The lasting influence of Jean Bourgain's work in the Study of Dispersive Equations

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Initial Remarks

* Jean Bourgain is a lasting presence in many fields of math!
* Proof of this claim is the many topics presented during the conference in his honor two years ago!
* this talk will only address how deep Jean's influence is in the study of nonlinear dispersive equations: KdV, Schrödinger equations, ---
* this talk is the story of how Jean became interested on Dispersive PDE. (thanks to C. Kenig + L. Vega for details.)

The beginning

- Spring 1990: C. Kenig is invited by L. Caffarelli at the IAS to give a talk. Carlos presented a not yet completed work on well-posedness of generalized KdV equations.

"Definition": Given an initial value problem (IVP)

$$(\mathrm{IVP}) \quad \begin{cases} \partial_t u + P(D)u = F(u) \\ u\big|_{t=0} = u_0 \end{cases}$$

We say that (IVP) is well-posed in an interval of time $[0,T]$ if $\exists!$ solution u and the solution is stable.

the work presented by Carlos was in collaboration with G. Ponce and L. Vega, later published as

Well-Posedness and Scattering Results for the Generalized Korteweg–de Vries Equation via the Contraction Principle

CARLOS E. KENIG
University of Chicago

GUSTAVO PONCE
University of California at Santa Barbara

AND

LUIS VEGA
Universidad Autonoma de Madrid

Dedicated to Professors Tosio Kato and Elias M. Stein.

Communications on Pure and Applied Mathematics, Vol. XLVI, 527–620 (1993)

CCC 0010–3640/93/040527–94

Main Point of K-P-V work: Prove well-posedness at the level of conservation laws, such as mass or energy, by using very sophisticated harmonic analysis tools to estimate the linear part.

Example: Consider the KdV initial value problem

$$(\text{KdV}) \begin{cases} \partial_t u + \partial^3_{xxx} u + u\,\partial_x u = 0 & u : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R} \\ u|_{t=0} = u_0 & t, x \in \mathbb{R} \end{cases}$$

Conservation Laws:

$$I_2(u) = \int_{\mathbb{R}} u^2(x,t)\,dx = I_2(u_0)$$

$$I_3(u) = \int_{\mathbb{R}} \left((\partial_x u)^2 - c\,u^3\right) dx = I_3(u_0)$$

$$\vdots$$

The fixed point argument

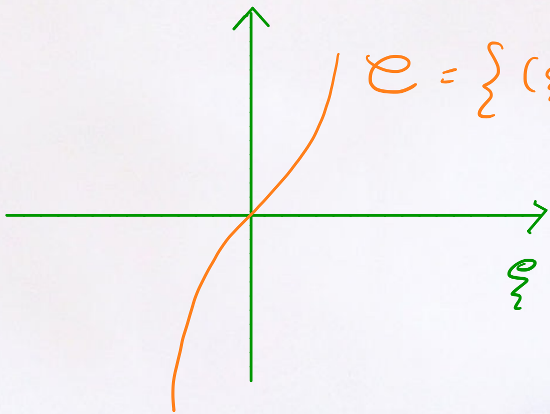
Define $\mathcal{U}(t)u_0(x) = \int_{\mathbb{R}} e^{i(x\cdot\xi + t\xi^3)}\hat{u}_0(\xi)\,d\xi \rightsquigarrow$ linear solution

$$(\mathrm{KdV}) \Longrightarrow \quad u = \underbrace{\mathcal{U}(t)u_0 + \int_0^t \mathcal{U}(t-t')\,u\,\partial_x u(t')\,dt'}_{=\,\Phi(u)}$$

<u>Goal:</u> If $u_0 \in H^s(\mathbb{R})$ find a space of functions $X_T^s \subseteq L_T^\infty H^s(\mathbb{R})$ s.t. $\Phi(u)$ has a fixed point in it.

Theorem [KPV] Let $s > \frac{3}{4}$, $u_0 \in H^s(\mathbb{R})$. Then $\exists\, T = T(\|u_0\|_{H^s})$ and a space $X_T^s \subseteq L_T^\infty H^s$ s.t. KdV has a unique solution $u \in X_T^s$.

Key Point of View: $\mathcal{V}(t)u_0 = R^* u_0$, $R =$ Restriction of FT on $\mathcal{C}$



$\mathcal{C} = \{(\xi, \xi^3) \,/\, \xi \in \mathbb{R}\}$

* During Carlos' talk at the IAS E. Speer was in the audience. With J. Lebowitz and H. Rose they had published a paper on the Gibbs measure associated to the 1D periodic quintic NLS:

Journal of Statistical Physics, Vol. 50, Nos. 3/4, 1988

Statistical Mechanics of the Nonlinear Schrödinger Equation

Joel L. Lebowitz,[1] Harvey A. Rose,[2] and Eugene R. Speer[3]

Received September 21, 1987

Facts:

- The Gibbs measure is defined for periodic NLS
- The Gibbs measure is supported on very rough spaces

Speer's Question: Is it possible to define a periodic NLS flow on the support of the Gibbs measure? Does this flow keep the measure invariant?

Carlos' Answer: "This is a very hard question, none of our estimates would work ... but I know who could make some progress on the problem: Jean Bourgain!

Explanation of the question of Speer

Consider the NLS equation on the torus $\mathbb{T}^d$ (periodicity)

Assume for now $\mathbb{T}^d$ is the square torus:

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u & p>1 \quad u : \mathbb{R}\times\mathbb{T}^d \longrightarrow \mathbb{C} \\ u|_{t=0} = u_0 \end{cases}$$

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 dx \pm \frac{1}{p+1}\int_{\mathbb{T}^d} |u|^{p+1} dx = H(u_0)$$

$$M(u) = \int_{\mathbb{T}^d} |u|^2 dx = M(u_0)$$

From NLS to an ∞ Hamiltonian system

If we set $\hat{u}(t,k) = a_k(t) + i\, b_k(t)$

$$(NLS) \iff \left\{ \begin{aligned} \dot{a}_k &= \frac{\partial H}{\partial b_k} \\ \dot{b}_k &= -\frac{\partial H}{\partial a_k} \end{aligned} \right. \qquad k \in \mathbb{Z}^d \quad \Bigg\} \text{ Infinite dimension Hamiltonian system.}$$

<u>Finite dim Gibbs measure</u>: If the Hamiltonian system were finite dim, say $|k| \le N$, then the Gibbs measure

$$d\mu = \frac{1}{Z} e^{-H(a_k, b_k)} \prod_{|k| \le N} da_k\, db_k \quad \text{is well-defined and invariant.}$$

The Gibbs measure of Lebowitz-Rose-Speer

Consider the defocusing, quintic NLS in $\mathbb{T}$. L-R-S were able to make sense of

$$\text{"} d\mu = \frac{1}{Z} e^{H(a_k, b_k)} \prod_{k\in\mathbb{Z}} da_k \, db_k \text{"}$$

by first introducing the Gaussian measure

$$d\rho = \frac{1}{\tilde{Z}} e^{-\sum_k \langle k\rangle^2 (|a_k|^2 + |b_k|^2)} \prod_k da_k \, db_k$$

with support on $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$, and then prove that $d\mu$ is absolutely continuous w.r.t. $d\rho$.

Question of Speer to Carlos

i) Is the defocusing, quintic, periodic 1D NLS flow $\Phi(t)$ defined in $H^{\sigma}(\mathbb{T})$, $\sigma < \frac{1}{2}$ for all times?

ii) Is $d\mu$ invariant w.r.t. $\Phi(t)$, i.e.

$$\forall A \subseteq H^{\sigma} \text{ is } \mu(A) = \mu(\Phi(t)(A)) \quad \forall t \in \mathbb{R} ?$$

Why was this question hard?

Define $S(t)u_0(x)$ to be the solution of $\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$

In $\mathbb{R}^d$

$$S(t)u_0(x) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\, \hat{u}_0(\xi)\, d\xi$$

oscillatory integral

(then good estimates)

In $\mathbb{T}^d$ (square)

$$S(t)u_0(x) = \sum_{k\in\mathbb{Z}^d} e^{i(x\cdot k + t|k|^2)}\, \hat{u}_0(k)$$

oscillatory series

(then no good estimates)

- Summer of 1990: Speer meets Jean probably in Paris and talks about the conversation with Carlos.

- Fall of 1990:
  - I start my Ph. D. at the University of Chicago
  - Jean visits Chicago and talks for hours with Carlos about KdV and NLS.

- May of 1991: Carlos and Jean meet at the 60th birthday Conference of E. Stein in Princeton. Jean wanted to make sure he knew all the references on the linear estimates for KdV and Schrödinger. Carlos' intuition was that he was making great progress.

June of 1991: Jean phones Carlos to ask few more questions.

Few weeks later Carlos receives the manuscript below where several periodic estimates were obtained.

I wrote some things down related to our last phone conversation.

Best regards,

Jean

FOURIER TRANSFORM RESTRICTION PHENOMENA FOR CERTAIN LATTICE SUBSETS AND APPLICATIONS TO NON-LINEAR EVOLUTION EQUATIONS

J. BOURGAIN(*)

1. INTRODUCTION.

The main purpose of this paper is to develop a harmonic analysis method for solving certain non-linear periodic (in space variable) evolution equations, such as the non-linear Schrödinger equation (NLSE)

$$\Delta_x u + i\partial_t u + u|u|^{p-2} = 0 \qquad (p \geq 3) \tag{1.1}$$

$$u = u(x,t) \text{ is } 1-\text{periodic in each coordinate of the } x-\text{variable} \tag{1.2}$$

with initial data

$$u(x,0) = \phi(x). \tag{1.3}$$

* Summer 1991: Carlos claims that he spent that whole summer studying Jean's manuscript, which later was published in two parts:

Geometric and Functional Analysis
Vol. 3, No. 2 (1993)

1016-443X/93/0200107-50$1.50+0.20/0
© 1993 Birkhäuser Verlag, Basel

FOURIER TRANSFORM RESTRICTION PHENOMENA FOR CERTAIN LATTICE SUBSETS AND APPLICATIONS TO NONLINEAR EVOLUTION EQUATIONS

Part I: Schrödinger Equations

J. Bourgain

$\mathbb{T}^d$ is a square torus!

Geometric and Functional Analysis
Vol. 3, No. 3 (1993)

1016-443X/93/0300209-54$1.50+0.20/0
© 1993 Birkhäuser Verlag, Basel

FOURIER TRANSFORM RESTRICTION PHENOMENA FOR CERTAIN LATTICE SUBSETS AND APPLICATIONS TO NONLINEAR EVOLUTION EQUATIONS

Part II: The KDV-Equation

J. Bourgain

The novel approach of Jean Bourgain

Fact: Every paper of Jean has layers after layers of amazing mathematics. In this case the overarching layer is the introduction of analytic number theory.

Example: To be able to perform a fixed point one needs a space and the space is defined via estimates of $S(t)u_0$. For the 1D periodic quintic NLS Jean used

$$\|S(t)u_0\|_{L^6_t L^6_x} \le C \|u_0\|_{H^s(\mathbb{T})} \quad \forall\, s>0.$$

For the 2D cubic periodic NLS he used

$$\|S(t)u_0\|_{L^4_{\mathbb{T}} L^4_{\mathbb{T}^2}} \leq C \|u_0\|_{H^s(\mathbb{T}^2)} \quad \forall s>0$$

periodic also in time.

$\mathbb{T}^2$ → square torus

Theorem [Bourgain] The quintic NLS on $\mathbb{T}$ and the cubic NLS on the square $\mathbb{T}^2$ are locally well-posed for data in H^s, $s>0$.

Remark: this theorem partially answers the question of Speer, namely there is a local flow in the support of the Gibbs measure for the 1D, periodic, quintic, defocusing NLS.

The analytic number theory in 1D and 2D

Let us consider the $\mathbb{T}^2$ (square) case: Assume $\operatorname{supp} \hat{u}_0 \subseteq B(0,N)$

$$\|S(t)u_0\|^2_{L^4(\mathbb{T}\times\mathbb{T}^2)} = \| S(t)u_0 \, S(t)u_0 \|^2_{L^2_{t,x}} = \| \widehat{S(t)u_0} * \widehat{S(t)u_0} \|^2_{\ell^2_{\lambda,k}}$$

and after using a Hölder inequality one has to estimate:

$$\# \{ k \in \mathbb{Z}^2 / \ k_1^2 + k_2^2 = N^2 \} \quad \text{for } N \in \mathbb{N}.$$

by Gauss lemma $\qquad \ll N^{\varepsilon}.$

this forces $u_0 \in H^s$, $s>0$!

* Jean in fact completely answered Speer's question in the positive in the following paper:

Commun. Math. Phys. 166, 1 – 26 (1994)

Communications in Mathematical Physics

Periodic Nonlinear Schrödinger Equation and Invariant Measures

J. Bourgain
I.H.E.S., 35, route de Chartres, F-911440 Bures-sur-Yvette, France

Received: 27 October 1993 / in revised form: 21 June 1994

answer to Speer's question.

He used the "deterministic" l.w.p in H^s; $s > 0$ and used the invariance of the measure by the flow to move from local to global well-posedness almost surely!

A much more challenging problem: the cubic NLS in $\mathbb{T}^2$:

Consider the equation $i\partial_t u + \Delta u = |u|^2 u \quad x \in \mathbb{T}^2$

with Hamiltonian $H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx$. One may wonder if

$$\text{“}\; d\mu = \frac{1}{Z} e^{-H(a_k, b_k)} \prod_{k \in \mathbb{Z}^2} da_k\, db_k \;\text{”}$$

is well defined. Glimm and Jaffe proved that this is the case if one replaces $H(u)$ with the Wick ordered $H_w(u)$, and the equation above with: $i\partial_t u + \Delta u = (|u|^2 u)_w$

Main Issue: Supp $\mu \subseteq H^s(\mathbb{T}^2)$, $s<0$. Using the Strichartz estimate $\|S(t)u_0\|_{L^4(\mathbb{T}\times\mathbb{T}^2)} \leq C\|u_0\|_{H^s}$ $s>0$ one only has a flow in H^s, $s>0$.

$\mathbb{T}^2$ is a square torus

Commun. Math. Phys. 176, 421 – 445 (1996)

Communications in Mathematical Physics

Invariant Measures for the 2*D*-Defocusing Nonlinear Schrödinger Equation

Jean Bourgain
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

Extremely influential paper!

Result: Almost sure definition of the global flow in the support of the measure and invariance.

"proof": Let $u_0^\omega = \sum_{k \in \mathbb{Z}^2} \frac{g_k(\omega)}{\langle k \rangle} e^{ik\cdot x}$ (initial data in supp μ)

i) Project the IVP onto $|k| \leq N$

ii) Instead of solving for u solve for $w = u - S(t)u_0^\omega$

iii) Prove that w is almost surely locally well defined in H^s, $s>0$, uniformly w.r.t. N.

iv) Use the invariance of the Gibbs measure to move from local to global!

Remarks:

i) The analysis here is much more complicated than the 1D quintic case. More probabilistic tools used.

ii) Jeon's work inspired few years later:

Burq - Tzvetkov (cubic NLW on 3D compact M)

T. Oh (Coupled KdV Systems)

iii) Ever more interaction between the Dispersive PDE Community and the SPDE community:

Looking for "generic" results!

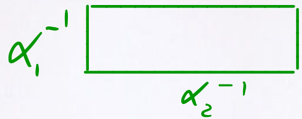
From square tori to any tori

The estimate $\|S(t)u_0\|_{L^4(\mathbb{T}\times\mathbb{T}^2)} \leq C\,\|u_0\|_{H^s(\mathbb{T}^2)}$ $\quad s>0$

$\hookrightarrow$ square torus

Was obtained by counting:

$$\#\{k\in\mathbb{Z}^2 / k_1^2+k_2^2=N^2\} \qquad N\in\mathbb{N}.$$

If $\mathbb{T}^2$ is not a square: 

and $\alpha_1/\alpha_2 \in \mathbb{R}\setminus\mathbb{Q}$

- $S(t)u_0$ is no longer periodic in time
- $\#\{k\in\mathbb{Z}^2 / \alpha_1 k_1^2+\alpha_2 k_2^2=N^2\}$ is not well estimated.

The ℓ^2-decoupling theorem

Annals of Mathematics 182 (2015), 351–389
http://dx.doi.org/10.4007/annals.2015.182.1.9

The proof of the l^2 Decoupling Conjecture

By Jean Bourgain and Ciprian Demeter

In this paper Jean and Ciprian proved the famous ℓ^2-decoupling conjecture ... and as a consequence All Strichartz Estimates, up to an ε, in any torus $\mathbb{T}^d$, for any d.

the proof does not use Analytic Number Theory.
On the contrary, arguments introduced in the proof set the stage for the proof of a main conjecture in ANT:

Annals of Mathematics 184 (2016), 633–682
http://dx.doi.org/10.4007/annals.2016.184.2.7

Proof of the main conjecture in Vinogradov's Mean Value Theorem for degrees higher than three

By Jean Bourgain, Ciprian Demeter, and Larry Guth

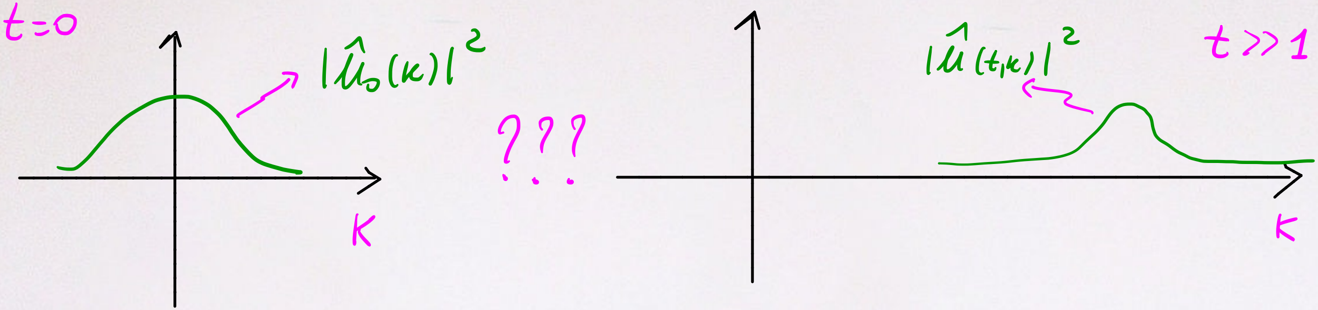
Jean's Influence in Weak Turbulence

Thanks to the availability of Strichartz estimates in all $\mathbb{T}^2$ and the conservation of mass and energy Jean proved that the flow associated to

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}^2), \; s \geq 1 \end{cases}$$

is globally well-defined.

What happens to $u(t)$ as $t \to \pm\infty$?



(Energy Transfer, Forward Cascade ---)

One way of checking if this phenomenon happens is to look at:

$s > 1$

$$\lim_{t\to\pm\infty} \sum_{k} |\hat{u}(t,k)|^2 \langle k \rangle^{2s} = \lim_{t\to\pm\infty} \|u(t)\|_{H^s}^2$$

Some Results

Theorem (Bourgain) If u is a smooth solution of the cubic, defocusing NLS in $\mathbb{T}^2$, then $\forall\, s>1$

$$\|u(t)\|_{H^s} \leq C\,|t|^{2(s-1)+\varepsilon}, \quad \varepsilon>0$$

Theorem (Colliander-Keel-S-Takaoka-Tao)
If $\mathbb{T}^2$ is rational, $s>1$, $0<\sigma \ll 1$ and $K \gg 1$, there exists a solution u to the cubic, defocusing NLS and $T \gg 1$ s.t.

$$\|u(0)\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T)\|_{H^s} \geq K.$$

Thank you Jean for planting so many seeds in mathematics so that future generations of mathematicians can keep attending to the trees that they generated, and keep cross pollinating them!

Gigliola