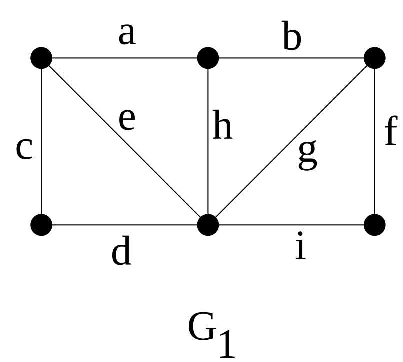
Massachusetts Institute of Technology
18.433: Combinatorial Optimization
Michel X. Goemans

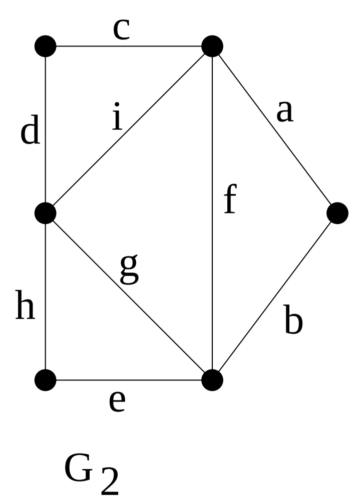
Handout 15
April 1st, 2009

Problem set 5

This problem set is due in class on Monday April 13th.

1. We are given the following two graphs G_1 and G_2 with edge set $E = \{a, b, c, d, e, f, g, h, i\}$.





Observe that $S = \{a, b, c, d\}$ is a forest in both G_1 and in G_2, so it is independent in $M_1 = M(G_1)$ and $M_2 = M(G_2)$. Construct the exchange graph corresponding to S, and indicate which elements are sources and sinks. Using the exchange graph, find a larger set of elements which is acyclic in both G_1 and in G_2.

2. Deduce König's theorem about the maximum size of a matching in a bipartite graph from the min-max relation for the maximum independent set common to two matroids.

3. Consider the spanning tree game. Show that player 2 has a winning strategy *if* the graph contains two edge-disjoint spanning trees.

   HINT. Let F_1 and F_2 be the edges selected by player 1 and 2, respectively, in the first k rounds (so player 2 has just played). Let $E' = E \setminus (F_1 \cup F_2)$ be the remaining edges. Try to maintain that there exists two disjoint subsets A and B in E' such that $F_2 \cup A$ and $F_2 \cup B$ are both spanning trees.

4. Consider a graph $G = (V, E)$. Let $E(M) = E$ and $\mathcal{I}(M) = \{F_1 \cup F_2 : F_1, F_2 \text{ are forests in } G\}$.

   (a) Show that M is a matroid by showing that property (I_2) is satisfied ((I_1) is trivially satisfied).

   (You shouldn't just refer to the matroid union Theorem in the lectures notes, but you can adapt its proof for the special case asked here.)

(b) Give an example which shows that the following algorithm does *not* find a maximum weight collection of edges which can be partitioned into 2 forests: Find a maximum weight forest F in G, delete the edges of F, and find again a maximum weight forest in the remaining graph.

5. Consider a graph $G = (V, E)$ with $|E| = 2(|V| - 1)$ and suppose the edges are partitioned into $|V| - 1$ blue edges (in B) and $|V| - 1$ red edges (in R). Suppose furthermore that G is the union of two edge-disjoint spanning trees (with no restriction on the colors of the edges). Show that G contains a tree with at most $\lceil \frac{|V|-1}{2} \rceil$ blue edges and with at most $\lceil \frac{|V|-1}{2} \rceil$ red edges.