# Solutions to Problem Set 1

1. (a) Consider the bipartite graph $G$ with a vertex for each square and two squares are adjacent if they share an edge. This graph is bipartite since the squares can be colored black and white in a checkerboard pattern.

   Any perfect tiling gives a perfect matching by simply selecting the edges corresponding to the dominoes selected. And vice versa.

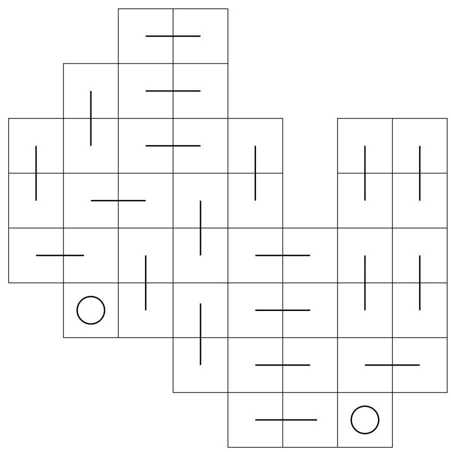

Figure 0.1: Maximum configuration of dominoes.

   (b) We claim that the configuration shown in Figure 0.1 is a maximum one and so no perfect tiling exists. We will prove that the matching $M$ corresponding to the configuration in Figure 0.1 is maximum by showing that there is no augmenting path as in the lecture. (Alternatively we could use Hall's theorem.)

   Let $A$ be the set of black squares and $B$ the set of white squares. Orient the edges of $G$ according to $M$, i.e. all the edges in $M$ are oriented from $B$ to $A$, and the edges not in $M$ are oriented from $A$ to $B$ as in Figure 0.3.

   Let $v$ be the only exposed vertex of $A$ and $w$ be the only exposed vertex of $B$, and consider $L$ to be the set of vertices reachable from $v$ (the enclosed area in Figure 0.3). Since $w$ is not in $L$ we obtain that no augmenting path exists.

   We can also deduce the fact that no perfect matching exists from Hall's theorem by observing that the 11 black vertices in $L$ (the enclosed region on the right of Figure 0.3) has only 10 (white) neighbors.

2. Let $\rho(G)$ be the size of a minimum edge cover and $\nu(G)$ the size of the maximum matching. A maximum matching covers $2\nu(G)$ vertices, and, because of the connectedness, the $n - 2\nu(G)$ remaining vertices can be covered by no more than $n - 2\nu(G)$ edges. This edges and the maximum matching are thus an edge cover of size $n - \nu(G)$.

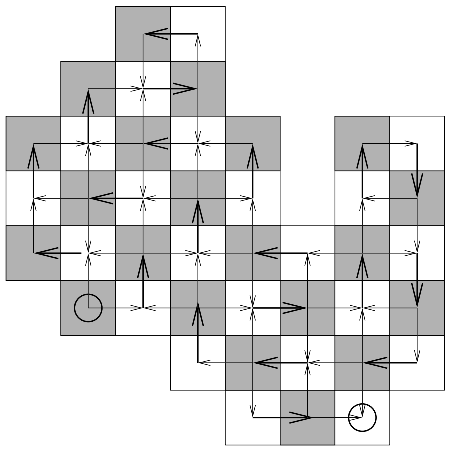

Figure 0.2: Oriented graph.

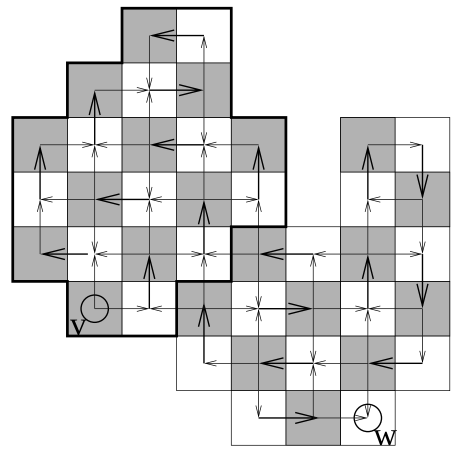


Figure 0.3: Set of reachable vertices from $v$.

On the other hand, a minimum edge cover has to be a forest (an acyclic graph). (Indeed, if it has any cycles then the removal of any edge of the cycle would still give an edge cover, of smaller cardinality.) The number of connected components of this forest is precisely $n - \rho(G)$ (because every component is a tree, and a tree on $k$ vertices has $k-1$ edges), and one can take one edge per component to get a matching. That is, $\nu(G) \geq n - \rho(G)$.

3. Let $A$, $B$ be the bipartition of $V$.

   (a) Because of $k$-regularity, we have $|A| = |B|$. Let $n = |A|$. By König's theorem, let $C$ be a minimum vertex cover of size equal to the maximum matching. Then, $N(A \setminus C) \subseteq B \cap C$, and because of $k$-regularity, $|A \setminus C| \leq |B \cap C|$. Similarly, $|B \setminus C| \leq |A \cap C|$. Adding the inequalities we get $|V \setminus C| \leq |C|$, which implies that $|C| \geq n$.

   (b) Any integer solution of the LP formulation

$$\text{Min} \quad \sum_{i,j} c_{ij} x_{ij}$$

subject to:

$$\begin{aligned} \sum_j x_{ij} &= 1 && i \in A \\ \sum_i x_{ij} &= 1 && j \in B \\ x_{ij} &\geq 0 && i \in A, j \in B \end{aligned}$$

is a perfect matching. Also, all the extreme points (if any) of the LP are integral (see lecture notes on bipartite matching). Thus, it is enough to prove that the LP is feasible (so it will have at least one extreme point), and $x_{ij} = 1/k$ is a feasible solution.

4. (a) Clearly the edge coloring number is at least $\delta$ since the $\Delta$ edges incident to a vertex of maximum degree have to be colored by different colors. To show the reverse inequality, first we will transform the graph $G$ into a $\Delta$-regular graph. For this purpose, first add vertices if needed so that both sides of the bipartition have the same number of vertices. Then add edges to the graph (in any way) so that every vertex has now degree $\Delta$. In the resulting graph $H$, we know by the previous exercise that there exists a perfect matching. Deleting this perfect matching, we still have a regular graph, now a $\Delta - 1$-regular graph. We can therefore again extract a perfect matching, delete it and proceed. In this process, we have partitioned $H$ into $\Delta$ perfect matchings,a nd thus the edges of $H$ can be colored with $\Delta$ colors. Since $G$ is a subgraph of $H$, restricting this coloring toi the edges of $G$ gives a valid coloring with (at most, and thus exactly) $\Delta$ colors.

   (b) Consider a cycle on 3 vertices.

5. (a) Let $Y \subset X \in \mathcal{I}$. Since $X$ is an independent set, there exists a matching $M_X$ that covers $X$. This matching also covers $Y$. Hence $Y$ is an independent set.

   (b) Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$. It follows that there exist matchings $M_X$ and $M_Y$ such that $M_X$ covers $X$ and $M_Y$ covers $Y$. Consider the graph $G' = (V, M_X \Delta M_Y)$. The set of edges of $G'$ is the union of paths and cycles.

   If $M_X$ covers some element $y$ in $Y \setminus X$. Then $X + y$ is an independent set.

   Otherwise, all the vertices in $Y \setminus X$ are of degree 1 in $G'$. Since $|Y| > |X|$, we have $|Y \setminus X| > |X \setminus Y|$. Therefore, by the previous observation, there are more degree 1 vertices in $Y \setminus X$ than in $X \setminus Y$. It follows that there exists a path $P$ in the decomposition of $G'$ starting in a vertex $y \in Y \setminus X$ and not ending in $X$. We conclude that $M_X \Delta P$ is a matching of $G$ that covers $X \cup \{y\}$. Thus, $X + y$ is an independent set.

6. (a) Clearly, the size of a maximum matching cannot be more than $|A| - \text{def}_{max}$ (since any matching can take have at most $|A| - |X|$ edges incident to $A - X$ and at most $|N(X)|$ edges incident to $X$).

   Conversely, consider the minimum vertex cover $C$ and let $X = A \setminus C$. Observe that $N(X) \subseteq C \cap B$, and thus:

   $$\text{def}(X) = |X| - |N(X)| \geq |A \setminus C| - |C \cap B| = |A| - |C \cap A| - |C \cap B| = |A| - |C|.$$

   Therefore $\text{def}_{max} \geq |A| - |C|$ and the result follows from König's theorem.

   (b) This is a simple counting argument. First of all,

   $$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

   Furthermore,

   $$|N(X \cup Y)| + |N(X \cap Y)| \leq |N(X)| + |N(Y)|,$$

since every vertex $b$ in $B$ contributes at least as much to the right-hand-side than to the left-hand-side. (Indeed, if $b \in N(X \cup Y) \setminus N(X \cap Y)$, it should be either in $N(X)$ or in $N(Y)$, while if $b \in N(X \cap Y)$, it should be in both $N(X)$ and in $N(Y)$.)

7. The greedy algorithm can provide solutions which are arbitrarily far away from the optimum. Reingold and Tarjan (SIAM J. on Computing, Vol. 10, 1981) show instances on a line for which the ratio between the greedy algorithm and the optimum cost matching is a factor more than $n^{0.58}$.