18.455 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 1

Feb 11, 2020

This problem set is due in class on Thu Feb 27, 2020.

1. Show that any 3-regular 2-edge-connected graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. (A 2-edge-connected graph has at least 2 edges in every cutset; a cutset being the edges between S and $V \setminus S$ for some vertex set S.)

2. A graph $G = (V, E)$ is said to be *factor-critical* if, for all $v \in V$, we have that $G \setminus \{v\}$ contains a perfect (i.e. covering all vertices) matching.

   Given a graph $H = (V, E)$, an *ear* is a path $v_0 - v_1 - v_2 - \cdots - v_k$ whose endpoints (v_0 and v_k) are in V and whose internal vertices (v_i for $1 \le i \le k-1$) are not in V. We allow that v_0 be equal to v_k, in which case the path would reduce to a cycle. Adding the ear to H creates a new graph on $V \cup \{v_1, \cdots, v_{k-1}\}$. The trivial case when $k = 1$ (a 'trivial' ear) simply means adding an edge to H. An ear is called *odd* if k is odd, and even otherwise; for example, a trivial ear is odd.

   (a) Let G be a graph that can be constructed by starting from an odd cycle and repeatedly adding odd ears. Prove that G is factor-critical.

   (b) Prove the converse that any factor-critical graph can be built by starting from an odd cycle and repeatedly adding odd ears.

3. Let $G = (V, E)$ be a graph and $T \subseteq V$. In this exercise, a path is called a T-path if its endpoints are *distinct* vertices of T and no internal vertex belongs to T. Notice that if $T = V$, a T-path is just a matching. Let τ be the maximum number of (vertex) disjoint T-paths.

   (a) Show that

   $$\tau \le \min_{U \subseteq V} |U| + \sum_{K \in \mathcal{C}(G \setminus U)} \left\lfloor \frac{|K \cap T|}{2} \right\rfloor$$

   where $\mathcal{C}(G \setminus U)$ denotes the connected components of $G \setminus U$.

   (b) Use the Tutte-Berge formula (in a modified graph $G' = (V', E')$) to prove equality:

   $$\tau = \min_{U \subseteq V} |U| + \sum_{K \in \mathcal{C}(G \setminus U)} \left\lfloor \frac{|K \cap T|}{2} \right\rfloor .$$

   (Corrected hint: Let $B = V \setminus T$. One construction for G' is to start from G and first add a disjoint copy of $G[B]$ on a new vertex set B'. Any $v \in B$ has a corresponding $v' \in B'$ and we connect each v' to v and to all neighbors of v in G. Thus $V' = V \cup B'$ and $E' = E \cup \{(u', v') | (u, v) \in G[B]\} \cup \{(u, v') |$ either $u = v$ or $(u, v) \in E\}$.)

4. Let $\mu(G)$ be the size of a max matching in G. Prove that G is factor-critical iff G is connected and $\mu(G) = \mu(G \setminus \{v\})$ for all $v \in V$.

5. Consider the problem of counting the number $\phi(G)$ of perfect matchings in a graph $G = (V, E)$. For any orientation $\vec{E}$, we can associate a skew-symmetric matrix A_s where

   $$A_s(i,j) = \begin{cases} 1 & (i,j) \in \vec{E} \\ -1 & (j,i) \in \vec{E}. \end{cases}$$

Show that

$$\mathbb{E}(\det(A_s)) = \phi(G),$$

when the orientation is chosen uniformly among all $2^{|E|}$ orientations. Deduce that there exists an orientation $\vec{E}$ with

$$\phi(G) \leq \det(A_s) \leq (\phi(G))^2.$$

(However, it is not known how to find such an orientation.)