18.455 Advanced Combinatorial Optimization	Michel X. Goemans

Problem Set 2

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This problem set is due on gradescope on April 14, 2020. You can either typeset your solutions or scan them in (the dropbox application for example can scan and collate up to 10 pages).

1. We have seen in lecture (before the break) that any rational polyhedral cone C has an integral Hilbert basis. Assume that C is also pointed (i.e. there exists a vector $b \in \mathbb{R}^n$ such that $b^T x > 0$ for all $x \in C \setminus \{0\}$). Show then that

$$H := \{a \in (C \setminus \{0\}) \cap \mathbb{Z}^n | a \text{ is not the sum of two other integral vectors in } C\}$$

   is the *unique minimal* Hilbert basis of C (i.e. it is a Hilbert basis, and every other Hilbert basis contains all vectors in H).

   (Observe this is not true if the cone is not pointed. For example, if you take $C = \mathbb{R}$ then the above definition gives $H = \emptyset$.)

2. Consider a function $r : 2^S \to \mathbb{Z}_+$ that satisfies the following rank axioms:

   (a) For all $U \subseteq S$: $r(U) \leq |U|$

   (b) For all $T \subseteq U$: $r(T) \leq r(U)$

   (c) Submodularity: For all $T, U \subseteq S$: $r(T) + r(U) \geq r(T \cup U) + r(T \cap U)$

   Now define $\mathcal{I} = \{I : |I| = r(I)\}$. Show that $(S, \mathcal{I})$ satisfies the independence axioms defining a matroid.

3. When is U_k^ℓ a minor of U_p^q?

4. Prove that a $0, \pm 1$ matrix which is minimally *not* totally unimodular (i.e. having all square subdeterminants in $\{-1, 0, 1\}$ except for the matrix itself) has determinant ± 2.

   (Hint: One option is to use induction on the size of the matrix. The base case is clear. For the inductive step, pivot on a nonzero element to zero out all the other elements in its column.)

5. Prove that if a matroid is representable over GF(2) and over GF(3) then it can be represented over any field by a totally unimodular matrix (it is thus regular).

   (Hint: Start with a representation $[I|B]$ of the matroid over GF(3) and interpret it as a real matrix with entries $0, \pm 1$. Also use the previous exercise.)

   (Remark 1: The statement is still true if one replaces GF(3) by any field of charateristic other than 2.)

   (Remark 2: Using the excluded minor characterizations for binary (U_4^2 to be excluded) and ternary (U_5^2, U_5^3, F_7 and F_7^* to be excluded) matroids, this gives an excluded minor characterization for regular matroids (namely excluding U_4^2, F_7 and F_7^* as minors, by the 3rd exercise.)