Minimum Bounded Degree Spanning Trees*

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Abstract

We consider the minimum cost spanning tree problem under the restriction that all degrees must be at most a given value k. We show that we can efficiently find a spanning tree of maximum degree at most $k+2$ whose cost is at most the cost of the optimum spanning tree of maximum degree at most k. This is almost best possible.

The approach uses a sequence of simple algebraic, polyhedral and combinatorial arguments. It illustrates many techniques and ideas in combinatorial optimization as it involves polyhedral characterizations, uncrossing, matroid intersection, and graph orientations (or packing of spanning trees). The result generalizes to the setting where every vertex has both upper and lower bounds and gives then a spanning tree which violates the bounds by at most two units and whose cost is at most the cost of the optimum tree. It also gives a better understanding of the subtour relaxation for both the symmetric and asymmetric traveling salesman problems. The generalization to l-edge-connected subgraphs is briefly discussed.

1 Introduction

The Minimum Bounded Degree Spanning Tree (MBDST) problem is given an undirected graph $G = (V, E)$, costs $c : E \rightarrow \mathbb{R}$ (not necessarily nonnegative) and an integer $k \geq 2$, find a spanning tree of maximum degree at most k and of minimum cost. For $k = 2$, this is the Hamiltonian path problem. The problem is NP-hard for any given k. We denote by $OPT(k)$ the minimum cost of any spanning tree of maximum degree $\leq k$.

In 1991, we formulated the conjecture:

Conjecture 1. *In polynomial time, one can find a spanning tree of maximum degree $\leq k+1$ whose cost is at most $OPT(k)$, the minimum cost of any spanning tree of maximum degree $\leq k$.*

The goal of this paper is to prove a weaker version of this conjecture with $k+1$ replaced by $k+2$; the original conjecture remains open.

Theorem 2. *The polynomial-time algorithm described in Figure 1 returns a spanning tree of maximum degree $\leq k+2$ whose cost is at most $OPT(k)$.*

This improves upon the results of Könemann and Ravi [18, 19] and of Chaudhuri et al. [3]. The latter one is currently the best known result and produces in polynomial time a tree of cost at most $OPT(k)$ whose maximum degree is $bk + 2(b+1)\log_b n$ where $b > 1$. Chaudhuri et al. also give a *quasi-polynomial* time algorithm producing a tree of cost at most $OPT(k)$ and of maximum degree $k + O(\log n / \log\log n)$. Ravi and Singh [24] consider a slight variant of MBDST in which the goal is to minimize the maximum degree k^* of a *minimum* spanning tree in a weighted graph. They describe an algorithm to find a minimum spanning tree of degree at most $k^* + p$ where p is the number of distinct costs in any minimum spanning tree. Our result actually implies that we can find a minimum spanning tree of maximum degree $k^* + 2$. We should also point out that Fürer and Raghavachari [9] consider the version without costs and provide an algorithm returning a tree of maximum degree at most $k+1$ in a graph with a tree of maximum degree k; a different algorithm (growing such a tree starting from a vertex) was also discovered by the author in 1991 but never published.

As Theorem 2 is surprisingly simple to present and analyze, we sketch it in this introduction.

Without the degree restrictions, the classical minimum spanning tree problem (MST) is the prototypical problem which the greedy algorithm solves exactly in polynomial time. More generally, the greedy algorithm allows to find a minimum-cost base*(a maximum cardinality independent set)* in any matroid $M = (E, \mathcal{I})$ with ground set E and family of independent sets[1] $\mathcal{I}$. In the case of the MST, the corresponding matroid is the graphic matroid $M(G)$ whose independent sets are all forests in G and whose bases are

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[1]For M to be a matroid, $\mathcal{I}$ needs to be such that (i) $A \in \mathcal{I}$, $B \subseteq A$ imply that $B \in \mathcal{I}$ and (ii) $A, B \in \mathcal{I}$, $|A| > |B|$ imply that there exists $e \in A \setminus B$ with $B \cup \{e\} \in \mathcal{I}$.

all spanning trees in G. One of the fundamental results in combinatorial optimization is that one can even optimize efficiently over the sets which are independent in *two* matroids (Edmonds [8] and Lawler [21]), or over the common bases of them, or over the bases of one matroid and the independent sets of the other. This is known as *matroid intersection*.

If the edge sets of maximum degree at most k were the independent sets of a matroid, we could solve MBDST in polynomial time by matroid intersection, but of course they are not. To attack the conjecture, we could *relax* the problem and try to construct a matroid M' whose independent sets contain all subgraphs of maximum degree at most k and also other sets (whose maximum degree is not too large), and then use matroid intersection to find a spanning tree of minimum cost which is also independent for M'. However, this "matroid relaxation" would fail to prove Conjecture 1 as one can see that, for a general graph G, any such matroid M' would also contain independent sets of maximum degree at least $3k/2$ (and probably even higher).

1. Solve the LP relaxation (LP) and obtain an extreme point x^* with support E^*.
2. Orient E^* into a directed graph A^* with maximum indegree at most 2.
3. Find a spanning tree T of minimum cost such that $\left|T \cap \delta^+_{A^*}(v)\right| \leq k$ for all $v \in V$.

Figure 1. The algorithm for Minimum Bounded Degree Spanning Trees.

Instead, our approach proceeds as follows (see Figure 1). We start by formulating a classical linear programming relaxation for the MBDST problem:

$$\text{Min} \quad c(x) = \sum_e c_e x_e$$

subject to:

$$(LP) \qquad \begin{array}{lrr} x(E(S)) \leq |S| - 1 & S \subset V & (1) \\ x(E(V)) = |V| - 1 & & (2) \\ x(\delta(v)) \leq k & v \in V & (3) \\ x_e \geq 0 & e \in E, & (4) \end{array}$$

where $x(F) = \sum_{e \in F} x_e$, $\delta(v)$ are the edges incident to v and $E(S)$ are the edges whose endpoints are both in S. Constraints (1), (2) and (4) form the spanning tree polytope (Edmonds [7]), the convex hull of all spanning trees in G. We solve the relaxation (see Section 2 for details) to obtain one of its extreme points x^* with $c(x^*) \leq OPT(k)$. Let $E^* = \{e \in E : x^*_e > 0\}$ denote the support of x^*. The main step is then to construct a matroid M^* with ground set E^* with the following properties:

(i). x^* can be seen as a convex combination of independent sets in M^*,

(ii). every independent set in M^* has maximum degree $k + 2$.

Using matroid intersection, we can find in polynomial time a minimum cost spanning tree T which is also independent in M^*. By (ii), this means that T has maximum degree $\leq k+2$. Furthermore, from the polyhedral characterization of matroid intersection (Edmonds [6]), one can see that (i) implies that the cost of the tree T, $c(T) = \sum_{e \in T} c_e$, is upper bounded by $c_e x^*_e$, which is a lower bound on $OPT(k)$.

The core of the approach is therefore in deriving the matroid M^*, which depends on the extreme point x^* or, more precisely, on E^*. For this, we first study properties of any extreme point. From standard uncrossing[2] arguments as applied to relaxations of many combinatorial optimization problems (see e.g. [4, 1] for the TSP), one can easily derive that $|E^*| \leq 2n - 1$ where $n = |V|$. One of the contributions of this paper is to observe that a simple algebraic argument implies that, for any set $U \subseteq V$, we have that $|E^*(U)| \leq 2|U| - 1$, where $E^*(U)$ denotes the set of edges entirely within U. We can even improve this slightly to

$$|E^*(U) \leq 2|U| - 3 \text{ for any } U \subseteq V. \tag{5}$$

This property allows us to derive that the support E^* of x^* can be oriented into A^* such that the indegree in A^* of any vertex is at most 2. Indeed, by a classical result of Hakimi [13], such an orientation exists if and only if, for every $U \subseteq V$, one has $|E^*(U)| \leq 2|U|$ (this follows easily from the max flow/min cut theorem or matching theory; also there is nothing special about the number '2', it can be replaced by p). The existence of the orientation A^* can also be derived by realizing that E^* can be partitioned into 2 forests, and orienting each of them with all indegrees at most 1; indeed, Nash-Williams [23] shows that the graph (V, E^*) can be partitioned into 2 forests (resp. p forests) if and only if $|E^*(U)| \leq 2(|U| - 1)$ for any set $U \subseteq V$ (resp. $|E^*(U)| \leq p(|U| - 1)$). (As an aside, the slightly stronger condition (5) is precisely the independence condition in 2-dimensional rigidity matroids, see Laman [20].)

Once we have the orientation A^* of E^* such that the indegree of every vertex is at most 2, the construction of the matroid $M^* = (E^*, \mathcal{I}^*)$ is rather straightforward. We define it as a *partition* matroid with independent sets:

$$\mathcal{I}^* = \{F \subseteq E^* : |F \cap \delta^+_{A^*}(v)| \leq k \text{ for all } v \in V\},$$

[2] Uncrossing is a fundamental technique in combinatorial optimization and many results follow from it, including matroid intersection (and many generalizations of it) or the Lucchesi-Younger theorem.

where $\delta^+_{A^*}(v)$ denotes the set of edges of E^* leaving v in the orientation A^*. This is a partition matroid as the sets $\delta^+_{A^*}(v)$ are disjoint. Observe that (ii) follows from the fact that A^* has indegrees at most 2, while (i) follows from the fact that the convex hull of independent sets of the partition matroid M^* is simply given by $\{x : x(\delta^+_{A^*}(v)) \leq k$ for all $v \in V, x_e \geq 0$ for all $e \in E^*\}$ and therefore contains in the set of solutions to (3) and (4).

The remainder of this extended abstract is structured as follows. In Section 2, we provide details of our algorithm, proof of Theorem 2 and several remarks. In Section 3, we generalize our result to the setting in which every vertex has an upper and a lower bound on its degree; Theorem 11 shows that we can relax these bounds by 2 units and obtain a spanning tree of cost no worse than optimum. In Section 4, we conjecture a property of extreme points of the relaxation that would imply Conjecture 1. The next two sections shed some new light on the Held-Karp relaxation for both the symmetric and asymmetric traveling salesman problem; these results may prove invaluable in obtaining improved approximation algorithms for these problems. Finally, the last section deals with the generalization to l-edge-connectivity.

2 Main result, Details and Proofs

The Uncrossing of Tight Sets. Let x^* be an extreme point to (1)–(4). By definition, x^* is uniquely determined by the *tight* constraints in the system, those satisfied at equalities. For any equality $x^*_e = 0$, we can remove the edge e and focus on the remaining support E^*. Let $\mathcal{F} = \{S : x^*(E(S)) = |S| - 1\}$ refer to the tight contraints among (1)-(2), and let $W = \{v \in V : x^*(\delta(v)) = k\}$ refer to the tight constraints among (3). For any set $F \subseteq E$, let $\chi(F) \in \mathbb{R}^{|E|}$ denote the characteristic vector of F.

Lemma 3. *If $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$ then $S \cap T$ and $S \cup T$ are both in $\mathcal{F}$, and furthermore $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cup T)) + \chi(E(S \cap T))$.*

Proof. As $S \cap T \neq \emptyset$, we have that:

$$\begin{aligned} |S| - 1 + |T| - 1 &= |S \cup T| - 1 + |S \cap T| - 1 \\ &\geq x^*(E(S \cup T)) + x^*(E(S \cap T)) \\ &\geq x^*(E(S)) + x^*(E(T)) \\ &= |S| - 1 + |T| - 1, \end{aligned}$$

and therefore we have equality throughout. This implies that $S \cup T$ and $S \cap T$ are also tight and furthermore that there are no edges e between $S \setminus T$ and $T \setminus S$ with $x^*_e > 0$. This shows the linear dependence $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cup T)) + \chi(E(S \cap T))$. $\square$

Lemma 3 show that if S and T *intersect* (i.e. $S \setminus T$, $T \setminus S$ and $S \cap T$ are non-empty) then one can derive two *other* tight sets $S \cap T$ and $S \cup T$. A *laminar* family $\mathcal{L}$ is a family of sets with no pair of *intersecting* sets, i.e. for any two $S, T \in \mathcal{L}$ we have that $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$. Standard uncrossing arguments (see Hurkens et al. [16], Cornuéjols et al. [4], Jain [17], Melkonian and Tardos [22] for details and illustrations) then show that we can obtain a *laminar* subfamily $\mathcal{L}$ of the family $\mathcal{F}$ and a subset T of W such that the system $Ax = b$ consisting of

$$\begin{cases} x(E(S)) = |S| - 1 & S \in \mathcal{L} \\ x(\delta(v)) = k & v \in T \end{cases}$$

is of full rank (and hence uniquely defines x^*). Observe that no singleton sets belong to $\mathcal{L}$ as the corresponding equalities are vacuous. The rows of A are $\chi(E(S))$ for $S \in \mathcal{L}$ and $\chi(\delta(v))$ for $v \in T$, all restricted to the support E^*.

For completeness, Jain's argument is included below. For a family $\mathcal{T} \subseteq 2^V$, let $span(\mathcal{T})$ denote the vector space spanned by $\chi(E(S))$ for $S \in \mathcal{T}$.

Theorem 4 ([17]). *If $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$ then $span(\mathcal{L}) = span(\mathcal{F})$.*

Proof. Let $\mathcal{L}$ be a maximal laminar subfamily of $\mathcal{F}$ and assume that $\chi(E(S)) \notin span(\mathcal{L})$ for some $S \in \mathcal{F}$. If there are several such sets S, choose one that intersects as few sets of $\mathcal{L}$ as possible. Suppose S intersects some $T \in \mathcal{L}$ (otherwise S can be added to $\mathcal{L}$, contradicting maximality). From Lemma 3, we have that $S \cap T$ and $S \cup T$ are also in $\mathcal{F}$ and that $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cup T)) + \chi(E(S \cap T))$. This means that either $\chi(E(S \cup T)) \notin span(\mathcal{L})$ or $\chi(E(S \cap T)) \notin span(\mathcal{L})$. In either case, we have a contradiction as $S \cup T$ and $S \cap T$ intersect fewer sets from $\mathcal{L}$ (as $\mathcal{L}$ is laminar). $\square$

Size of a Laminar Family. A laminar family (without the empty set) on a ground set U of size n has cardinality at most $2n-1$ (by induction on n, and summing over the maximal proper subsets of U in the family). Furthermore, the inductive argument shows that this bound is attained only if both U and two complementary sets S and $U \setminus S$ are in the family; S and $U \setminus S$ are the two maximal proper subsets of U in the family. If the laminar family has no singleton sets then the maximum cardinality of such a family is $n - 1$.

Everywhere Sparse. The classical argument at this point is to observe that $|E^*| = |\mathcal{L}| + |T|$ and since a laminar family $\mathcal{L}$ with no singleton sets has at most $n - 1$ sets, we have that $|E^*| = |\mathcal{L}| + |T| \leq n - 1 + n = 2n - 1$ and thus is rather sparse. The novelty here is to use a simple algebraic argument to show that *any induced subgraph* of E^* is sparse.

Theorem 5. *For any set $U \subseteq V$, we have $|E^*(U)| \leq 2|U| - 1$.*

Proof. Let B be the submatrix of A consisting of all the columns of A corresponding to edges in $E^*(U)$. Since A is of full rank, we have that $rank(B) = |E^*(U)|$. We will now get an upper bound on the (row) rank of B by computing the number of *distinct non-zero* rows of B.

- Rows of B corresponding to vertices $v \in T$ are identically 0 if $v \notin U$ as $\delta(v) \cap E^*(U) = \emptyset$. Thus only $|U|$ rows of B corresponding to vertices in T are non-zero.
- Rows of B corresponding to sets $S \in \mathcal{L}$ depend only on $S \cap U$ (as $E^*(U) \cap E^*(S) = E^*(S \cap U)$) and furthermore are identically 0 if $|S \cap U| \leq 1$. Thus the number of *distinct, non-zero* rows of B corresponding to sets $S \in \mathcal{L}$ is upper bounded by the cardinality of $\mathcal{L}_U = \{S \cap U : S \in \mathcal{L} \text{ and } |S \cap U| \geq 2\}$. As $\mathcal{L}_U$ is a laminar family of 2^U with no singleton sets, we have that $|\mathcal{L}_U| \leq |U| - 1$.

Over both types of rows, we get that the total number of distinct non-zero rows is upper bounded by $|U| + |U| - 1 = 2|U| - 1$, proving the result. □

This result can be slightly improved, although this is not necessary for deriving Theorem 2.

Theorem 6. *For any set $U \subseteq V$, we have $|E^*(U)| \leq 2|U| - 3$.*

Proof. We continue the proof of Theorem 5 and use a better bound for the rank than simply the number of distinct non-zero rows. First observe that in order to have $|E^*(U)| = 2|U|-1$ or $2|U|-2$ we need to have either (i) $|\mathcal{L}_U| = |U|-1$ and $|T \cap U| \geq |U| - 1$, or (ii) $|\mathcal{L}_U| \geq |U| - 2$ and $U \subseteq T$.

There are two linear relations we exploit. First,

$$2\chi(E^*(U)) = \sum_{v \in U} \chi(\delta(v) \cap E^*(U)).$$

Secondly, for any set $S \subset U$, we have that

$$\chi(E^*(U)) + \chi(E^*(S)) - \chi(E^*(U \setminus S)) = \sum_{v \in S} \chi(\delta(v) \cap E^*(U)). \tag{6}$$

This means that we lose one unit in our bound on the rank whenever all the sets in one of these relations are present.

In case (ii), it means that since $U \subseteq T$, we lose one unit in the rank whenever $U \in \mathcal{L}_U$ and a second unit whenever two complementary sets S and $U \setminus S$ are also in $\mathcal{L}_U$. As a laminar family of 2^U with no singleton sets, without U and without two complementary sets has size at most $|U| - 3$, we obtain the claim in case (ii).

On the other hand, for case (i), $|\mathcal{L}_U| = |U| - 1$ implies that $U \in \mathcal{L}_U$ and there is a set $S \in \mathcal{L}_U$ with $U \setminus S \in \mathcal{L}_U$. (6) implies that at most $|S| - 1$ of $\chi(\delta(v) \cap E^*(U))$ for $v \in S$ are linearly independent with all $\chi(E^*(T))$ for $T \in \mathcal{L}_U$, and similarly for S replaced by $U \setminus S$ in (6). This decreases the bound on the rank by two units. □

The Orientation of E^*. As discussed in the introduction, Theorem 5 implies that E^* can be oriented into the directed graph (V, A^*) such that the indegree of every vertex in A^* is at most 2. This follows from:

Theorem 7 (Hakimi [13]). *An undirected graph $G = (V, E)$ can be oriented into a directed graph (V, A) with indegree of vertex v at most $i(v)$ for every $v \in V$ if and only if $|E^*(U)| \leq \sum_{i \in U} i(v)$ for all $U \subseteq V$.*

This orientation result can be easily derived from König's Theorem for bipartite matchings (in the bipartite graph with E as one side of the bipartition, $i(v)$ copies of vertex v on the other side, and edges between $(u, v) \in E$ and every copy of u and v), from the max flow/min cut theorem or from matroid intersection. Theorem 6 and Hakimi's result actually show that we could choose three vertices a, b and c and impose that they have indegree at most 1 ($i(\cdot) = 1$) while all the other vertices have indegree at most 2.

There are alternate ways to find the orientation. One is presented now and another one at the end of this section. Theorem 6 implies that E^* can be partitioned into two forests by Nash-Williams' theorem [23] or [25, Chapter 51] (or, once again, by matroid intersection!). We can then orient each forest to insure that every indegree in any of these two forests is at most 1, resulting in an orientation A^* of maximum indegree 2. (Here, we can impose that two vertices have indegree 1, or one has indegree 0.)

The Matroid M^*. We define the independence system $M^* = (E^*, \mathcal{I}^*)$ by the following independent sets:

$$\mathcal{I}^* = \{F \subseteq E^* : |F \cap \delta^+_{A^*}(v)| \leq k \text{ for all } v \in V\},$$

where $\delta^+_{A^*}(v)$ denotes the set of (undirected) edges of E^* leaving v in the orientation A^*. As the sets $\delta^+_{A^*}(v)$ are disjoint, this defines a partition matroid. As every vertex has indegree at most 2 in A^*, we obtain the following.

Lemma 8. *Let $F \in \mathcal{I}^*$ be an independent set in M^*. Then every vertex has degree at most $k + 2$ in F.*

If we use an orientation A^* in which the indegree of 3 vertices is constrained to be at most 1 (as is possible by Hakimi's result) then we obtain that these three vertices have degree at most $k + 1$ in any independent set $F \in \mathcal{I}^*$.

Our algorithm uses matroid intersection to find in polynomial-time (see 'Algorithmic Considerations' below for details) the minimmum cost set T which is both a base of the graphic matroid $M((V, E^*))$ defined on E^* and an independent set in M^*. T is therefore a spanning tree of maximum degree $\leq k + 2$.

Analysis. We are now ready to prove Theorem 2. For any matroid $M = (E, \mathcal{I})$, the independent set polytope $P(M)$ is defined as the convex hull of the characteristic vectors of independent sets and is given by [7, 6]:

$$P(M) = \{x \in \mathbb{R}^E : \begin{array}{ll} x(F) \le r(F) & F \subseteq E \\ x_e \ge 0 & e \in E\}, \end{array}$$

where $r(F) = \max\{|U| : U \in \mathcal{I} \text{ and } U \subseteq F\}$ is the rank function of the matroid. Similarly, the convex hull of characteristic vectors of the bases of M, or base polytope $B(M)$, is given by the face of $P(M)$ induced by $x(E) \le r(E)$, i.e. $B(M) = P(M) \cap \{x : x(E) = r(E)\}$. For example, the *spanning tree polytope* or base polytope $B(M(V, E^*))$ of the graphic matroid $M(V, E^*)$ can be seen to be given by (1), (2) and (4) (with x restricted to E^*). Similarly, the independent set polytope $P(M^*)$ of matroid M^* is given by

$$P(M^*) = \{x : \begin{array}{ll} x(\delta^+_{A^*}(v)) \le k & v \in V \\ x_e \ge 0 & e \in E^*. \end{array} \tag{7}$$

Observe that x^* belongs to both $B(M(V, E^*))$ and $P(M^*)$ (as (7) is implied by (3)).

Edmonds [6] shows that the convex hull of independent sets common to two matroids is precisely the intersection of their independent set polytopes (and similarly if we take the base polytopes). This implies that the convex hull of spanning trees of E^* that are independent for M^* is defined by $B(M(V, E^*)) \cap P(M^*)$:

$$\{x : \begin{array}{ll} x(E(S)) \le |S| - 1 & S \subset V \\ x(E(V)) = |V| - 1 & \\ x(\delta^+_{A^*}(v)) \le k & v \in V \\ x_e \ge 0 & e \in E^*\}. \end{array}$$

The characteristic vector of the spanning tree T returned by our algorithm is thus an optimum solution to the linear program $\min\{\sum_e c_e x_e : x \in B(M(V, E^*)) \cap P(M^*)\}$. As x^* is also a feasible solution to this linear program, we have that $c(T) = \sum_{e \in T} c_e \le \sum_e c_e x^*_e \le OPT(k)$, and this completes our proof of Theorem 2.

Polyhedral result. Our result can also be restated polyhedrally.

Corollary 9. *Let $Q(k)$ be the polytope defined by (1), (2), (3) and (4), and let $P(k)$ be the convex hull of characteristic vectors of spanning trees of maximum degree at most k. Then*

$$P(k) \subseteq Q(k) \subseteq P(k+2).$$

In other words, any convex combination of spanning trees such that the average degree of any vertex is at most k can be viewed as a convex combination of spanning trees each having maximum degree $\le k + 2$.

Proof. $Q(k)$ is a relaxation of $P(k)$, and hence $P(k) \subseteq Q(k)$. We have shown that for an extreme point x^* of $Q(k)$, we have that $x^* \in B(M(V, E^*)) \cap P(M^*) \subseteq P(k + 2)$, and therefore $Q(k) = conv(\{x^*\}) \subseteq P(k + 2)$. □

Algorithmic Considerations. Optimizing over (1)-(4) can be done in polynomial time. This can be done, for example, by using the ellipsoid algorithm as the separation problem over (3) is clearly polynomial while the separation over the base polytope (1), (2) and (4) of the graphic matroid can be done in strongly polynomial time (for any matroid, not just the graphic matroid) by an algorithm of Cunningham [5] (see also [25, Section 40.3]). An alternative is to derive a compact formaulation of the spanning tree problem (by bidirecting edges and using a flow reformulation) and using any polynomial-time algorithm for linear programming. Yet another approach is to use Lagrangean relaxation.

The orientation of E^* with maximum degree at most 2 can be done by finding a maximum matching in a graph with $O(n)$ vertices where $n = |V|$ (as $|E^*| = O(n)$) and $O(n)$ edges, and therefore can be obtained in $O(n^{1.5})$ time.

For our matroid intersection, we can exploit the fact that one of the matroids is a partition matroid. Brezovec, Cornuéjols and Glover [2] present an algorithm for precisely this case and even specialize it when the other matroid is a graphic matroid. In that setting, the complexity of their algorithm is $O(nm + n^2p + np^2)$ where $m = |E^*|$ and p is the number of sets defining the partition matroid; in our case (as $m = O(n)$ and $p = O(n)$), this gives $O(n^3)$. We can also exploit the fact that our matroids (both the graphic and the partition matroids) are linear and use an efficient matroid intersection algorithm for linear matroids due to Gabow and Xu [10]. Their running time is $O(mr^{1.77} \log(nW))$ where r is the sum of the ranks of the matroid and W is an upper bound on the largest cost; in our setting, this gives $O(n^{2.77} \log(nW))$. If W is small, we can also use a recent algorithm of Harvey [14] for the linear case whose running time is $O(mn^{1.38}W)$, or $O(n^{2.38}W)$ in our case.

The use of a weighted matroid intersection algorithm is actually not completely necessary. Indeed, we can simply return an extreme point of $B(M(V, E^*)) \cap P(M^*)$ whose cost is at most $\sum_e c_e x^*_e$. This can be done by decomposing x^* as a convex combination of characteristic vectors of spanning trees which are also independent in M^*. This can be done in strongly polynomial time by an algorithm of Cunningham [5].

The bottleneck for our algorithm is actually to solve the linear program in order to obtain x^*. Observe, however, that our algorithm as stated does not need x^* but only needs to know E^* (or a superset of it guaranteed to be orientable with maximum indegree 2). We would like to raise the question of whether a fast "combinatorial" algorithm can be designed which only finds E^* (and not x^*). This is an interesting open question.

The orientation revisited. The orientation A^* of E^* with all indegrees at most 2 can also be obtained purely algebraically (instead of using for example bipartite matchings). This is sketched below for completeness. This is not necessary for the bounded degree spanning tree result and can therefore be skipped.

For a matrix A, row index set I, column index set J, we denote by $A[I,J]$ the submatrix induced by the rows in I and the columns in J.

Lemma 10. *Let A be a nonsingular matrix with row index set and column index set $[n] = \{1, 2 \cdots, n\}$. Then, for any partition $\{I_1, I_2, \cdots, I_k\}$ of $[n]$, there exists a partition $\{J_1, J_2, \cdots, J_k\}$ of $[n]$ such that $A[I_p, J_p]$ is nonsingular for $p = 1, \cdots, k$.*

Proof. The result is an easy consequence of the matroid union theorem (see [25, Corollary 42.1a]). Another elementary proof goes as follows. First, by induction, we can assume that $k = 2$. By the generalized Laplace expansion (Laplace, 1772), we have that, for any $I \subset [n]$, we can express $\det(A)$ as:

$$\sum_{J \subseteq [n] : |J| = |I|} sign(I,J) \det(A[I,J]) \det(A[[n] \setminus I, [n] \setminus J]),$$

for suitably defined $sign(I,J) \in \{+1, -1\}$. As $\det(A) \neq 0$, there must be an index set J with $\det(A[I,J]) \neq 0$ and $\det(A[[n] \setminus I, [n] \setminus J]) \neq 0$, proving the lemma. □

The partition $\{J_1, J_2, \cdots, J_k\}$ can be found efficiently as follows. Assume, by induction on l, that we have an *nonsingular* matrix A' obtained from A by replacing every entry of column $j \leq l - 1$ by 0 except those corresponding to precisely one of the I_p's. As the determinant of A' is linear as a function of the entries of column l, there must be an index p such that replacing every entry of column l by 0 except those in this I_p gives a nonsingular matrix. This completes the inductive step. The final matrix A' is block diagonal and its nonsingularity implies that every block $A'[I_p, J_p] = A[I_p, J_p]$ is nonsingular. This shows that by simply checking linear independence, we can find a partition $\{J_1, J_2, \cdots, J_k\}$ as stated in Lemma 10.

We apply Lemma 10 for the MBDST in the following way. Consider the nonsingular matrix A defining the extreme point x^*. Let I_1 be the index set of the rows of A corresponding to $\chi(E(S))$ for $S \in \mathcal{L}$ and I_2 those corresponding to $\chi(\delta(v))$ for $v \in T$. Lemma 10 shows that E^* can be partitioned into E_1 and E_2 such that (i) $\chi(E_1(S))$ for $S \in \mathcal{L}$ are linearly independent and (ii) $\chi(\delta_{E_2}(v))$ for $v \in T$ are linearly independent. It is easy to see that (i) implies that E_1 must be a forest (as there must be precisely one edge in $E_1(S) \setminus \cup_{T \subset S, T \in \mathcal{L}} E_1(T)$ for every $S \in \mathcal{L}$) and that (ii) implies that every connected component of E_2 is either a tree or a tree plus an edge (i.e. a unique cycle with trees attached to it). Thus, both E_1 and E_2 can be oriented such that the indegree of every vertex is 1, and this gives an orientation A^* of E^* with indegrees at most 2 for every vertex.

Maximum degree of an MST. Ravi and Singh [24] consider the problem of minimizing the maximum degree of a minimum spanning tree (MST) in a weighted graph. Our approach also applies to their setting. Let C^* be the cost of an optimum spanning tree (of any maximum degree). Consider the optimum face of the spanning tree polytope; this face is defined by

$$F = \{x \in \mathbb{R}^{|E|} : (1), (2), (4) \text{ and } \sum_e c_e x_e = C^*\}.$$

By binary search, one can find the smallest integer $\hat{k} \in \mathbb{N}$ such that $F \cap \{x : x(\delta(v)) \leq \hat{k}$ for all $v \in V\} \neq \emptyset$. Clearly $\hat{k}$ is a lower bound on the smallest maximum degree of any MST. Furthermore, for $k = \hat{k}$, we know that the optimum of the linear program (LP) given in the introduction has value precisely equal to C^* (by definition of C^*), and our algorithm given in Figure 1 outputs a tree of maximum degree $\leq \hat{k} + 2 \leq k^* + 2$ and of cost less or equal to C^*. Thus it outputs an MST with maximum degree at most $k^* + 2$.

3 General Upper and Lower Bounds

The approach also works for the case in which we are given upper and lower bounds on the degree of every vertex in the spanning tree. In the General Minimum Bounded Degree Spanning Tree (GMBDST) problem, we are given a graph $G = (V,E)$, costs $c : E \to \mathbb{R}$ and degree bounds $u : V \to \mathbb{N}$ and $l : V \to \mathbb{N}$, and the goal is to find a spanning tree T such that $l(v) \leq d_T(v) \leq u(v)$ for all $v \in V$ and of minimum cost. An extension of our algorithm gives:

Theorem 11. *There exists a polynomial-time algorithm which outputs a tree T satisfying the weaker degree bounds $l(v) - 2 \leq d_T(v) \leq u(v) + 2$ for all $v \in V$ and of cost at most the cost of the optimum tree satisfying the original degree bounds.*

Proof. To derive this, we first solve the LP relaxation to obtain an extreme point x^* with support E^*.

$$\text{Min} \quad c(x) = \sum_e c_e x_e$$

subject to:

$$
\begin{array}{lll}
x(E(S)) \leq |S|-1 & S \subset V & \\
x(E(V)) = |V|-1 & & \\
x(\delta(v)) \leq u(v) & v \in V & (8) \\
x(\delta(v)) \geq l(v) & v \in V & (9) \\
x_e \geq 0 & e \in E. &
\end{array}
$$

Notice that among the tight sets, we need to consider at most one of (8) or (9) for a given vertex v. So, from uncrossing and Theorems 5 or 6, we derive that E^* can be oriented into A^* so that every vertex indegree is at most 2 (and we could again impose that the indegree of 3 vertices be at most 1).

Now, we can find a minimum cost spanning tree T in (V, E^*) satisfying

$$
l(v) - 2 \leq |T \cap \delta^+_{A^*}(v)| \leq u(v), \tag{10}
$$

for all $v \in V$. For this purpose, we first define a (generalized partition) matroid $\hat{M} = (E^*, \hat{\mathcal{I}})$ whose bases are precisely all subgraphs of cardinality $n-1$ satisfying (10). The independent sets of our matroid $\hat{M}$ are

$$
\begin{aligned}
\hat{\mathcal{I}} \;\; = \;\; & \{F \subseteq E^* : |F \cap \delta^+_{A^*}(v)| \leq u(v) \text{ for all } v \in V \\
& \text{and } \sum_v \max(l(v)-2, |F \cap \delta^+_{A^*}(v)|) \leq n-1\}.
\end{aligned}
$$

An independent set F of cardinality $n-1$ satisfies

$$
\begin{aligned}
n-1 \;\; &= \;\; |F| = \sum_v |F \cap \delta^+_{A^*}(v)| \\
&\leq \;\; \sum_v \max(l(v)-2, |F \cap \delta^+_{A^*}(v)|) \leq n-1
\end{aligned}
$$

with equality throughout and thus satisfies (10).

We can now find a spanning tree T in (V, E^*) satisfying (10) by finding a common base between the graphic matroid on E^* and matroid $\hat{M}$. Again, this is precisely the problem that the algorithm of Brezovec et al. [2] applies to, and therefore we can find in $O(n^3)$ a spanning tree T of minimum cost satisfying (10).

Polyhedrally, the convex hull P of characteristic vectors of spanning trees satisfying (10) is given by:

$$
P = \left\{ x \in \mathbb{R}^{|E^*|} : \begin{array}{ll}
x(E(S)) \leq |S|-1 & S \subset V \\
x(E(V)) = |V|-1 & \\
x(\delta^+_{A^*}(v)) \leq u(v) & v \in V \\
x(\delta^+_{A^*}(v)) \geq l(v)-2 & v \in V \\
x_e \geq 0 & e \in E^*
\end{array} \right\}.
$$

This can either be proved by using the connection to matroid intersection discussed above (and the generalized partition matroid) or more simply by observing that any extreme point of P is also an extreme point of the polytope of common independent sets to a graphic matroid and a partition matroid whose independent sets F satisfy $|F \cap \delta^+(v)| \leq b(v)$ where $b(v)$ is suitably chosen in $\{l(v)-2, u(v)\}$.

The analysis follows from the description of P. Indeed, x^* satisfies all inequalities describing P; for (10), we have that

$$
x^*(\delta^+_{A^*}(v)) \geq x^*(\delta(v)) - 2 \geq l(v) - 2.
$$

Therefore, the cost of T, $c(T)$ is upper bounded by $\sum_{e \in E} c_e x^*_e$ and thus by the optimum value. □

4 Towards $k+1$?

One could try to improve the degree bound from $k+2$ to $k+1$ and prove Conjecture 1. One possibility would be to prove additional properties of any extreme point x^* and use a different matroid in place of M^*. In particular, if Conjecture 12 below was true (and the orientation could be obtained efficiently!) then this would settle Conjecture 1 positively.

Conjecture 12. *For any extreme point x^* to (1)–(4) with support E^*, there exists an orientation A^* of E^* such that*

$$
\sum_{e \in \delta^-_{A^*}(v)} (1 - x^*_e) \leq 1 \tag{11}
$$

for all $v \in V$.

Whenever there exists an orientation satisfying (11), we can get a spanning tree of maximum degree $k+1$ and of cost at most $OPT(k)$ by using the matroid $\tilde{M} = (E^*, \tilde{\mathcal{I}})$ with

$$
\tilde{\mathcal{I}} = \{F \subseteq E^* : |F \cap \delta^+_{A^*}(v)| \leq \lceil x^*(\delta^+_{A^*}(v)) \rceil \text{ for all } v \in V\}.
$$

Indeed, for an $F \in \tilde{\mathcal{I}}$, we have:

$$
\begin{aligned}
& |F \cap \delta(v)| \\
& \quad = \;\; |F \cap \delta^-_{A^*}(v)| + |F \cap \delta^+_{A^*}(v)| \\
& \quad \leq \;\; |\delta^-_{A^*}(v)| + \lceil x^*(\delta^+_{A^*}(v)) \rceil \\
& \quad = \;\; \sum_{e \in \delta^-_{A^*}(v)} (1 - x^*_e) + x^*(\delta^-_{A^*}(v)) + \lceil x^*(\delta^+_{A^*}(v)) \rceil \\
& \quad < \;\; 1 + k + 1 = k + 2,
\end{aligned}
$$

the last inequality following from (11), (3) and $\lceil y \rceil < y + 1$. As the degree $|F \cap \delta(v)|$ of v is an integer, it must be at most $k+1$.

5 Symmetric TSP and the Subtour Polytope

The approach is quite general and applies to many other settings where uncrossing can be applied. For example, consider the subtour polytope of the symmetric traveling salesman problem:

$$
SUB = \left\{ x \in \mathbb{R}^{|E|} : \begin{array}{ll}
x(\delta(S)) \geq 2 & S \subset V \\
x(\delta(v)) = 2 & v \in V \\
x_e \geq 0 & e \in E
\end{array} \right\}.
$$

Held and Karp [11, 12] show that any point in SUB can be viewed as a convex combination of *1-trees* such that every vertex has average degree 2. A *1-tree* is a spanning tree on $V \setminus \{1\}$ together with 2 edges incident to vertex 1 (for some fixed vertex 1). The family of 1-trees forms the bases of a matroid M_{1t}.

The approach developed here leads to the following refinement:

Theorem 13. *Any point in SUB can be decomposed into a convex combination of 1-trees, each of maximum degree 4, such that every vertex has average degree 2.*

Again, we believe that the bound of 4 on the degree can be replaced by a bound of 3. The proof below actually shows that we can impose the degree of 3 on any 3 vertices of V.

Theorem 13 can be proved in the same way as our MBDST result. First, any extreme point x^* of SUB can be uniquely determined by

$$\begin{cases} x^*(\delta(S)) = 2 & S \in \mathcal{L} \\ x^*_e = 0 & e \notin E^* \end{cases}$$

where $\mathcal{L}$ is a laminar family as was shown by Cornuéjols et al. [4]. As mentioned earlier, a laminar family on a ground set of size n, without the entire set and without two complementary sets (which would give the same equality) has at most $2n-3$ sets. The same argument as in Theorem 5 (as $\delta(S) \cap E^*(U) = \delta(S \cap U) \cap E^*(U)$) then shows that

$$|E^*(U)| \leq 2|U| - 3,$$

for any $U \subseteq V$. Hence E^* can also be oriented into A^* such that the indegree of any vertex is at most 2. Again we can define a partition matroid M^* whose independent sets F satisfy: $|F \cap \delta^+_{A^*}(v)| \leq 2$ for all $v \in V$. Instead of considering 1-trees, we can consider *restricted 1-trees* T which are defined to be independent in M^*. Observe that any restricted 1-tree has maximum degree at most 4. As x^* belongs to the matroid polytope for M^* (and to the base polytope for M_{1t}), we have that x^* can be decomposed into a convex combination of restricted 1-trees such that every vertex has degree 2 on average, proving Theorem 13.

Algorithmically, we can use matroid intersection to find a restricted 1-tree of minimum cost in polynomial time, and this shows:

Corollary 14. *For any instance of the symmetric traveling salesman problem, one can find in polynomial time a 1-tree of maximum degree 4 and of cost less or equal to the subtour bound (and hence of the minimum cost tour).*

6 Asymmetric TSP

The same approach can also be applied to a classical relaxation of the *asymmetric* traveling salesman problem also introduced by Held and Karp [11]. Held and Karp have defined a 1-arborescence (sometimes called directed 1-tree) in a directed graph (V, A) as a directed subgraph whose undirected counterpart is a 1-tree and with indegree equal to 1 at every vertex; so these are the common bases to two matroids, the 1-tree matroid M_{1t} and a partition matroid M_{in}. Held and Karp show that any solution to:

$$ASUB = \left\{ x \in \mathbb{R}^{|A|} : \begin{array}{ll} x(\delta^+(S)) \geq 1 & S \subset V \\ x(\delta^+(v)) = 1 & v \in V \\ x(\delta^-(v)) = x(\delta^+(v)) & v \in V \\ x_e \geq 0 & e \in A \end{array} \right\}$$

can be decomposed as a convex combination of 1-arborescences of average outdegree 1. Here, A denotes the arc set of our directed graph (V, A).

Similar to Theorem 5, we can prove the following properties of extreme points of $ASUB$.

Theorem 15. *For any extreme point x^* of $ASUB$ with support A^*, we have that*

$$|A^*(U)| \leq 3|U| - 4,$$

for any $U \subseteq V$.

Proof. If S, T are such that $x^*(\delta^+(S)) = x^*(\delta(T)) = 1$ with $S \cap T \neq \emptyset$ and $S \cup T \neq V$ then $S \cap T$ and $S \cup T$ also give tight cuts and furthermore there are no arcs e with $x^*_e > 0$ between $S \setminus T$ and $T \setminus S$ (in either direction). From the standard uncrossing argument, we then get that any extreme point x^* with support A^* can be defined by a cross-free family $\mathcal{C}$ (i.e. for $S, T \in \mathcal{C}$, either $S \subseteq T$, or $T \subseteq S$ or $S \cap T = \emptyset$ or $S \cup T = V$) of linearly independent tight cuts $x^*(\delta^+(S)) = 1$ as well as $n-1$ of the balance constraints (we can discard one of the balance constraints since they are not all linearly independent).

By adding the balance constraints (6) over $v \in S$, we can derive that $x^*(\delta^+(S)) = x^*(\delta^-(S))$ and therefore (by complementing) we can assume that none of the sets in $\mathcal{C}$ contain a given vertex r. Thus $\mathcal{C}$ is now a *laminar family* not containing V and not containing two complementary sets, and therefore has cardinality at most $2n-3$. This implies that $|A^*| \leq 2n - 3 + n - 1 = 3n - 4$, a slight improvement over the bound of $3n-2$ in Vempala and Yannakakis [26].

If we restrict our attention to the arcs in $A^*(U)$, our tight cut equalities again lead to row vectors $\chi(\delta^+(S))$ which are idential to $\chi(\delta^+(S \cap U))$ over $A^*(U)$. Similarly for the balance constraints (6). Therefore, the reader can observe that the same argument as in Theorem 5 shows that $|A^*(U)| \leq 3|U| - 4$. □

This means that A^* can be "reoriented" into O^* such that the new indegree in O^* is at most 3 (in addition, we have up to 4 units to play with; for example we can impose that the indegree of 2 given vertices is at most 1). Reorienting means that an arc $(u,v) \in A^*$ may now become $(v,u) \in O^*$. This reorientation gives a way to assign each arc (u,v) either to u or v but not both. We now define a new matroid M^* whose independent sets F satisfy

$$|F \cap \delta^+_{O^*}(v) \cap \delta^+_{A^*}(v)| \le 1 \qquad (12)$$

and

$$|F \cap \delta^+_{O^*}(v) \cap \delta^-_{A^*}(v)| \le 1 \qquad (13)$$

for all $v \in V$. This is again a partition matroid (with $2n$ parts now). Observe that any extreme point x^* of $ASUB$ belongs to the independent set polytope of M^*. Therefore, by Edmonds' polyhedral characterization of matroid intersection, we know that x^* can be viewed as a convex combination of bases of M_{1t} which are also independent in M^*.

Now consider such a base T of M_{1t} which is also independent in M^*. T is a (weakly) 1-tree (as a base of M_{1t}) when viewed as undirected, and for every vertex v, we have at most 1 outgoing arc from (12), at most 1 incoming arc from (13) and at most 3 additional arcs (incoming or outgoing) from arcs in $\delta^-_{O^*}(v)$. So the total degree of v is at most 5, and its indegree and outdegree are at most 4. In summary, this shows the following.

Theorem 16. *Any point in $ASUB$ can be decomposed into a convex combination of weakly 1-trees, each having (i) maximum indegree ≤ 4, (ii) maximum outdegree ≤ 4 and (iii) maximum total degree ≤ 5, such that every vertex has on average indegree 1 and outdegree 1. Furthermore, we can find in polynomial time (through matroid intersection) a weakly 1-tree of maximum total degree ≤ 5 and maximum indegree and outdegree ≤ 4 of cost less or equal to the Held-Karp bound obtained by optimizing over $ASUB$.*

7 l-edge-connectivity

The approach can also be applied to the problem of finding an l-edge-connected subgraph ($l \ge 2$) with maximum degree at most k and of minimum cost. However, the results are weaker and therefore are only sketched in this section. We should emphasize that we are *not* assuming the triangle inequality (and not even assuming we have a complete graph) or allowing multiple edges in the solution, since otherwise the splitting off technique of Mader can transform any solution into one with all degrees equal to l or $l+1$ without any increase in cost. We are assuming general costs, and the restriction that an edge can be selected at most once.

A relaxation of the problem is given by the following linear program:

$$\text{Min} \quad c(x) = \sum_e c_e x_e$$

subject to:

$$\begin{aligned} x(\delta(S)) &\ge l && S \subset V \\ x(\delta(v)) &\le k && v \in V \\ 0 \le x_e &\le 1 && e \in E. \end{aligned}$$

Let x^* be an extreme point optimizing the above linear program. Let $E^* = \{e : 0 < x^*_e < 1\}$ and let $E_1 = \{e : x^*_e = 1\}$. The uncrossing argument shows that $|E^*(U)| \le 2|U| - 3$ for every $U \subseteq V$ and therefore E^* can be oriented into A^* such that all indegrees are at most 2. (E_1, however, can have many edges; in fact it will have at least $(\frac{l}{2} - 2)n$ edges.)

We now define two matroids both on the ground set $E^* \cup E_1$. Matroid M_1 is defined as the $\lfloor l/2 \rfloor$-fold union (see Chapters 42 and 51 in [25]) of the graphic matroid on $E^* \cup E_1$. Any *spanning set* F for M_1 (i.e. a set containing a base) is therefore $\lfloor l/2 \rfloor$-edge-connected. Furthermore, the spanning set polytope [25, Corollary 40.2f] for M_1, or the convex hull of characteristic vectors of spanning sets, can be shown to contain x^*, and therefore x^* dominates (componentwise) an element y^* of the base polytope of M_1.

Matroid M_2 is a partition matroid similar to M^* for MBDST except that we can take all of E_1; its collection of independent sets is given by:

$$\{F \subseteq E^* \cup E_1 : |F \cap \delta^+_{A^*}(v)| \le k - |\delta_{E_1}(v)| \text{ for all } v \in V\}.$$

Any independent set for M_2 has degree at most $k+2$ and furthermore x^* belongs to its independent set polytope (and so does any $y^* \le x^*$).

Using matroid intersection, we can find a minimum cost base T of M_1 which is also independent in M_2. Its cost will be upper bounded by $c(y^*) \le c(x^*)$, and hence by the optimum solution. We have thus shown:

Theorem 17. *In polynomial time, one can find a $\lfloor l/2 \rfloor$-edge-connected subgraph of maximum degree $\le k+2$ whose cost is at most the cost of the optimum l-edge-connected subgraph of maximum degree at most k.*

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