18.455 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 3

April 23, 2020

This problem set is due on gradescope on April 30, 2020.

1. Given a full rank matrix $A \in \mathbb{R}^{n\times n}$ (true also for any field F), let R and C denote the indices of the rows and columns of A. Given $I \subset R$, show using matroid intersection that there exists $J \subset C$ with $|I| = |J|$ such that both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank.

   (Another way of proving this without matroid intersection would be through the generalized Laplace expansion of the determinant.)

2. Given a graph $G = (V, E)$ and a partition of V into $V_1, V_2, \cdots, V_\ell$, we denote by

   $$\delta(V_1, V_2, \cdots, V_\ell) := \{(u, v) \in E : u \in V_i, j \in V_j, i \neq j\}.$$

   Use matroid union to prove that a graph contains k edge-disjoint spanning trees if and only if

   $$\forall \ell, \forall \text{ partition } \rho = (V_1, \cdots, V_\ell) \text{ of } V : |\delta(V_1, \cdots, V_\ell)| \geq (\ell - 1)k.$$

3. Consider a matroid $M = (S, \mathcal{I})$, and let B be a given basis of M. Let $A = S \setminus B$. From M and B, we define a *linking system* $\mathcal{P}$ by

   $$\mathcal{P} = \{(B' \setminus B, B \setminus B') \subset A \times B : B' \text{ is a basis of } M\}.$$

   $\mathcal{P}$ consists of pairs $(X, Y) \subset A \times B$ which corresponds to valid basis exchanges for B in the matroid M. For such a pair (X, Y), we say that X is linked to Y. Observe that

   (a) $(\emptyset, \emptyset) \in \mathcal{P}$
   (b) $(X, Y) \in \mathcal{P} \Rightarrow |X| = |Y|$

   (We could add some additional axioms that would then characterize linking systems but we won't do it here.)

   (a) Given a bipartite graph $G = (V, E)$ with bipartition (A, B), let $\mathcal{P}$ be the pairs (X, Y) with $X \subseteq A$ and $Y \subseteq B$ such that there exists a perfect matching between X and Y. Show that $\mathcal{P}$ define a linking system.
   (b) Given a matrix L (over some field F), let A index the rows of L and B index the columns of L. Say that $X \subseteq A$ is linked to $Y \subseteq B$ is the corresponding submatrix $L(X, Y)$ is of full rank (over F). Show that this also define a linking system.
   (c) Let $\mathcal{P}_1 \subseteq 2^{A\times B}$ and $\mathcal{P}_2 \subseteq 2^{B\times C}$ be two linking systems. Define

   $$\mathcal{P}_1 * \mathcal{P}_2 = \{(X, Z) \subseteq A \times C : \exists Y \subseteq B \text{ with } (X, Y) \in \mathcal{P}_1 \text{ and } (Y, Z) \in \mathcal{P}_2\}.$$

   Show that $\mathcal{P}_1 * \mathcal{P}_2$ is also a linking system.
   (d) Suppose we are given disjoint sets $V_0, V_1, \cdots, V_k$ and, for $i = 1, \cdots, k$, a linking system $\mathcal{P}_i$ on (V_{i-1}, V_i). This constitutes a linking network. Define a flow to be $(X_0, X_1, \cdots, X_k) \subseteq (V_0, V_1, \cdots, V_k)$ where X_{i-1} is linked to X_i in $\mathcal{P}_i$ for $i = 1, \cdots, k$. The value of the flow is $|X_0| = |X_1| = \cdots = |X_k|$. (If all the linking systems involved are of the matching type given above, a flow corresponds to a set of vertex-disjoint directed paths in a layered network. But the beauty here is that you can have many different types of linking systems involved.) How would you efficiently find a maximum flow (i.e. one of maximum value) in such a linking network (given access to matroid independence oracles for all the matroids defining the linking systems)? (Can you do it with matroid union/partition?)

(One can also derive a max-flow min-cut type result, but I won't formulate it here.)

4. Suppose we are given an undirected graph $G = (V, E)$, and additional vertex $s \notin V$, an integer k, and we would like to add the minimum number of edges between s and vertices of V (multiple edges are allowed) such that the resulting graph H on $V + s$ has k edge-disjoint paths between any two vertices of V (i.e. the only cut that could possibly have fewer than k edges is the cut separating s from V).

   (a) Argue that this problem is equivalent to finding $x : V \to \mathbb{Z}_+$ minimizing $x(V)$ such that $\forall \emptyset \neq S \subset V$:
   $$x(S) \geq k - d_E(S),$$
   where $d_E(S) = |\delta_E(S)|$ corresponds to the number of edges between S and $V \setminus S$ in G.

   (b) Add k edges between s and each vertex of V. Let A be these $k|V|$ newly added edges. Say that $F \subseteq A$ is feasible if the graph $(V + s, E \cup (A \setminus F))$ has at least k edge-disjoint paths between any two vertices of V. Prove that the feasible sets form the independence sets of a matroid.

   (c) How would you efficiently solve the original problem (with k part of the input)?

   (This result is useful to solve the following k-edge-connectivity augmentation problem efficiently. Given a graph[1] $G = (V, E)$ and $k \geq 2$ (doesn't work with $k = 1$), construct a graph $H = (V, F)$ with as few edges as possible such that the graph $G \cup H = (V, E \cup F)$ is k-edge-connected. The connection to the exercise here is that one needs p edges here if and only if in the exercise one needs $2p - 1$ or $2p$ edges.)

[1] in this statement, by graph, we really mean a multigraph in which there might be several edges between the same two endpoints.