18.438 Advanced Combinatorial Optimization October 20, 2009

Lecture 10

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1 Matroid Optimization

We consider the problem of finding a maximum weight independent set in a matroid. Let $M = (S, \mathcal{I})$ be a matroid and $w : S \to \mathbb{R}$ be a weight function on the ground set. As usual, let r be the matroid's rank function. We want to compute $\max_{I \in \mathcal{I}} w(I)$, where $w(I) = \sum_{e \in \mathcal{I}} w(e)$. For this problem, we have an independence oracle that given a set $I \subseteq S$ determines whether or not $I \in \mathcal{I}$. This problem can be solved by the following greedy algorithm. Wlog, we assume all elements have nonnegative weights by discarding those with negative weights (using the fact that they won't appear in an optimum solution given that any subset of an independent set is independent).
Algorithm:

1. Order elements in non-increasing weights: $w(s_1) \geq w(s_2) \geq \ldots \geq w(s_n) \geq 0$.
2. Take $I = \{s_i : r(U_i) > r(U_{i-1})\}$, where $U_i = \{s_1, \ldots, s_i\}$ for $i = 1, \ldots, n$.

Claim 1 *$I \in \mathcal{I}$. In fact, for all i, $I \cap U_i \in \mathcal{I}$ and $|I \cap U_i| = r(U_i)$.*

Proof: The proof is by induction on i. It is certainly true for $i = 0$. Assume the statement holds for $i - 1$. So $I \cap U_{i-1} \in \mathcal{I}$ and $|I \cap U_{i-1}| = r(U_{i-1})$. There are two cases:

1. $r(U_i) = r(U_{i-1})$: Then $s_i \notin I$ and $I \cap U_{i-1} \in \mathcal{I}$. Also $|I \cap U_i| = |I \cap U_{i-1}| = r(U_{i-1}) = r(U_i)$.
2. $r(U_i) > r(U_{i-1})$: There exists an independent set $J \subseteq U_i$ such that $|J| > |I \cap U_{i-1}|$. Note $|J| = r(U_i)$ and $|I \cap U_{i-1}| = r(U_{i-1})$. There is an element in J not in $I \cap U_{i-1}$ that we can add to $I \cap U_{i-1}$ to get an independent set. It must be s_i. So $I \cap U_{i-1} + s_i = I \cap U_i \in \mathcal{I}$ and $|I \cap U_i| = r(U_i) = r(U_{i-1}) + 1$.

This completes the proof. □

The way the greedy algorithm is described seems to indicate that we need access to a rank oracle (which can be simply constructed from an independence oracle by constructing a maximal independent subset). However, the proof above shows that we can simply maintain I by starting from $I = \emptyset$ and adding s_i to it if the resulting set continues to be independent. This algorithm thus makes n calls to the independence oracle.

We still need to show that $w(I)$ has the maximum weight. To do this we consider the matroid polytope due to Edmonds.

Definition 1 (Matroid Polytope) *$\mathcal{P} = \text{conv}(\{\chi(I) : I \in \mathcal{I}\})$, where χ is the characteristic vector.*

Theorem 2 *Let $\mathcal{Q} = \left\{ \begin{array}{ll} x \in \mathbb{R}^{|S|} : & \sum_{s \in U} x_s \leq r(U), \quad \forall U \subseteq S \\ & x \geq 0 \end{array} \right\}$. The system of inequalities defining $\mathcal{Q}$ is TDI. Therefore, $\mathcal{Q}$ integral (as the right-hand-side is integral), and hence $\mathcal{Q} = \mathcal{P}$. Furthermore, the greedy algorithm produces an optimum solution.*

Proof: Consider the following linear program and its dual. Let O_P and O_D be their finite optimal values. Note $x(U) = \sum_{s \in U} x_s$.

$$\begin{array}{lll} O_P = & \text{Max} & w^T x \\ & \text{s.t.} & x(U) \leq r(U), \quad \forall U \subseteq S \\ & & x \geq 0. \end{array} \tag{1}$$

$$
\begin{aligned}
O_D = \quad & \text{Min} \quad \sum_{U \subseteq S} r(U) \cdot y_U \\
& \text{s.t.} \quad \sum_{U : s \in U} y_U \geq w_s, \quad \forall s \in S \\
& \qquad\quad y_U \geq 0, \qquad\qquad \forall U \subseteq S.
\end{aligned} \tag{2}
$$

Let $w \in \mathbb{Z}^{|S|}$. We need to show that there exists an optimum solution to (2) that is integral. Wlog, we assume w is nonnegative by discarding elements with negative weights (as the corresponding constraints in the dual are always satisfied (given that $y \geq 0$) and thus discarding such elements will not affect the feasibility/integrality of the dual). So $w \in \mathbb{Z}_+^{|S|}$. Let J be the independent set found by the greedy algorithm. Note that $w(J) \leq \text{Max}_{I \in \mathcal{I}} w(I) \leq O_P = O_D$. We find y such that the min value O_D equals $w(J)$. Define y as follows:

$$
\begin{aligned}
& y_{U_i} = w(s_i) - w(s_{i+1}) \text{ for } i = 1, \dots, n-1, \\
& y_{U_n} = w(s_n), \\
& y_U = 0, \text{ for other } U \subseteq S.
\end{aligned}
$$

Clearly, y is integral and $y \geq 0$ by construction. To check the first condition in Equation (2), we note that for any $s_i \in S = \{s_1, \dots, s_n\}$,

$$
\sum_{U : s_i \in U} y_U = \sum_{j=i}^{n} y_{U_j} = w(s_i), \text{ by telescoping sum.}
$$

Hence y is a feasible solution of the dual linear program. Now we compute the objective function of the dual:

$$
\begin{aligned}
\sum_{U \subseteq S} r(U) \cdot y_U &= \sum_{i=1}^{n-1} r(U_i) \cdot (w(s_i) - w(s_{i+1})) + r(U_n) \cdot w(s_n) \\
&= w(s_1) r(U_1) + \sum_{i=2}^{n} w(s_i)(r(U_i) - r(U_{i-1})) \\
&= w(J).
\end{aligned}
$$

It follows that $w(J) = \text{Max}_{I \in \mathcal{I}} w(I) = O_P = O_D$ and that y gives the minimum value O_D. Since w was an arbitrary integral vector and there is a corresponding optimal integral solution y, it follows that the linear system defining $\mathcal{Q}$ is TDI. Since b is integral (as in $Ax \leq b$ when Equation 1 is written this way), it follows that $\mathcal{Q}$ is integral. Furthermore, any integral solution x in Q is the characteristic vector of an independent set I (since $x(U) = |I \cap U| \leq r(U)$ for all U including circuits). Thus $\mathcal{P} = \mathcal{Q}$. Furthermore, the greedy algorithm is optimal. $\square$

2 Matroid Intersection

Let $\mathcal{M}_1 = (S, \mathcal{I}_1), \mathcal{M}_2 = (S, \mathcal{I}_2)$ be two matroids on common ground set S with rank functions r_1 and r_2. Many combinatorial optimization problems can be reformulated as the problem of finding the maximum size common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. Edmonds and Lawler studied this problem and proved the following min-max matroid intersection characterization.

Theorem 3

$$
\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \in S} \left(r_1(U) + r_2(S \setminus U) \right).
$$

The fact that the min $\leq$ max is easy. Indeed, for any $U \subseteq S$ and $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we have

$$\begin{aligned} |I| &= |I \cap U| + |I \cap (S \setminus U)| \\ &\leq r_1(U) + r_2(S \setminus U), \end{aligned}$$

since $I \cap U$ is an independent set in $\mathcal{I}_1$ and $I \cap (S \setminus U)$ is an independent set in $\mathcal{I}_2$. Therefore, $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| \leq \min_{U \in S} (r_1(U) + r_2(S \setminus U))$. We will prove the other direction algorithmically.

There is no equivalent theorems for three or more matroids. In fact, the problem of finding the independent set of maximum size in the intersection of three matroids is NP-Hard.

Theorem 4 *Given three matroids $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ where $\mathcal{M}_i = (S, \mathcal{I}_i)$, it is NP-hard to find the independent set I with maximum size in $\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$.*

Proof: The reduction is from the Hamiltonian path problem. Let $D = (V, E)$ be a directed graph and s and t are two vertices in D. Given an instance $(D = (V, E), s, t)$ of the Hamiltonian path problem, we construct three matroids as follows: $\mathcal{M}_1$ is equal to the graphic matroid of the undirected graph G which is the undirected version of D. $\mathcal{M}_2 = (E, \mathcal{I}_2)$ is a partition matroid in which a subset of edges is an independent set if each vertex has at most one incoming edge in this set (except s which has none), i.e, $\mathcal{I}_2 = \{F \subseteq E : |\delta^-(v) \cap F| \leq f_s(v)\}$ where $f_s(v) = 1$ if $v \neq s$ and $f_s(s) = 0$. Similarly, we define $\mathcal{M}_3 = (E, \mathcal{I}_3)$ such that $\mathcal{I}_3 = \{F \subseteq E : |\delta^+(v) \cap F| \leq f_t(v)\}$ where $f_t(v) = 1$ if $v \neq t$ and $f_t(t) = 0$. It is easy check that any set in the intersection of these matroids corresponds to the union of vertex-disjoint directed paths with one of them starting at s and one (possibly a different one) ending at t. Therefore, the size of this set is $n - 1$ if and only if there exists a Hamiltonian path from s to t in D. $\square$

Examples

The following important examples illustrate some of the applications of the matroid intersection theorem.

1. For a bipartite graph $G = (V, E)$ with partition $V = V_1 \cup V_2$, consider $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ where $\mathcal{I}_i = \{F : \forall v \in V_i, \deg_F(v) \leq 1\}$ for $i = 1, 2$. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are partition matroids, while $\mathcal{I}_1 \cap \mathcal{I}_2$, the set of bipartite matchings of G, does not define a matroid on E. Also, note that the rank $r_i(F)$ of F in M_i is the number of vertices in V_i covered by edges in F. Then by Theorem 3, the size of a maximum matching in G is

$$\begin{aligned} \nu(G) &= \min_{U \in E} (r_1(U) + r_2(E \setminus U)) & (3) \\ &= \tau(G) & (4) \end{aligned}$$

   where $\tau(G)$ is the size of a minimum vertex cover of G (as the vertices from V_1 covered by U together with the vertices of V_2 covered by $E \setminus U$ form a vertex cover). Thus, the matroid intersection theorem generalizes Kőnig's matching theorem.

2. As a corollary to Theorem 3, we have the following min-max relationship for the minimum common spanning set in two matroids.

$$\begin{aligned} \min_{F \text{ spanning in } M_1 \text{ and } M_2} |F| &= \min_{B_i \text{ basis in } \mathcal{M}_i} |B_1 \cup B_2| \\ &= \min_{B_i \text{ basis in } \mathcal{M}_i} |B_1| + |B_2| - |B_1 \cap B_2| \\ &= r_1(S) + r_2(S) - \min_{U \subseteq S} [r_1(U) + r_2(S \setminus U)]. \end{aligned}$$

   Applying this corollary to the matroids in example 1, it follows that the minimum edge cover in a bipartite graph G is equal to the maximum of $|V| - r_1(F) - r_2(E \setminus F)$ over all $F \subseteq E$. Since this is exactly the maximum size of a stable set in G (consider the vertices of V_1 not covered by F and those of V_2 not covered by $E \setminus F$), the corollary is a generalization of the Kőnig-Rado theorem.

3. Consider a graph G with a k-coloring on the edges, i.e., edge set E is partitioned into (disjoint) color classes $E_1 \cup E_2 \cup \ldots \cup E_k$. The question of whether or not there exists a *colorful* spanning tree (i.e. a spanning tree with edges of different colors) can be restated as a matroid intersection problem on $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ with

$$\begin{aligned} \mathcal{I}_1 &= \{F \subseteq E : F \text{ is acyclic}\} \\ \mathcal{I}_2 &= \{F \subseteq E : |F \cap E_i| \leq 1 \ \forall i\} \end{aligned}$$

Since $\mathcal{I}_1 \cap \mathcal{I}_2$ is the set of colorful forests, there is a colorful spanning tree of G if and only if

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = |V| - 1.$$

By Theorem 3, this is equivalent to the condition

$$\min_{U \subseteq E} (r_1(U) + r_2(E \setminus U)) = |V| - 1.$$

Since $r_1(U) = |V| - c(U)$ (where $c(U)$ denotes the number of connected components of (V, U)), it follows that there is a colorful spanning tree of G if and only if the number of colors in $E \setminus U$ is at least $c(U) - 1$ for any subset $U \subseteq E$. In other words, a colorful spanning tree exists if and only if removing the edges of any t colors leaves a graph with at most $t + 1$ components.

4. Given a digraph $G = (V, A)$, a branching D is a subset of arcs such that

   (a) D has no directed cycles

   (b) For every vertex $v, \deg_{\text{in}}(v) \leq 1$ in D.

   Branchings are the common independent sets of matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2)$, where

$$\begin{aligned} \mathcal{I}_1 &= \{F \subseteq E : F \text{ is acyclic in the underlying undirected graph } G\} \\ \mathcal{I}_2 &= \{F \subseteq E : \deg_{\text{in}}(v) \leq 1 \ \forall v \in V\}. \end{aligned}$$

   Note that $\mathcal{M}_1$ is a graphic matroid on G and $\mathcal{M}_2$ is a partition matroid. Therefore, the problem of finding a maximum branching of a digraph can be solved by a matroid intersection algorithm.

To prove the matroid intersection theorem, we need some exchange properties of bases. Let $\mathcal{B}$ be the set of bases of matroid M. Then for bases $B, B' \in \mathcal{B}$,

1. $\forall x \in B' \setminus B$, there exists $y \in B \setminus B'$ such that $B + x - y \in \mathcal{B}$.
2. $\forall x \in B' \setminus B$, there exists $y \in B \setminus B'$ such that $B' - x + y \in \mathcal{B}$.

Note that the two statements are not saying the same thing. The first statement says that adding x to B creates a unique circuit and this circuit is distroyed by removing a single element; the second statement says that if we remove an element x of B', we can find an element of B to add to $B' - e$ and keep a basis (this is one of the matroid axioms). The following lemma is a stronger basis exchange statement.

Lemma 5 *For all $x \in B' \setminus B$, there exists $y \in B \setminus B'$ such that $B + x - y, B' - x + y \in \mathcal{B}$.*

We will prove the matroid intersection theorem next time.