On dispersive equations and their importance in mathematics

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What is a dispersive equation

The simplest possible evolution partial differential equation that is not either hyperbolic or parabolic is the Airy Equation, which initial value problem (IVP) can be written as

$$\begin{cases} (-i\partial_t + \partial_{xxx})u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

where $x \in \mathbb{R}$. The wave solution of this IVP is the simplest example of a solution to a dispersive equation. We will compute this solution explicitly and we will see that it satisfies the following, rather informal, definition:

Definition

An evolution partial differential equation is dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time.

To find a solution for the Airy Cauchy problem let's look for plane-wave solutions: for fixed A and k we write

$$v_k(x,t) = Ae^{(kx-\omega t)} = Ae^{k(x-\omega/kt)}$$

where k =wave number and ω =angular frequency. If we substitute v_k into our equation we obtain the relationship

$$Ae^{k(x-\omega/kt)}[-i\omega + (ik)^3] = 0$$

and from here

$$\omega = \frac{(ik)^3}{i} \quad \Longleftrightarrow \quad \frac{\omega}{k} = -k^2.$$

The equation

$$\frac{\omega}{k} = -k^2$$

is called the $\boxed{\text{Dispersive Relation}}$ for the Airy equation.

Remark

The dispersive relations says that "plane waves with large wave number travel faster than those with a smaller one". This is the reason why there is "spreading". In mathematical terms this phenomenon is called broadening of the wave packet.

To understand this better let's use the Fourier transform: for the initial data we have

$$u_0(x) = \int \hat{u}_0(k) e^{ikx} dk$$

Now think of $\int$ as a sum and, for fixed k, of $v_k(x) = \hat{u}_0(k)e^{ikx}$ as a wave. Then each wave $v_k(x)$ evolves into

$$v_k(x,t) = \hat{u}_0(k) e^{ik(x+tk^2)},$$

where the wave with larger k travels faster.

By “adding up” all these waves we obtain the solution to the Airy IVP

$$u(x,t) = \int \hat{u}_0(k) e^{ik(x+k^2 t)} dk.$$

(We will see that in the periodic case this will be different since there boundary conditions are imposed!) It is instructive to contrast the Airy equation with the transport equation

$$\begin{cases} (\partial_t + C\partial_x)u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

where $x \in \mathbb{R}$, and let's pick $C > 0$. The dispersive relation is $\frac{\omega}{k} = C$, that is the velocity is constant, so the wave packet travels with the same speed and there is no dispersion.

Definition (More formal)

We say that an evolution equation (defined on $\mathbb{R}^n$), is dispersive if its dispersive relation $\omega(k)/|k| = g(k)$ is a real function such that

$$|g(k)| \to \infty \text{ as } \ |k| \to \infty.$$

There is also a more geometric/analytic definition of dispersion. Back to the solution u of the Airy IVP, we recall that

$$u(x,t) =: W(t)u_0(x) = \int \hat{u}_0(k) e^{i(xk+k^3 t)} dk.$$

If we define the curve

$$S = \{(k,\tau)/\tau = k^3, k \in \mathbb{R}\},$$

then one can also write

$$u(x,t) =: W(t)u_0(x) = R^* u_0(x,t),$$

where R^* is the adjoint of R, the Fourier restriction operator on S:

$$R(f) := \hat{f}(k, k^2).$$

Remark

Since in general $\hat{f}$ belongs only to $L^2(\mathbb{R} \times \mathbb{R})$ and S is of Lebesgue measure zero, it is not obvious that $\hat{f}$ restricted to S even makes sense!

Theorem (Informal)

If S is a "curved" graph then for any $\hat{f} \in L^2(\mathbb{R} \times \mathbb{R})$ its restriction on S is well defined and moreover "good" estimates can be proved.

See for example *Harmonic Analysis* by E. Stein.

Examples of Dispersive Equations

- The (generalized) KdV equation:

$$\partial_t u + \partial_{xxx} u + \gamma u^k \partial_x u = 0,$$

- Nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u + N(u, Du) = 0,$$

- Boussinesq equation:

$$\partial_{tt} u - \partial_{xx} u - \partial_{xx}(1/2u^2 + \partial_{xx} u) = 0,$$

These equations were all introduced in order to describe a certain wave phenomena. As a consequence the first obvious questions that one would like to address are: existence, uniqueness and stability of solutions (local well-posedness), maximum time of existence, blow up, scattering, existence of solitons etc.
It turned out that while investigating these questions ones steps out of the field of harmonic or Fourier analysis and enters others fields like symplectic geometry, analytic number theory, probability and dynamical systems.
To illustrate these interactions I will first look at KdV type equations and then at Schrödinger ones.

The KdV equation

The (generalized) KdV initial value problem takes the form of

$$\begin{cases} \partial_t u + \partial_{xxx} u + u^k \partial_x u = 0 \\ u(0, x) = u_0(x), \end{cases}$$

This problem models long waves along a shallow channel. Linked to this problem is the discovery of solitons: In 1834 a naval architect *John Scott Russel*, while riding his horse along a canal observed the first recorded soliton. The "event" was repeated in a "controlled" manner in 1995:

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Definition

A soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at a constant speed.

Remark

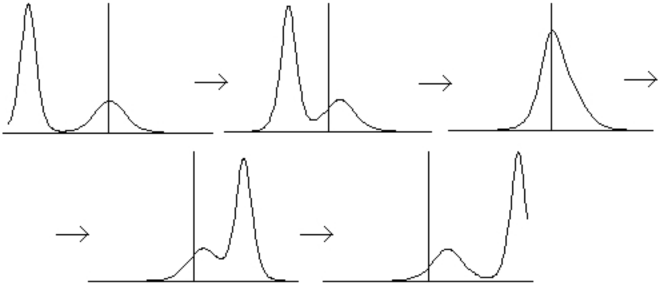
Clearly a soliton cannot be solution to the Airy equation (remember dispersion!!). In fact solitons are caused by a perfect cancellation of nonlinear and dispersive effects in the medium. An interesting feature of solitons is that in spite of the fact that they are nonlinear phenomena, they behave "linearly" when they interact!

We actually have an explicit form of solitons to a (generalized) KdV equation:

$$u_{c,k}(x,t) = \Phi_{c,k}(x - ct), \quad \text{for} \quad c > 0$$

where $\Phi_{c,k} = c(k+2)/2\,\mathrm{sech}^2(k/2\sqrt{c}x)^{1/k}$.

Interaction of two solitons



KdV as an integrable system

Consider the equation (k=1)

$$\partial_t u + \partial_{xxx} u + u \partial_x u = 0.$$

Fermi, Pasta and Ulam used this equation to justify the apparent paradox in chaos theory that many “complicated enough” physical systems exhibit almost exactly periodic behavior instead of ergodic behavior! There are many (equivalent) ways of formulating this fact:

- The system is integrable
- The equation admits Lax pairs
- Inverse scattering completely solves the IVP.
- The system admits infinitely many conservation laws

Some conservation laws

The first 3 for the KdV equation are:

$$\int u(x,t)\,dx$$

$$\int u^2(x,t)\,dx \quad \text{mass}$$

$$\int u_x^2(x,t) - \frac{u^3}{3}(x,t)\,ds \quad \text{Hamiltonian}$$

The first complete algorithm to compute all the conservation laws is due to Miura, Gardner and Kruskal. In principle at this point we don't even know if the integrals above actually are finite, we don't know yet if the solution $u(x,t)$ exist!

Well-posedness for (generalized) KdV

We consider the IVP

$$\begin{cases} \partial_t u + \partial_{xxx} u + u^k \partial_x u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

We introduce the Sobolev space H^s by recalling that the norm of a function f in this space is

$$\left(\int |\hat{f}(k)|^2 (1+|k|)^{2s}\, dk \right)^{1/2} < \infty$$

Definition

The IVP is locally well-posed in H^s if for any $u_0 \in H^s$ there exists $T = T(u_0)$ and a unique solution u in a Banach space $X_T^s \subset C([0,T], H^s)$. Moreover there is continuity with respect to the initial data. If T can be taken arbitrerely large we say that we have global well-posed.

The "classical" method to prove well-posedness for dispersive equations is by a priori estimates. This is the Energy Method. These a priori bounds are found by using the equation and integration by parts. Using this methods the KdV equation can be proved to be locally well posed in H^s, $s > 3/2$.

The second method, developed by Kenig, Ponce and Vega is based on Oscillatory Integrals. To understand this method we first observe that by the Duhamel Principle one can rewrite the IVP as an "equivalent" integral equation:

$$u(x,t) = W(t)u_0(x) + c\int_0^t W(t-t')(u\partial_x u)(x,t')\,dt',$$

where, as we know, $W(t)u_0(x) = R^*(u_0)(x,t)$ is the solution of the linear IVP.

This methods is based on the following steps:

- The proof of several estimates for $W(t)u_0(x)$ using the Fourier restriction operator R and other estimates based on oscillatory integrals.
- The definition of a Banach space X_T^s of space time functions where the norms of the above estimates are considered.
- The use of the Banach space $X_T^s \subset C([0,T], H^s)$ as a space where to look for the fixed point of the operator

$$Lv(x,t) = W(t)u_0(x) + c\int_0^t W(t-t')(v\partial_x v)(x,t')\,dt'.$$

- The use of the integral equation above to claim that the fixed point is the unique solution of the equation.

This method allowed Kenig, Ponce and Vega to improve local well-posedness for the KdV IVP to $H^s,\ s > 3/4$, and global well-posedness in H^1. Similar results for the generalized $(k > 1)$ KdV equations.

Bourgain's Contribution

Bourgain continues with the fixed point idea, but introduces in this context another type of space: $X^{s,b}$. The norm of a function $f \in X^{s,b}$ is defined by

$$\|f\|_{X^{s,b}} = \left(\iint |\hat{f}(k,\tau)|^2 < k >^{2s} < \tau - k^3 >^{2b} \, dk \, d\tau \right)^{1/2}.$$

Again we see reappearing the cubic

$$S = \{(k,\tau)/\tau = k^3, k \in \mathbb{R}\}.$$

With this space Bourgain was able to now attack also the periodic KdV equation. Here oscillatory integrals cannot be used since Forier transform gives oscillatory series, not integrals! These spaces were also later used by Kenig, Ponce and Vega to prove local well posedness for negative Sobolev spaces $H^{-\rho}$. More precisely on the line $\rho < 3/4$ and on the circle $\rho \leq 1/2$.

The symplectic KdV flow

Why do we care about negative Sobolev spaces? One very good reason is described below. If we consider the periodic case and we take Forier transform in space of the solution u

$$u(x,t) \quad \Longleftrightarrow \quad (\hat{u}(k,t))_{k\in\mathbb{Z}},$$

we can view the IVP as an Hamiltonian system of infinite dimension for the infinite vector $(\hat{u}(k))_{k\in\mathbb{Z}}$. For this system we can define the symplectic form

$$(u,v) = \int u\partial_x^{-1} v\, dx$$

on the Sobolev space $H^{-1/2}$. It is then reasonable to ask if certain theorems (see Gromov) that are proved in the finite dimension setting are still true here.

Theorem

(Colliander, Keel, S, Takaoka and Tao) The symplectic KdV flow is global in time on $H^{-1/2}$ and the Gromov non-squeezing theorem holds.

This theorem has two major parts. The first deals with extending local well-posedness in $H^{-1/2}$ to global well-posedness by the use of the *I-Method*. The second part deals with proving the non-squeezing theorem by approximating the system with a finite dimension one in an appropriate way and then by taking the limit. Similar results where obtained earlier by Kuksin for compact perturbation of certain linear systems and by Bourgain for a certain Schrödinger equation.

The Schrödinger equation

The Schrödinger equation describes for example how quantum states of a physical system change in time. One example is the IVP

$$\text{NLS} \begin{cases} i\partial_t u + \Delta u + \sigma |u|^{p-1} u = 0 \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^n \end{cases}$$

with $p > 1$ and $\sigma = \pm 1$.
The solution of its linear IVP is

$$S(t)u_0(x) = \int e^{i(x\cdot k + t|k|^2)} \hat{u_0}(k)\, dk = R^*(u_0),$$

where now $R(f) = \hat{f}(k, |k|^2)$, the operator that restrict the Fourier transform on the surface given by

$$P = \{(k,\tau)/\tau = |k|^2,\ k \in \mathbb{R}^n\}.$$

The Strichartz estimates in $\mathbb{R}^n$

In order to prove (local) well-posedness for Schrödinger equations the Strichartz Estimates are fundamental.
We call admissible couple any pare of exponents (q, r) such that

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right) \quad q > 2, \ r < \infty.$$

Then for any admissible couple (q, r)

$$\|S(t)u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_x^2},$$

and for any other admissible couple $(\tilde{q}, \tilde{r})$

$$\left\| \int_0^t S(t-t')F(t')\,dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Thanks to these estimates one can then prove well-posedness results for Schrödinger type IVP via fixed point theorems.

How difficult is it to prove well-posedness? It is certainly easier if

- the interval of time $[0, T]$ is short,
- the initial data are small,
- the nonlinearity is weak.

In fact, from the Duhamel principle u is solution to the NLS equation above if and only if

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')\sigma|u|^{p-1}u(t')\,dt'$$

and if one could claim that the non-linear perm is a “small” perturbation, then a fixed point theorem in the space of the Strichartz norms will provide well-posedness, at least for short times. The question of long time well-posedness or blow up is far more complex.

Scaling

There are many important player in the game of well-posedness for NLS on $\mathbb{R}^n$: scaling invariance and conservation laws, monotonicity formula (i.e. Morawetz type estimates), Viriel identities and other kind of symmetries. Here we only consider the first one as an example: if u solves NLS above then

$$u_\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$$

solves the same equation with initial datum $u_{0,\lambda} = u_0(\frac{x}{\lambda})$. We have that

$$\|u_{0,\lambda}\|_{\dot{H}^s} \sim \lambda^{s_c - s} \|u_0\|_{\dot{H}^s}$$

where $s_c = n/2 - 2/(p-1)$ is the critical exponent. We can now "classify" the difficulty of the NLS problem above in terms of s_c.

So for $s_c = n/2 - 2/(p-1)$ and since $\|u_{0,\lambda}\|_{\dot{H}^s} \sim \lambda^{s_c - s}$, we have

- If $s < s_c$ the space H^s is supercritical
  (as $\lambda \to \infty$ the norm of $\|u_{0,\lambda}\|_{\dot{H}^s}$ grows)
- If $s = s_c$ the space H^s is critical
  (as $\lambda \to \infty$ the norm of $\|u_{0,\lambda}\|_{\dot{H}^s}$ does not change)
- If $s > s_c$ the space H^s is subcritical
  (as $\lambda \to \infty$ the norm of $\|u_{0,\lambda}\|_{\dot{H}^s}$ gets smaller).

Local well-posedness is by now very well understood, (see also Cazenave, Weissler, Kato, Tsutsumi, Ginibre,Velo etc). In recent years a lot of progress as been made to prove global well-posedness at the level of the energy (H^1) or mass (L^2) norms, even when they are "critical spaces" in the sense above.

ENERGY CRITICAL NLS IN $\mathbb{R}^3$

These are two types of theorems now available:

THEOREM (DEFOCUSING CASE $\sigma = -1$)

Assume that the energy of the quintic defocusing NLS

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 \, dx - \frac{2\sigma}{6} \int_{\mathbb{R}^n} |u|^6 \, dx$$

is finite. Then the IVP is globally well-posed and at infinity (in time) the solution approximate a linear one (scattering).

For the proof see Bourgain and Grillakis in the radial case, Colliander-Keel-S-Takaoka-Tao for the general case. Also see Ryckman-Visan for $n = 4$ and Visan for $n > 4$.

Theorem (Focusing case $\sigma = 1$)

Assume that u is the solution of the quintic focusing NLS and

$$\sup_t \|\nabla u_0(t)\|_{L^2_x} < \|\nabla W\|_{L^2_x},$$

where W is the stationary solution. Then if we also assume radial symmetry the same conclusion as above holds.

For the proof see Kenig-Merle, See also Killip-Visan for $n \geq 5$, where the radial assumption has been removed.
In a certain sense the complement of this last theorem is a collection of recent and strong results of blow up rate and blow up profile due to Merle-Raphael.

Schrödinger equations on manifolds

Given a manifold (M, g) equipped with its Laplace-Beltrami operator Δ_g one can certainly define a Schrödinger equation on it. The interesting questions here are related to the understanding of the influence of the geometry on the behavior of the solutions...when they exist. I will list some results by concentrating on Strichartz estimates:

- On the sphere $\mathbb{S}^n$: There is a loss of $\frac{1}{n}$ derivative.
  Burq-Gerard-Tzvetkov
- On the hyperbolic space $\mathbb{H}^n$: There is a larger family of Strichartz estimates.
  Banica, Banica-Carles-S, Ionescu-S, Banica-Carles-Duyckaerts, Ancher-Pierfelice
- On the torus $\mathbb{T}^n$: Limited Strichartz estimates.
  Bourgain

Here $\mathbb{T}^n$ is the Torus for which the symbol of Δ is $\sum_{i=1}^n k_i^2$!

The cubic defocusing Schrödinger equation in $\mathbb{T}^2$

Well-posedness in H^s, $s > 0$ follows from the Strichartz estimate due to Bourgain:

$$\|S(t)u_0\|_{L^4_{\mathbb{T}} L^4_T} \leq \|u_0\|_{H^\epsilon_x}$$

for any $\epsilon > 0$. The interesting part of the proof of this estimate is that it is based on counting the lattice points on a "thin" sphere

$$k_1^2 + k_2^2 = R^2 + \theta, \;\; |\theta| < 1, R >> 1.$$

Here Gauss lemma is used to claim that this number is smaller than R^ϵ for any $\epsilon > 0$.

What about irrational tori? If one wants to use the same method of Bourgain then one needs to have a good estimate for the number of lattice points on a "thin" ellipsoid. This is basically an open question except for a partial result of Bourgain.

Notion of Weak Turbulence

Definition

Weak turbulence is the phenomenon of global-in-time solutions shifting their mass toward increasingly high frequencies.

This shift is also called forward cascade.

- One way of measuring weak turbulence is to consider the function in time

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,k)|^2 |k|^{2s} d\xi$$

  for $s \gg 1$ and prove that it grows for large times t.
- Weak turbulence is incompatible with scattering or complete integrability.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^n$ DO NOT exhibit weak turbulence. There are solutions to dispersive equations on $\mathbb{T}^n$ that exhibit weak turbulence. In particular for $NLS(\mathbb{T}^2)$ there exists $u(x,t)$ s. t. $\|u(t)\|_{\dot{H}^s}^2 \to \infty$ as $t \to \infty$.

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the defocusing IVP

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \ \textit{where} \ x \in \mathbb{T}^2, \end{cases} \qquad (NLS(\mathbb{T}^2))$$

and $T > 0$ such that $\|u_0\|_{H^s} \le \sigma$ and $\|u(T)\|_{\dot{H}^s}^2 \ge K$.

The toy model

The idea of the proof is to make the ansatz

$$v(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{i(\langle n,x\rangle + |n|^2 t)},$$

and rewrite the equation as an ODE in terms of the infinite vector $(a_n(t))$. We also consider only the resonant part of the ODE and we construct a special finite set of frequencies Λ that is closed under resonance and has several other "good" properties. Thanks to these properties we arrive to a finite dimension toy model

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)},$$

for $j = 0, ..., M+1$, with the boundary condition

$$b_0(t) = b_{M+1}(t) = 0.$$

Remark

This new IVP conserves the momentum, the mass ($\sum_{j=1}^{M} |b_j(t)|^2 = 1$) *and the energy!*

Global well-posedness for this system is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^M \ / \ |x|^2 = 1\} \text{ and } \tilde{W}(t) : \Sigma \to \Sigma,$$

where $\tilde{W}(t)b(0) = b(t)$ for any solution $b(t)$ of our system. It is easy to see that if we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_M) \in \Sigma \,/\, |b_j| = 1,\ b_k = 0,\ k \neq j\}$$

then

$$\tilde{W}(t)\mathbb{T}_j = \mathbb{T}_j \text{ for all } j = 1, ..., M$$

($\mathbb{T}_j$ is invariant).

At this point the problem has been set up in such a way that if we could show that once we start "near" one of the first tori (low frequencies) we end up at a certain time T near one of the last tori (high frequencies) then we are done. In fact we have the following result:

Theorem

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time T such that

$$W(T)x_3 = x_{M-2}.$$

Remark

Our theorem does not show that one can find a solution u which H^s norm grows in time. We cannot even prove that it grown as a $log|t|$!

I would like to conclude by listing other topics of great interest and intense mathematical activity:

- Wave maps
  Nahmod-Stefanov-Uhlenbeck, Klainerman-Rodnianski, Krieger-Schlag, Shatah-Struwe, Sterbenz-Tataru, Tao.
- Schrödinger maps
  Bejenaru-Kenig-Ionescu, Chang-Shatah-Uhlenbeck, Ding-Wang, Kenig-Lamm-Pollack-S-Toro, McGahagan, Nahmod-Stefanov-Uhlenbeck, Rodnianski-Rubenstain-S, Terng-Uhlenbeck.
- "Almost surely" well-posedness
  Oh, Bourgain, Burq-Tzvetkov, Oh-Rey Bellet-Nahmod-S
- Regularity theorems for supercritical dispersive equations (similar to Navier-Stokes)