N(u)| - 1$ triangles, $v$ is contained in at most $|N(v)| - 1$ triangles while the number of triangles containing both $u$ and $v$ is exactly $|N(u) \cap N(v)|$. Hence, the total number of triangles containing $u$ or $v$ is bounded above by $|N(u)| - 1 + |N(v)| - 1 - |N(u) \cap N(v)| = |N(u) \cup N(v)| - 2$.

The assumption in the Lemma implies that $|N(u) \cup N(v)| \leq |V(G) \setminus N(w)| + s$. Combining this bound with the previous observation completes the proof. $\square$

**Proof of Theorem 6:** We prove the bound $(n^2 + 9n)/7$ by induction on the number of vertices $n$. Let $G$ be a graph in $\mathrm{WFG}_n$ with maximum degree $\Delta$, and let $w \in V(G)$ be such that $d_w = \Delta$. We consider three cases.

If $\Delta < \frac{3}{7}n + 5$, i.e. $\Delta \leq (3n + 34)/7$, then the result follows from equation (2).

If there exists a vertex cover of $G[N(w)]$ of cardinality less than or equal to $\Delta/8$ then Lemma 8 implies that

$$t(G) \leq \frac{1}{2}(\Delta - 1)(n - \Delta + \frac{\Delta}{8}) < \frac{\Delta(n - \frac{7}{8}\Delta)}{2} \leq n^2/7.$$

In these two cases, we have not used the inductive hypothesis.

Finally, assume that $\Delta \geq \frac{3}{7}n + 5$ and that there is no vertex cover of $G[N(w)]$ of cardinality less than or equal to $\Delta/8$. By Lemma 9, there exist two adjacent vertices $u$ and $v$ in $N(w)$ whose total degree in $G[N(w)]$ is at most 9. Lemma 10 now implies that there are at most $n - \Delta + 7 \leq \frac{4}{7}n + 2$ triangles containing $u$ or $v$. By deleting $u$ and $v$, we obtain a graph in $\mathrm{WFG}_{n-2}$ which contains at most

$$\frac{(n-2)^2}{7} + \frac{9}{7}(n-2)$$

triangles by the inductive hypothesis. Hence,

$$t(G) \leq \frac{(n-2)^2}{7} + \frac{9}{7}(n-2) + \frac{4n}{7} + 2 = \frac{n^2}{7} + \frac{9}{7}n.$$

$\square$

## 4 Triangles from a matching

In order to prove the slight additional improvement described in Theorem 2, we first prove a weaker form of Gallai's conjecture. This result is also crucial in improving the upper bound in the case of regular graphs. Let $d_{uv}$ denote $|N(u) \cap N(v)|$, the number of triangles containing the edge $uv$.

**Proposition 11** *Let $G$ be a graph that contains no wheel with 3 or 4 spokes and let $M$ be a matching in it. Then*

$$\sum_{(u,v) \in M} d_{uv} \leq \frac{n^2}{8}. \tag{5}$$

**Proof:** For simplicity, throughout this proof, a *triangle* always refers to a triangle containing an edge in $M$. Thus, the left hand side of (5) is the number of triangles in $G$. Let $m$ denote the cardinality of $M$, let $P$ denote the set of unmatched vertices, and let $p = |P| = n - 2m$.

4.1 For two edges $(u, v)$ and $(x, y)$ in $M$, the induced graph $G[u, v, x, y]$ can contain at most two triangles (Figure 2). Indeed, more than two triangles would imply that $\{u, v, x, y\}$ induces a $K_4$, i.e. a wheel with 3 spokes.

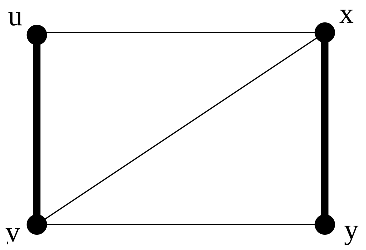


Figure 2: Observation 4.1 in the proof of Proposition 11

Consider first the graph $H$ with a vertex $uv$ for each edge $(u, v)$ of $M$, and with an edge $(uv, xy)$ whenever $\{u, v, x, y\}$ induces two triangles. Let $Q$ be a maximum matching in $H$. Let $S$ be the set of vertices in $G$ belonging to edges of $M$ that are saturated by $Q$, and let $R$ be the remaining vertices in $M$. Hence $(S, R, P)$ is a partition of $V(G)$. Let $q$ denote the cardinality of $Q$ and let $r$ denote the number of unmatched vertices in $H$, i.e. $r = m - 2q$. Clearly, $|S| = 4q$ and $|R| = 2r$. Thus, $p + 2r + 4q = n$.

4.2 If $(uv, xy) \in Q$ and $w$ is any vertex of $G$ then $\{u, v, x, y\}$ and $w$ can connect with one another in at most 1 triangle (Figure 3). Indeed, if $\{u, v, x\}$ is one of the two triangles induced by $\{u, v, x, y\}$ and $w$ forms two triangles with $\{u, v, x, y\}$ then $\{u, v, x, w\}$ induces a $K_4$.

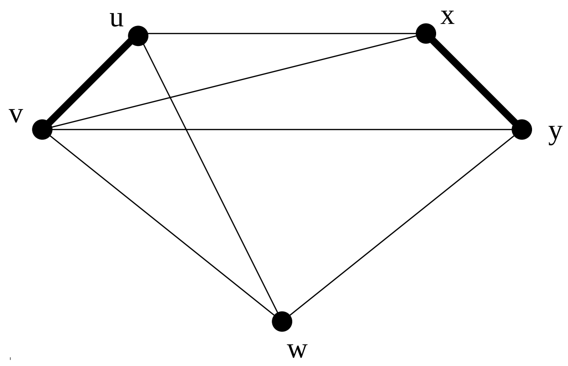


Figure 3: Observation 4.2 in the proof of Proposition 11

4.3 If $(uv, xy) \in Q$ and $(a, b) \in M$ then we claim that $\{u, v, x, y\}$ and $\{a, b\}$ connect with one another in at most two triangles (Figure 4). Assume without loss of generality that $(u, x), (v, x)$ and $(v, y)$ are in $E(G)$. If $\{u, v, x, y\}$ and $\{a, b\}$ connect with one another in three or more triangles then we can assume without loss of generality that $\{u, v\}$ and $\{a, b\}$ connect with one another in two triangles. Without loss of generality, we can assume that $(u, a)$ and $(v, b)$ are in $E(G)$. We consider two cases. If $(v, a) \in E(G)$ (see Figure 4.a) then $\{x, y, a, b\}$ is included in $N(v)$ implying that any triangle between $\{x, y\}$ and $\{a, b\}$ would create a $K_4$. If $(u, b) \in E(G)$ (see Figure 4.b) then a triangle of the form $\{a, b, z\}$ with $z \in \{x, y\}$ would create the cycle $u - a - z - v - u$ in the neighborhood of $b$ while a triangle of the form $\{x, y, c\}$ with $c \in \{a, b\}$ would create the cycle $u - c - y - v - u$ in the neighborhood of $x$.

From 4.2 and 4.3, there are at most $q(n - 4q)$ triangles that connect $S$ to $V - S$. Within $S$, in addition to the $2q$ triangles corresponding to the edges in $Q$, there are at most $\frac{1}{2}4q(q-1)$ triangles by 4.3. Therefore, the number of triangles containing vertices in $S$ is at most

$$2q(q - 1) + q(n - 4q) + 2q = q(n - 2q). \qquad (6)$$

We now concentrate on the number of triangles induced by $V - S$. Recall that $V - S$ corresponds to vertices in $P$ (which are unmatched in $M$) and to vertices in $R$ (which are incident to edges in $M$ that are unmatched in $Q$).

4.4 If $uv$ and $xy$ are unmatched vertices in $Q$ then $\{u, v\}$ and $\{x, y\}$ can connect with one another in at most one triangle by the maximality of $Q$.

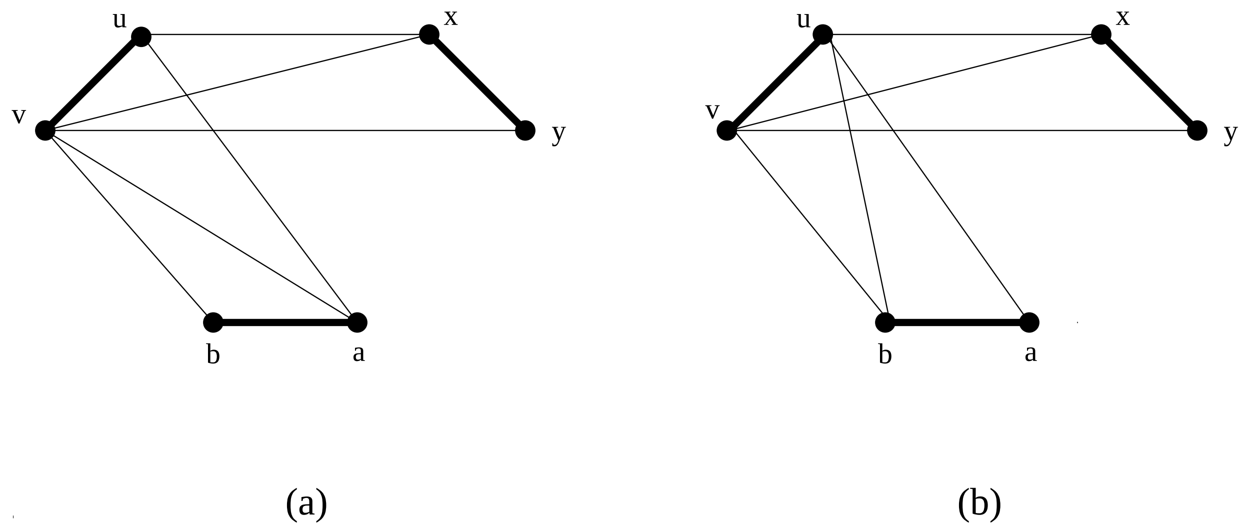


Figure 4: Observation 4.3 in the proof of Proposition 11

4.5 Consider a vertex $w \in P$. We say that two edges $(u, v)$ and $(x, y)$ in $M$ are *independent* if $\{u, v, x, y\}$ does not induce any triangle. Since the neighborhood of $w$ cannot contain any triangle, $w$ can induce triangles only with independent edges in $M$.

Let $s$ be the cardinality of a largest set of independent edges in $G[R]$. The number of triangles in $G[R]$ is at most $s(r-s) + \frac{1}{2}(r-s)(r-s-1) \leq (r^2 - s^2)/2$ by 4.4. Moreover, the number of triangles between $P$ and $R$ is at most $ps$ by 4.5. Therefore, there are at most $T = ps + (r^2 - s^2)/2$ triangles in $G[V - S]$. We have that $ps + (r^2 - s^2)/2 \leq (2r + p)^2/8$ for all real $r \geq s$, $p \geq 0$; (because this is equivalent to $4p(s-r) \leq (p-2s)^2$). Hence, the number of triangles in $G[V - S]$ is at most $(n - 4q)^2/8$. Combined with (6), this implies that the number of triangles containing edges in $M$ is at most $q(n - 2q) + (n - 4q)^2/8 = n^2/8$. $\square$

# 5 Graphs with large minimum degree

In this section we prove Theorem 3.

Partition the edge set of $G$ into two classes. We say that an edge $(u, v)$ of $E(G)$ is a *fat* edge if at least $\sqrt{n}$ triangles contain it, i.e. $d_{uv} \geq \sqrt{n}$. An edge which is not fat is said to be *lean*. In the next two lemmas, we show that the set of fat edges is an ideal candidate to play the role of the matching $M$ in Proposition 11.

**Lemma 12** *Let $G$ be a graph in* $\mathrm{WFG}_n$ *with $d_u + d_v + d_w \geq n + 3\sqrt{n}$ for every triangle $uvw$. (E.g., $\delta \geq n/3 + \sqrt{n}$). Then every triangle of $G$ contains a fat edge.*

**Proof:** Consider a triangle with vertices $u$, $v$ and $w$. Observe that $N(u) \cap N(v) \cap N(w) = \emptyset$. Indeed, a vertex adjacent to $u$, $v$ and $w$ would have a cycle in its neighborhood. Hence,

$$n \geq |N(u) \cup N(v) \cup N(w)| = d_u + d_v + d_w - d_{uv} - d_{uw} - d_{vw}. \tag{7}$$

Since $d_u + d_v + d_w \geq n + 3\sqrt{n}$, at least one of the quantities $d_{uv}$, $d_{uw}$ and $d_{vw}$ is greater than or equal to $\sqrt{n}$. □

**Lemma 13** *Every fat edge of $G \in \mathrm{WFG}_n$ belongs to a triangle with two lean edges.*

**Proof:** Let $(u, v)$ be a fat edge. Suppose, on the contrary, that for each $x \in N(u) \cap N(v)$ at least one of the edges $(u, x)$ and $(v, x)$ is fat. This implies, that

$$\sum_{x \in N(u) \cap N(v)} ((d_{ux} - 1) + (d_{vx} - 1)) \geq |N(u) \cap N(v)|(\sqrt{n} - 1) \geq n - d_{uv}. \tag{8}$$

Consider all the triangles of $G$ of the forms $uxy$ and $vxy$ where $x \in N(u) \cap N(v)$ and $y \notin N(u) \cap N(v)$, $y \neq u, v$. The number of these triangles is exactly the left hand side of (8). However, all of these triangles have a distinct third vertex outside $(N(u) \cap N(v)) \cup \{u, v\}$, so their number is at most $n - 2 - d_{uv}$, contradicting (8). Indeed, for example, if $uxy$ and $ux'y$ are triangles with $x, x' \in N(v)$, then we have a wheel with center $u$ and cycle $xyx'v$. □

**Proof of Theorem 3**: Let $G$ be a graph in $\mathrm{WFG}_n$ with $d_u \geq \frac{2}{5}n + \frac{17}{5}$ for each vertex $u$. For $n \geq 96$ Lemma 12 implies that each triangle contains a fat edge.

We claim that the fat edges form a matching.

Let $m$ be the maximum number of fat edges incident to a vertex of $G$. We shall prove that $m = 1$. Consider a lean edge $(v, w)$. By the above argument, any triangle containing $(v, w)$ must contain a fat edge. Since there are at most $2m$ fat edges incident to either $v$ or $w$, we obtain that

$$d_{vw} \leq 2m \tag{9}$$

for any lean edge $(v, w)$.

Let $u$ be a vertex with $m$ fat edges incident to it, say $(u, v_1), (u, v_2), \ldots, (u, v_m)$. Since $G$ is wheel-free, there exist at most $m - 1$ triangles containing two fat edges incident to $u$. By summing $d_{uv_i}$ over $i = 1, \ldots, m$, we count every triangle containing $u$ at most once, except for those containing two fat edges incident to $u$. Hence,

$$\sum_{i=1}^{m} d_{uv_i} \leq d_u - 1 + (m - 1). \tag{10}$$

The left hand side is at least $m\sqrt{n}$, hence we get $m < \sqrt{n}+1$. For any fat edge $(u,v)$, Lemma 13 implies that there is a triangle $uvw$ with two lean edges. Then (7) and (9) give

$$d_{uv} \geq d_u + d_v + d_w - n - 4m. \tag{11}$$

We derive that

$$\sum_{i=1}^{m} d_{uv_i} \geq md_u + \sum_{i}^{m} d_{v_i} + \sum_{i}^{m} d_{w_i} - nm - 4m^2. \tag{12}$$

Comparing (10) and (12) we get $m-2 \geq (3m-1)\delta - nm - 4m^2$. If $2 \leq m < \sqrt{n}+1$, $n \geq 500$, $\delta > (2n+16)/5$ this leads to a contradiction. If $100 \leq n < 500$, we can use $d_{vw} < \sqrt{n}$ instead of (9) to get a contradiction in exactly the same way. Therefore $m$ must be equal to 1, i.e. the fat edges form a matching.

Every triangle contains a fat edge, so by Proposition 11, there are at most $n^2/8$ triangles in $G$. □

# 6 Proof of the upper bound

Here we prove Theorem 2 by induction on $n$. If $n \leq 9126$, then it follows from Theorem 6. Let $c = 1/2457 = 1/7 - 1/7.02$.

If $\Delta > \left(\frac{3}{7}+4c\right)n$ then the proof is similar to the proof of Theorem 6. Either there exists a vertex cover of $G[N(w)]$ of cardinality less or equal to $\Delta/9$ in which case $t(G) < 9/64n^2 < (1/7-c)n^2$ or there exist two adjacent vertices $u$ and $v$ in $N(w)$ whose total degree in $G[N(w)]$ is at most 10. In the latter case, Lemma 10 implies that we destroy less than

$$\left(\frac{4}{7}-4c\right)n+8$$

triangles by deleting $u$ and $v$. The claim therefore follows by induction.

If $|E(G)| < \frac{3}{2}\left(\frac{1}{7}-c\right)n^2$ then the result follows from (1).

Assume now that

6.1 $\Delta \leq \left(\frac{3}{7}+4c\right)n$ and that

6.2 $|E(G)| \geq \frac{3}{2}\left(\frac{1}{7}-c\right)n^2$.

If $G$ satisfies the hypotheses of Theorem 3 then we are done. Otherwise, we must have that $\delta \leq \frac{2}{5}n + \frac{16}{5}$. Consider the graph $G'$ obtained from $G$ by repeatedly deleting a vertex of minimum degree until

$$\delta(G') > \frac{2}{5}|V(G')| + \frac{16}{5}.$$

Let $v_0, v_1, \dots, v_{t-1}$ be the sequence of vertices that we delete and let $G_i$ be the graph obtained from $G$ by deleting $\{v_0, \dots, v_{i-1}\}$. In particular, $G_0 = G$ and $G_t = G'$. By definition, we have that

$$d_{v_i}(G_i) \leq \frac{2}{5}|V(G_i)| + \frac{16}{5}$$

and that

$$d_{v_i}(G) \leq \frac{2}{5}(n-i) + \frac{16}{5} + f_i$$

where $f_i$ denotes the number of edges in $G$ joining $v_i$ to $\{v_0, \dots, v_{i-1}\}$. Let $\varepsilon = t/n$. In order to give an upper bound on $\varepsilon$, we consider the number of edges of $G$. Using our assumptions 6.1. and 6.2., we derive that

$$\begin{aligned} 3\left(\frac{1}{7} - c\right) \leq \frac{2|E(G)|}{n^2} &= \frac{1}{n^2}\sum_{v\in V} d_v(G) \\ &\leq \left(\frac{3}{7} + 4c\right)(1-\varepsilon) + \frac{1}{n^2}\sum_{i=0}^{t-1}\left(\frac{2}{5}n - \frac{2}{5}i + \frac{16}{5} + f_i\right) \\ &= \left(\frac{3}{7} + 4c\right)(1-\varepsilon) + \frac{2}{5}\varepsilon - \frac{1}{5}\varepsilon^2 + \frac{|E(G[\{v_0, \dots, v_{t-1}\}])|}{n^2} + \frac{17}{5}\frac{\varepsilon}{n} \\ &= \frac{3}{7} + 4c - \left(\frac{1}{35} + 4c\right)\varepsilon + \frac{1}{20}\varepsilon^2 + \frac{73}{20}\frac{\varepsilon}{n}, \end{aligned}$$

where we have used (3), that the number of edges of $G[\{v_0, \dots, v_{t-1}\}]$ is at most $(\varepsilon n)^2/4 + \varepsilon n/4$. For $n$ sufficiently large, say larger than $n_0 = 9126$, and given the value of $c$, the above inequality can be seen to imply $\varepsilon < 0.12$. Since $G'$ satisfies the hypotheses of Theorem 3, we have that $t(G') \leq (n^2/8)(1-\varepsilon)^2$. Moreover,

$$\begin{aligned} t(G) - t(G') &\leq \sum_{i=0}^{t-1}(d_{v_i}(G_i) - 1) \\ &\leq \sum_{i=0}^{t-1}\left(\frac{2}{5}n - \frac{2}{5}i + \frac{11}{5}\right) \\ &= \frac{2}{5}\varepsilon n^2 - \frac{1}{5}\varepsilon^2 n^2 + \frac{12}{5}\varepsilon n. \end{aligned}$$

Therefore,

$$\begin{aligned} t(G) &\leq \left(\frac{1}{8}(1-\varepsilon)^2 + \frac{2}{5}\varepsilon - \frac{\varepsilon^2}{5}\right)n^2 + \frac{12}{5}\varepsilon n \\ &< \frac{1}{7.02}n^2 + 5n. \end{aligned}$$

□

**Remark.** The result can be improved to $t(n) \leq n^2/7.03 + O(n)$ as shown below. Let $c = 1/1540$. We execute the first two steps of the previous proof, so from now on we may suppose that 6.1. and 6.2. hold. We delete from $G$ a vertex $x$ if $d_x < (n-i)/3 + \sqrt{n}$. Lemma 13 implies that we obtain a graph where each triangle contains a fat edge. Delete a fat-lean-lean triangle, $uvw$, if $d_u + d_v + d_w \leq 1.2(n-i) + 12$. We obtain, that for each fat edge $d_{uv} > 0.2(n-i) + O(1)$. If there exists two adjacent fat edges, $(u, x_1)$ and $(u, x_2)$, then for some fat-lean-lean triangles $ux_1w_1$ and $ux_2w_2$ we get that $d_u + d_{x_1} + d_{x_2} + d_{w_1} + d_{w_2} \leq 2(n-i) + O(1)$. Delete these 5 points. Repeat these steps.

The upper bound for the number of edges implies $\varepsilon \leq .235$. At each of the above steps, by deleting $\ell$ vertices $(1 \leq \ell \leq 5)$ we have distroyed only at most $\ell(n-i)/3 + O(1)$ triangles. We obtain that $t(G)/n^2 \leq \frac{1}{8}(1-\varepsilon)^2 + \frac{1}{3}\varepsilon - \frac{1}{6}\varepsilon^2 + o(1)$. □

# 7 Upper bound for regular graphs

In this last section, we prove Theorem 4, the upper bound for $d$-regular graphs.

If $d \leq \frac{12}{31}n$, equation (2) implies that $t(G) < n^2/7.75$.

If $d > \frac{2}{5}n + \frac{16}{5}$, then Theorem 3 implies that $t(G) \leq n^2/8$, (for $n \geq 100$).

Assume now that $\frac{12}{31}n < d \leq \frac{2}{5}n + \frac{16}{5}$. From (12) we can deduce that any vertex of $G$ is incident to at most two fat edges. (The details are left to the reader.) If no vertex is incident to two fat edges, the result follows from Proposition 11. Otherwise, let $r$ be the maximum, over all vertices $u$, of the number of triangles containing a fat edge which is incident to $u$, and let $w$ be a vertex attaining this maximum. From the definition of $r$, we must have $d_{uv} \leq r$ for all edges $(u, v) \in E$. Moreover, since a fat edge is contained in at least $3d - n - 8$ triangles by (11) and since there exists a vertex incident to two fat edges, $r$ must be at least $6d - 2n - 17$. Let $S = V - \{w\} - N(w)$ and let $T_i$ for $i = 0, \ldots, 3$ be the number of triangles having exactly $i$ vertices of $S$. Clearly, $T_0 \leq d - 1$ since any such triangle must involve $w$. Moreover, by summing $d_v - 1$ over all vertices $v \in S$, we observe that $3T_3 + 2T_2 + T_1 \leq (d-1)(n-d)$. Finally, we claim that $T_1 \leq r(d-r) + O(n)$. To see this, observe that the number of lean edges contained in $N(w)$ is at least $r - 1$ if $w$ is incident to just one fat edge or $r - 2$ if $w$ is incident to two fat edges. Thus the number of fat edges contained in $N(w)$ is at most $d - r + 1$. To compute an upper bound on $T_1$, we sum $d_{uv} - 1$ over all edges contained in $N(w)$ (the $-1$ term comes from the fact that we don't need to count triangles involving vertex $w$):

$$T_1 \leq 3(r-2) + (r-1)(d-r+1) = r(d-r) + O(n)$$

since lean edges are contained in at most 4 triangles by (9) while fat edges are contained in at most $r$ triangles by definition of $r$. Therefore,

$$t(G) \le T_0 + \frac{1}{2}(T_1 + 2T_2 + 3T_3) + \frac{1}{2}T_1 \le \frac{1}{2}(d(n-d) + r(d-r)) + O(n).$$

Under the constraint that $r \ge 6d - 2n - 17$, the right-hand-side is maximized for $r = 6d - 2n - O(1)$ giving $t(G) \le \frac{1}{2}(d(n-d) + (6d-2n)(2n-5d)) + O(n)$. Under the constraint $\frac{12}{31}n < d$, this is in turn maximized for $d = \frac{12}{31}n + O(1)$ proving that $t(G) \le n^2/7.75 + O(n)$. $\square$

## 8 Wheel-free triple systems

A family of 3-element sets is called *wheel-free* if it contains no $k$ triples isomorphic to $\{\{0,1,2\},\{0,2,3\},\dots,\{0,i,(i+1)\},\dots\{0,k,1\}\}$, where $k \ge 3$ is an arbitrary integer. For example, the vertex sets of the triangles in a wheel-free graph form such a system. But the general case is different. Let $\mathrm{ex}(n;W)$ denote the largest cardinality of a wheel-free triple system on an $n$-element set. V. T. Sós, Erdős and Brown [9] proved that $\lim_{n\to\infty} \mathrm{ex}(n;W)/n^2 = 1/3$.

For further problems of this type see, e.g., [4] and the references therein.

Another interesting question is whether (and how) our results can be extended to $\{W_3, W_4\}$-free graphs, or even more generally for $(5,8)$-free graphs.

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