Balanced Shifted Tableaux

Yibo Gao
Joint work with: Jiyang Gao and Shiliang Gao

Massachusetts Institute of Technology

UCB Combinatorics Seminar, Spring 2022
Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0)$ be a partition.

**Definition**

A **standard Young tableau** of shape $\lambda$ is a filling of $\lambda$ using $1, \ldots, |\lambda|$ such that each row and each column form increasing sequences.

For each box $(i, j)$ in its Young diagram, let its **hook** $H_\lambda(i, j)$ consist of all the boxes directly to the right or the bottom of $(i, j)$, including itself.

**Theorem (Hook length formula)**

The number of standard Young tableaux of shape $\lambda$ equals

$$f^\lambda = \frac{|\lambda|!}{\prod_{(i,j) \in \lambda} |H_\lambda(i,j)|}.$$
Balanced tableaux

For a box $(i, j) \in \lambda$, let $r_{k\lambda}(i, j)$ be the size of the right arm of $H_{\lambda}(i, j)$, i.e. the number of boxes to the right of $(i, j)$, including itself.

Definition (Edelman-Greene 1987)

A **balanced tableau** of shape $\lambda$ is a filling $T$ of $\lambda$ using $1, \ldots, |\lambda|$ such that $T(i, j)$ is the $r_{k\lambda}(i, j)$-th largest entry in its hook.

\[
\begin{array}{cccc}
3 & 7 & 4 & 2 \\
5 & 8 & 6 \\
1 & 9 \\
\end{array}
\]

Theorem (Edelman-Greene 1987)

For a partition $\lambda$, the number of balanced tableau of shape $\lambda$ equals the number of standard Young tableaux of shape $\lambda$. 
Fix $\lambda$. Let $rk_\lambda : \lambda \rightarrow \mathbb{Z}_{>0}$ be a function on the boxes in $\lambda$.

Define a **balanced tableau with respect to** $rk_\lambda$ to be a filling $T$ of $\lambda$ using $1, \ldots, |\lambda|$ such that $T(i,j)$ is the $rk_\lambda(i,j)$-th largest entry in its hook.

- $rk_\lambda = \text{size of the right arm}$ gives balanced tableaux
- $rk_\lambda = 1$ gives standard Young tableaux

**Question**

*Given $\lambda$, what are all the possible functions $rk_\lambda$ such that the number of balanced tableaux with respect to $rk_\lambda$ can be enumerated by the hook length formula?*
The Edelman-Greene insertion algorithm provides a bijection between reduced words of $w \in \mathfrak{S}_n$ and pairs of tableaux $(P, Q)$ such that

- the insertion tableau $P$ is increasing in rows and columns, whose reverse reading word is a reduced word of $w$, and
- the recording tableau $Q$ is standard of the same shape.

**Theorem (Edelman-Greene 1987)**

*The number of reduced words of the longest permutation $w_0 \in \mathfrak{S}_n$ equals the number of standard Young tableaux of the staircase shape $(n - 1, n - 2, \ldots, 1)$.*

**Theorem (Edelman-Greene 1987)**

*The Stanley symmetric function $F_w$ is Schur-positive.*
Example: \( w = s_2 s_1 s_2 s_3 s_2 \)

\[
\begin{array}{c|ccccc}

P & \emptyset & 2 & 1 & 1 & 1 \\
& & 2 & 2 & 2 & 2 \\
Q & \emptyset & 1 & 1 & 1 & 1 \\
& & 2 & 2 & 2 & 2 \\
\end{array}
\]

And we see that the reading word of \( P \) equals

\[
s_2 s_3 s_1 s_2 s_3 = s_2 s_1 s_3 s_2 s_3 = s_2 s_1 s_2 s_3 s_2 = w.
\]
Let $\lambda = (\lambda_1 > \cdots > \lambda_d)$ be a strict partition, which corresponds to a shifted shape by shifting the $i$-th row $i$ steps to the right.

Example: the shifted shape $\lambda = (6, 2, 1)$

Definition
A standard Young tableau of shifted shape $\lambda$ is a filling of $\lambda$ using $1, \ldots, |\lambda|$ that is increasing in each row and column.

Let $\text{SYT}(\lambda)$ be the set of standard Young tableaux of $\lambda$. 
Hook length formula for shifted shapes

Let \( \lambda = (\lambda_1 > \cdots > \lambda_d) \) be a shifted shape. The hook \( H_\lambda(i, j) \) contains:
- boxes to the right and below, if \( j \geq 0 \);
- boxes to the right and below, and then turn again to the right with a “broken leg”, if \( j \leq 0 \).

Hooks for shifted shapes

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Theorem (Hook length formula for shifted shapes)

The number of standard Young tableaux of shifted shape \( \lambda \) equals

\[
| \text{SYT}(\lambda) | = \frac{|\lambda|!}{\prod_{(i, j) \in \lambda} |H_\lambda(i, j)|}.
\]
If we let $r_{k_{\lambda}}(i,j)$ be the size of the right arm of $H_{\lambda}(i,j)$ and define balanced shifted tableaux analogously, then they are not equinumerous to standard shifted tableaux.

We make a definition for balanced shifted tableaux as close to this idea as possible, and show that they are equinumerous to standard shifted tableaux.
Extended shifted shapes

**Definition (Gao, Gao and G. 2022)**

For a filling $B$ of $\lambda$, its **extended filling** $\tilde{B}$ is a filling of the extended shape

$$\tilde{\lambda} = \lambda \cup \{(1, \bar{d}), (2, \bar{d-1}), \ldots (d, \bar{1})\}$$

which agrees with $B$ on $\lambda$ and equals $B(i, 0)$ on the newly added boxes $(i, -(d+1-i))$ for $i = 1, \ldots, d$.

The **extended hook** $\tilde{H}_\lambda(i, j) \subset \tilde{\lambda}$ is the hook $H_\lambda(i, j)$ for $j \geq 0$, and is $H_\lambda(i, j) \cup \{(d+1+j, j)\}$ if $j < 0$.

**Example: extended filling and extended hooks**

Intuitively, column 0 correspond to roots $e_i$’s, while the extended boxes correspond to roots $2e_i$’s.
A balanced condition

Define the following rank function

- \( \text{rk}_\lambda(i, j) = \# \text{boxes in row } i \text{ of } H_\lambda(i, j), \ j \geq 0, \)
- \( \text{rk}_\lambda(i, \bar{j}) = \# \text{boxes with non-negative column index of } H_\lambda(i, \bar{j}), \ j > 0. \)

An alternative description of \( \text{rk}_\lambda(i, \bar{j}) \):
Let \( \text{rk}_\lambda(i,j) \) be defined as above.

**Definition (Gao, Gao and G. 2022)**

A **balanced shifted tableau** of shape \( \lambda \) is a filling \( B \) of \( \lambda \) using 1, \ldots, \( |\lambda| \) such that for all \( (i,j) \in \lambda \), \( B(i,j) \) is the \( \text{rk}_\lambda(i,j) \)-th largest entry in the extended hook \( \tilde{H}_\lambda(i,j) \).

Let \( \text{BS}(\lambda) \) be the set of balanced shifted tableaux of shape \( \lambda \).

**Example: a balanced shifted tableau**

\[
\begin{array}{cccccc}
4 & 6 & 3 & 4 & 2 & 5 & 9 \\
8 & 7 & 8 & & & \\
1 & 1 & & & &
\end{array}
\]

**Theorem (Gao, Gao and G. 2022)**

For a shifted shape \( \lambda \), \( |\text{SYT}(\lambda)| = |\text{BS}(\lambda)| \).
Proof sketch

Our proof is bijective, with the following strategy:

- We provide a bijection for the trapezoid \( Z(d, r) \).
- For any \( \lambda \), pad it to \( Z(d, r) \) and apply the bijection for \( Z(d, r) \).
- This framework is largely the same as the original arguments by Edelman and Greene, with the main difference that double staircases (the type \( B \) analogue of staircases) are not enough.

The trapezoid \( Z(3, 2) \)

\[
\begin{align*}
\text{SYT}(\lambda) & \leftrightarrow \ \text{SYT}(Z(d, r))|_{\lambda} \leftrightarrow \ \text{Red}(w^{\lambda}) \leftrightarrow \ \text{BS}(Z(d, r))|_{\lambda} \leftrightarrow \ \text{BS}(\lambda) \\
& \downarrow \quad \downarrow \quad \downarrow \\
\text{SYT}(Z(d, r)) & \leftrightarrow \ \text{Red}(w^{(d, r)}) \leftrightarrow \ \text{BS}(Z(d, r))
\end{align*}
\]
Definition (Root system)

Let $E = \mathbb{R}^d$. A root system $\Phi \subset E$ is a finite set of vectors, such that

- $\Phi$ spans $E$;
- for $\alpha \in \Phi$, $k\alpha \in \Phi$ iff $k \in \{\pm 1\}$;
- for $\alpha, \beta \in \Phi$, $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$;
- for $\alpha, \beta \in \Phi$,

$$
\sigma_\alpha(\beta) := \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi.
$$

$$
\begin{array}{rcl}
\end{array}
$$
Root systems and Weyl groups

Let $\Phi \subset E$ be a root system.

- We can partition $\Phi$ into **positive roots** $\Phi^+$ and negative roots $\Phi^-$. 
- Given $\Phi = \Phi^+ \sqcup \Phi^-$, there is a unique choice of **simple roots** $\Delta = \{\alpha_0, \ldots, \alpha_{d-1}\} \subset \Phi^+$ such that each $\alpha \in \Phi^+$ can be written as a unique non-negative integral linear combination of $\Delta$.
- The **Weyl group** $W(\Phi) \subset GL(E)$ is generated by reflections $\{\sigma_\alpha | \alpha \in \Phi\}$, or equivalently, by $\{\sigma_\alpha | \alpha \in \Delta\}$.
- $\{s_i := \sigma_{\alpha_i} | \alpha_i \in \Delta\}$ is the set of **simple reflections**.

For $w \in W(\Phi)$,

- let $\ell(w)$ be its **Coxeter length**, i.e. $\ell(w)$ is the minimal $\ell$ such that $w = s_{a_1} \cdots s_{a_\ell}$ is a product of $\ell$ simple reflections;
- such a word $a = (a_1, \ldots, a_\ell)$ is called a **reduced word** of $w$;
- the **inversion set** is $\text{Inv}(w) := \{\alpha \in \Phi^+ | w\alpha \in \Phi^-\}$;
- it's a classical fact that $\ell(w) = |\text{Inv}(w)|$. 
Root systems and Weyl groups

We adopt the following convention for type $A$, $B$, $C$ root systems.

- **Type $A_{n-1}$:**
  - $\Phi(A_{n-1}) = \{e_j - e_i \mid 1 \leq i \neq j \leq n\}$,
  - $\Phi^+(A_{n-1}) = \{e_j - e_i \mid 1 \leq i < j \leq n\}$,
  - $\Delta(A_{n-1}) = \{e_{i+1} - e_i \mid 1 \leq i \leq n - 1\}$,
  - $W(A_{n-1}) = \mathfrak{S}_n$.

- **Type $B_n$:**
  - $\Phi(B_n) = \{\pm e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$,
  - $\Phi^+(B_n) = \{e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$,
  - $\Delta(B_n) = \{\alpha_0 = e_1\} \cup \{\alpha_i = e_{i+1} - e_i \mid 1 \leq i \leq n - 1\}$,
  - $W(B_n) = \{\text{permutations on } 1, \ldots, n, \bar{1}, \ldots, \bar{n} \mid w(i) = -w(\bar{i}), \forall i\}$.

- **Type $C_n$:**
  - $\Phi(C_n) = \{\pm e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}$,
  - $\Phi^+(C_n) = \{e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}$,
  - $\Delta(C_n) = \{2e_1\} \cup \{e_{i+1} - e_i \mid 1 \leq i \leq n - 1\}$,
  - $W(C_n) = W(B_n)$.
Reflection order of positive roots

**Definition (Reflection order)**

Given a reduced word \( a \in \text{Red}(w) \), its corresponding **reflection order** is an ordering \( \text{ro}(a) = \gamma_1, \ldots, \gamma_{\ell(w)} \) of \( \text{Inv}(w) \) where \( \gamma_j = s_{a_1} \cdots s_{a_{j-1}} \alpha_j \in \Phi^+ \).

A reduced word \( a \in \text{Red}(w) \) can be viewed as a chain

\[
  w^{(0)} = \text{id} \rightarrow w^{(1)} \rightarrow \cdots \rightarrow w^{(\ell)} = w
\]

where \( w^{(i)} = w^{(i-1)} s_{a_i} \). Then the root \( \gamma_i \) satisfies \( w^{(i)} = \sigma \gamma_i w^{(i-1)} \).

**Example: reflection order for** \( a = (2, 1, 2, 3, 2) \) in \( \mathfrak{S}_4 \)

\[
  1234 \xrightarrow{e_3-e_2} 1324 \xrightarrow{e_3-e_1} 3124 \xrightarrow{e_2-e_1} 3214 \xrightarrow{e_4-e_1} 3241 \xrightarrow{e_4-e_2} 3421 = w.
\]

We have \( \gamma_1 = e_3-e_2, \gamma_2 = e_3-e_1, \gamma_3 = e_2-e_1, \gamma_4 = e_4-e_1, \gamma_5 = e_4-e_2 \).

**Example: reflection order for** \( a = (2, 1, 0, 3, 1) \) in \( W(B_3) \)

\[
  1234 \xrightarrow{e_3-e_2} 1324 \xrightarrow{e_3-e_1} 3124 \xrightarrow{e_3} 3124 \xrightarrow{e_4-e_2} 3142 \xrightarrow{e_3+e_1} 1342 = w.
\]
Reflection order of positive roots

The following proposition is classical and well-known, which is basically equivalent to the biconvexity classification of inversion sets.

**Proposition (Björner 1984)**

Let $\gamma = \gamma_1, \ldots, \gamma_{\ell(w)}$ be an ordering of $\text{Inv}(w)$. Then $\gamma$ is a reflection order if and only if for all the triples $\alpha, \beta, \alpha + \beta \in \Phi^+$ such that $\alpha, \alpha + \beta \in \text{Inv}(w)$,

1. if $\beta \notin \text{Inv}(w)$, then $\alpha$ appears before $\alpha + \beta$ in this sequence;
2. and if $\beta \in \text{Inv}(w)$, then $\alpha + \beta$ appears in the middle of $\alpha$ and $\beta$. 
BS(Z(d, r)) ↔ Red(w^{(d,r)}) via reflection order

Recall that $d = \ell(\lambda)$ is the number of parts. Let $r \geq 0$. The **trapezoid** is

$$Z(d, r) := (r + 2d - 1, r + 2d - 3, \ldots, r + 3, r + 1)$$

with height $d$ and base lengths $r + 2d - 1$ and $r + 1$. We provide an extended label of $Z(d, r)$ by roots:

**Example: label of $Z(3, 2)$ by roots**

<table>
<thead>
<tr>
<th>2e_3</th>
<th>e_3+e_2</th>
<th>e_3+e_1</th>
<th>e_3</th>
<th>e_4+e_3</th>
<th>e_5+e_3</th>
<th>e_3−e_1</th>
<th>e_3−e_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2e_2</td>
<td>e_2+e_1</td>
<td>e_2</td>
<td>e_4+e_2</td>
<td>e_5+e_2</td>
<td>e_2−e_1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2e_1</td>
<td>e_1</td>
<td>e_4+e_1</td>
<td>e_5+e_1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
BS\(\mathcal{Z}(d, r)\) $\leftrightarrow$ Red\(\mathcal{w}^{(d,r)}\) via reflection order

Write a signed permutation \(w \in \mathcal{W}(B_n)\) by its one-line notation \(w(1)w(2) \cdots w(n)\). Define a signed permutation \(w^{(d,r)} \in \mathcal{W}(B_{d+r})\) by

\[
w^{(d,r)}(i) := \begin{cases} 
d + i & \text{if } 0 < i \leq r, \\
\frac{i}{i-r} & \text{if } i > r.
\end{cases}
\]

For example, \(w^{(3,2)} = 45\overline{1}\overline{2}\overline{3}\).

Since Inv\(\left(w^{(d,r)}\right)\) is the set of root labels of \(\mathcal{Z}(d, r)\), the following proposition makes sense:

**Proposition (Gao, Gao and G. 2022)**

The reflection order bijects Red\(\mathcal{w}^{(d,r)}\) to BS\(\mathcal{Z}(d, r)\).
**BS(Z(d, r)) ↔ Red(w^{(d,r)}) via reflection order**

**Example: label of Z(3, 2) by roots**

<table>
<thead>
<tr>
<th>2e_3</th>
<th>e_3 + e_2</th>
<th>e_3 + e_1</th>
<th>e_3</th>
<th>e_4 + e_3</th>
<th>e_5 + e_3</th>
<th>e_3 - e_1</th>
<th>e_3 - e_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2e_2</td>
<td>e_2 + e_1</td>
<td>e_2</td>
<td>e_2</td>
<td>e_4 + e_2</td>
<td>e_5 + e_2</td>
<td>e_2 - e_1</td>
<td></td>
</tr>
<tr>
<td>2e_1</td>
<td>e_1</td>
<td>e_4 + e_1</td>
<td>e_5 + e_1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Consider a reduced word \( a = (2, 0, 1, 0, \ldots \) with reflection order

\[
12345 \xrightarrow{e_3 - e_2} 13245 \xrightarrow{e_1} \overline{13245} \xrightarrow{e_3 + e_1} 3\overline{1245} \xrightarrow{e_3} \overline{3\overline{1245}} \longrightarrow \ldots
\]

<table>
<thead>
<tr>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
</tr>
</thead>
</table>
BS(Z(d, r)) ↔ Red(w^{(d,r)}) via reflection order

It is easy to see that a reflection order gives a balanced shifted tableaux. Recall that in a reflection order, \( \alpha + \beta \) appears between \( \alpha \) and \( \beta \).

Example: label of \( Z(3, 2) \) by roots

\[
\begin{array}{cccccccc}
2e_3 & e_3 + e_2 & e_3 + e_1 & e_3 & e_4 + e_3 & e_5 + e_3 & e_3 - e_1 & e_3 - e_2 \\
2e_2 & e_2 + e_1 & e_2 & e_4 + e_2 & e_5 + e_2 & e_2 - e_1 \\
2e_1 & e_1 & e_4 + e_1 & e_5 + e_1
\end{array}
\]

The other direction BS(Z(d, r)) \( \rightarrow \) Red(w^{(d,r)}) is weirdly very technical.
Kraśkiewicz’s insertion is the type $B$ analogue of Edelman-Greene. 

For a unimodal sequence of non-negative integers $R = (r_1 > r_2 > \ldots > r_k < r_{k+1} < \ldots < r_m)$, 

- the decreasing part is $R^\downarrow = (r_1 > r_2 > \ldots > r_k)$, and 
- the increasing part is $R^\uparrow = (r_{k+1} < r_{k+2} < \ldots < r_m)$.

Let $w \in W(B_n)$. Kraśkiewicz’s insertion maps $a = (a_1, \ldots, a_\ell) \in \text{Red}(w)$ to a pair of shifted tableaux $(P(a), Q(a))$ of the same shape by 

$$(P^{(i)}, Q^{(i)}) := (P^{(i-1)}, Q^{(i-1)}) \leftarrow a_i,$$

starting with $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$. 

Yibo Gao (MIT)  
Balanced Shifted Tableaux  
April 11, 2022 23 / 46
Kraśkiewicz’s insertion

Step 1: Set $R$ to be the first row of $P^{(i-1)}$ and $a = a_i$.

Step 2: Insert $a$ into $R$:

- **Case 0 ($R = \emptyset$):** Insert $a$ into the left-most box. Stop.
- **Case 1 ($Ra$ is unimodal):** Append $a_i$ to the right of $R$. Stop.
- **Case 2 ($Ra$ is not unimodal):** Let $b$ be the smallest number in $R^\uparrow$ such that $b \geq a$.
  - **Case 2.0 ($a = 0$ and $R$ contains 101 as a subsequence):** We leave $R$ unchanged and return to start of Step 2 with $a = 0$ and $R$ equals the next row.
  - **Case 2.1.1 ($b \neq a$):** Replace $b$ with $a$ and set $c = b$.
  - **Case 2.1.2 ($b = a$):** Keep $R^\uparrow$ unchanged and set $c = a + 1$.

Now insert $c$ into $R^\downarrow$. Let $d$ be the largest integer such that $d \leq c$.

- **Case 2.1.3 ($d \neq c$):** Replace $d$ with $c$ and set $a' = d$.
- **Case 2.1.4 ($d = c$):** Keep $R^\downarrow$ unchanged and set $a' = c - 1$.

Step 3: Repeat Step 2 with $a = a'$ and $R$ the next row.
Kraśkiewicz’s insertion

Let \( a = (3, 1, 2, 1, 0, 3, 4, 3) \).

\[
\begin{align*}
P^{(0)} &= \emptyset \\
Q^{(0)} &= \emptyset \\
P^{(3)} &= \begin{array}{ccc} 3 & 1 & 2 \\ & & 1 \\ \end{array} \\
Q^{(3)} &= \begin{array}{ccc} 1 & 2 & 3 \\ & & 4 \end{array} \\
P^{(4)} &= \begin{array}{ccc} 3 & 2 & 1 \\ & & 1 \end{array} \\
Q^{(4)} &= \begin{array}{ccc} 1 & 2 & 3 \\ & & 4 \end{array} \\
P^{(7)} &= \begin{array}{cccc} 3 & 2 & 1 & 0 \end{array} \begin{array}{ccc} 3 & 4 \\ 1 \end{array} \\
Q^{(7)} &= \begin{array}{cccc} 1 & 2 & 3 & 5 & 6 & 7 \\ 4 \end{array} \\
P^{(8)} &= \begin{array}{cccc} 4 & 2 & 1 & 0 \end{array} \begin{array}{ccc} 3 & 4 \\ 1 & 3 \end{array} \\
Q^{(8)} &= \begin{array}{cccc} 1 & 2 & 3 & 5 & 6 & 7 \\ 4 & 8 \end{array}
\]
Kraśkiewicz’s insertion

**Definition (Kraśkiewicz 1989)**

A shifted tableaux \( T \) with \( d \) rows is a **standard decomposition tableaux** of \( w \in W(B_n) \) if

- \( \pi(T) = T_d T_{d-1} \ldots T_1 \) is a reduced word of \( w \),
- \( T_i \) is a unimodal subsequence of maximal length in \( T_d T_{d-1} \ldots T_i \).

**Theorem (Kraśkiewicz 1989)**

*The Kraśkiewicz’s insertion gives a bijection between \( \{a \in \text{Red}(w)\} \) and the pairs of tableaux \( (P(a), Q(a)) \) where \( P(a) \) is a standard decomposition tableaux of \( w \) and \( Q(a) \) is a standard tableaux of the same shape.*

**Corollary (Kraśkiewicz 1989)**

*For any \( w \in W(B_n) \),

\[
|\text{Red}(w)| = \sum_{P \in \text{SDT}(w)} f^{sh}(P).
\]*
Kraśkiewicz’s insertion

**Definition**

A signed permutation \( w \in W(B_n) \) is **vexillary** if \( \text{SDT}(w) \) consists of exactly one shifted tableau. We denote this tableau as \( P(w) \).

In this case, the **type C Stanley symmetric function** of \( w \) equals a single **Schur-Q function**.

**Theorem (Billey-Lam 1998)**

A signed permutation \( w \in W(B_n) \) is vexillary if and only if \( w \) pattern avoids the following:

\[
\begin{align*}
\bar{3}2\bar{1} & \quad \bar{3}21 & \quad 32\bar{1} & \quad 321 & \quad 3\bar{1}2 & \quad \bar{2}31 & \quad \bar{1}32 & \quad \bar{4}\bar{1}\bar{2}3 & \quad \bar{4}1\bar{2}3 \\
\bar{3}\bar{4}\bar{1}\bar{2} & \quad \bar{3}\bar{4}1\bar{2} & \quad 3\bar{4}\bar{1}\bar{2} & \quad 34\bar{1}\bar{2} & \quad 3\bar{1}42 & \quad \bar{2}\bar{3}\bar{4}\bar{1} & \quad 2413 & \quad 2\bar{3}\bar{4}\bar{1} & \quad 2143.
\end{align*}
\]

This is equivalent to \( w \) avoiding 2143 as a permutation in \( \mathfrak{S}_{2n} \).
An enumeration problem interlude

For \( \pi \in \mathfrak{S}_k \), write \( B_n(\pi) \) as the set of signed permutations \( w \in B_n \) which avoid \( \pi \) as if \( w \in \mathfrak{S}_{2n} \).

Warning

This is not the same as Billey-Postnikov pattern avoidance.

Theorem (G. and Hänni 2020)

For \( n \geq 1 \), \( |B_n(2143)| = |B_n(1234)| \).

This settled a conjecture by Anderson and Fulton.

Our technique also shows that, in the Weyl groups of type \( D_n \),

Corollary (G. and Hänni 2020)

For \( n \geq 1 \), \( |D_n(2143)| = |D_n(1234)| \).
An enumeration problem interlude

These numbers have very nice enumeration formula.

**Theorem (Egge 2010)**

\[ |B_n(1234)| = \sum_j \binom{n}{j}^2 C_j \]

where \( C_j \) is the \( j \)-th Catalan number.

There should be more Wilf-equivalent families in this sense:

**Conjecture (G. and Hänni 2020)**

For \( m \geq 2k \),

\[ |B_n(12 \cdots m)| = |B_n(k \cdots 1 (k+1) \cdots (m-k) m \cdots (m-k+1))| \].
SYT(Z(d, r)) ↔ Red(w^{d,r}) via Kraśkiewicz’s insertion

It’s straightforward to check that w^{d,r} is vexillary. The shifted tableau P(w^{d,r}) has shape Z(d, r) and can be nicely described.

The insertion tableau for w^{3,2}

\[
P(w^{3,2}) = \begin{array}{cccccc}
4 & 3 & 0 & 1 & 2 & 3 & 4 \\
3 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 \\
\end{array}
\]

Corollary

By restricting to the recording tableaux, Kraśkiewicz’s insertion gives a bijection between Red(w^{d,r}) and SYT(Z(d, r)).
Proof sketch

We have now finished the second row of

\[
\begin{align*}
\text{SYT}(\lambda) \leftrightarrow \text{SYT}(Z(d, r))|_\lambda & \quad \leftrightarrow \quad \text{Red}(w^\lambda) \leftrightarrow \text{BS}(Z(d, r))|_\lambda \leftrightarrow \text{BS}(\lambda) \\
\downarrow \subseteq & \quad \downarrow \subseteq & \quad \downarrow \subseteq \\
\text{SYT}(Z(d, r)) & \leftrightarrow \text{Red}(w^{(d, r)}) \leftrightarrow \text{BS}(Z(d, r))
\end{align*}
\]

For an arbitrary \( \lambda \), choose \( r \) large enough such that \( \lambda \subset Z(d, r) \).
The choice of \( r \) will not matter for the bijection.
We now describe \( \text{SYT}(\lambda) \leftrightarrow \text{SYT}(Z(d, r))|_\lambda \) and \( \text{BS}(\lambda) \leftrightarrow \text{BS}(Z(d, r))|_\lambda \).
SYT(\(\lambda\)) \leftrightarrow SYT(Z(d, r))|_{\lambda}

For \(T \in SYT(\lambda)\), we pad it to obtain \(T^+ \in SYT(Z(d, r))\).

**Example: padding a standard shifted tableau**

\[
T = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 & 9 \\
4 & 7 & & & & \\
8 & & & & & \\
\end{array}, \quad \text{then} \quad T^+ = \begin{array}{cccccccc}
1 & 2 & 3 & 5 & 6 & 9 & 10 \\
4 & 7 & 11 & 12 & 13 & & & \\
8 & 14 & 15 & & & & & \\
\end{array}.
\]

Define \(SYT(Z(d, r))|_{\lambda}\) to be the set of all such \(T^+\) obtained from some \(T \in SYT(\lambda)\).
SYT($Z(d, r))|_\lambda \leftrightarrow \text{Red}(w^\lambda)$

For every $T^+ \in \text{SYT}(Z(d, r))|_\lambda$,
- entries $|\lambda| + 1, \ldots, |Z(d, r)|$ are at fixed positions;
- so the first $|Z(d, r)| - |\lambda|$ steps of inverse Kraśkiewicz’s insertion are the same.

**Warning**

Inverse Kraśkiewicz’s insertion may not be well-defined.

**Inverse Kraśkiewicz’s insertion**

<table>
<thead>
<tr>
<th>Insertion Tableau $P$</th>
<th>Recording Tableau $Q$</th>
<th>Letter $a_i$’s</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Tableau P" /></td>
<td><img src="image2" alt="Tableau Q" /></td>
<td>$a_{15} = 2$</td>
</tr>
</tbody>
</table>

Yibo Gao (MIT)  
Balanced Shifted Tableaux  
April 11, 2022  
33 / 46
Once the padded entries are gone, we have a fixed insertion tableaux $P(w^\lambda)$ which determines $w^\lambda$.

We can read off $w^\lambda$ explicitly:

- draw a right triangle with sidelength $d + r$ over $\lambda$ at $(1, 1)$;
- rotate by $45^\circ$ and consider the Dyck path bordering $\lambda$;
- label the upsteps in order by $d + r, d + r - 1, \ldots, d + 1, \bar{1}, \bar{2}, \ldots, \bar{d}$;
- read the corresponding downsteps in order to obtain $w^\lambda$.

Example: $w^\lambda = \bar{2}14\bar{3}5$ for $\lambda = (6, 2, 1) \subset \mathbb{Z}(3, 2)$.
There exists a way to pad balanced shifted tableaux as well.

**Lemma (Gao, Gao and G. 2022)**

Let $B \in \text{BS}(\lambda)$ and $i \in [d]$ such that $i = 1$ or $\lambda_{i-1} - \lambda_i \geq 3$. Let $\lambda^\#$ be obtained from $\lambda$ by adding a box to the $i$-th row and let $j$ be the column index of the added box. Define $B^\#$ obtained from $B$ by

- interchange column $j$ and $j+1$ of $B$,
- and set $B^\#(i,j) = |\lambda| + 1$.

Then $B^\# \in \text{BS}(\lambda^\#)$. Moreover, this is a bijection between $\text{BS}(\lambda)$ and $\{T \in \text{BS}(\lambda^\#) \mid T(i,j) = |\lambda| + 1\}$.

Repeatedly applying this lemma from top to bottom, we can pad any $B \in \text{BS}(\lambda)$ to $B^+ \in \text{BS}(Z(d, r))$. 
Here is an example.

Example: padding a balanced shifted tableau

\[
B = \begin{array}{cccccc}
6 & 3 & 4 & 5 & 9 & 10 \\
7 & 8 & 11 & 12 & 13 \\
2 & & & & & \\
\end{array}, \quad \text{then} \quad B^\# = \begin{array}{cccccc}
6 & 3 & 4 & 9 & 5 & 10 \\
7 & 8 & 12 & 11 & 13 \\
2 & 14 & & & & \\
\end{array}.
\]

Similarly, let \( BS(Z(d, r))|_\lambda \) be the set of balanced shifted tableaux \( B^+ \) that can be obtained in this way from some \( B \in BS(\lambda) \).
For every $B^+ \in BS(Z(d, r))|_\lambda$, 
- entries $|\lambda|+1, \ldots, |Z(d, r)|$ are at fixed positions;
- so the last $|Z(d, r)| - |\lambda|$ roots of the corresponding reflection order are fixed;
- this means $BS(Z(d, r))|_\lambda$ is in bijection with some $Red(u^\lambda)$;
- we can check that $w^\lambda = u^\lambda$.

Now all steps of bijection are completed.
An example

Let’s go the other way and start with a balanced shifted tableau

\[
B = \begin{array}{cccccc}
6 & 3 & 4 & 1 & 5 & 9 \\
7 & 8 & \ \\
2 & \ \\
\end{array}
\]

We have \( \lambda = (6, 2, 1) \) and choose \( Z(3, 2) \).
An example: $\text{BS}(\lambda) \to \text{BS}(Z(d, r))|_\lambda$

We pad it to $B^+ \in \text{BS}(Z(d, r))|_\lambda$:

$$
\begin{array}{cccccc}
6 & 3 & 4 & 1 & 5 & 9 \\
7 & 8 & \circ & \circ & & \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 1 & 5 & 9 & 10 \\
7 & 8 & \circ & \circ & \circ & \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 5 & 1 & 9 & 10 \\
7 & 8 & 11 & \circ & \circ & \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 5 & 9 & 1 & 10 \\
7 & 8 & 11 & 12 & \circ & \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 5 & 9 & 1 & 10 \\
7 & 8 & 12 & 11 & 13 & \circ \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 9 & 5 & 10 & 1 \\
7 & 8 & 12 & 11 & 13 & \circ \\
2 & & & & & \\
\end{array} \to
\begin{array}{cccccc}
6 & 3 & 4 & 9 & 10 & 5 & 1 \\
7 & 8 & 12 & 13 & 11 & \circ \\
2 & & & & & \\
\end{array} = B^+
$$
An example: $\text{BS}(Z(d, r))|_{\lambda} \rightarrow \text{Red}(w^{(d, r)})$

\[
B^+ = \begin{array}{cccc}
6 & 3 & 4 & 9 \\
7 & 8 & 12 & 13 \\
2 & 14 & 15 & 1 \\
\end{array}
\]

This gives a reflection order

\[
12345 \xrightarrow{e_3 - e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3 + e_1} 3\bar{1}245 \xrightarrow{e_3} 3\bar{1}245 \xrightarrow{e_3 - e_1} \bar{1}3245
\]

\[
\xrightarrow{e_3 + e_2} \bar{1}2\bar{3}45 \xrightarrow{e_2 + e_1} 2\bar{1}\bar{3}45 \xrightarrow{e_2} 2\bar{1}\bar{3}45 \xrightarrow{e_4 + e_3} 2\bar{1}\bar{4}35 \xrightarrow{e_5 + e_3} \bar{2}\bar{1}\bar{4}53
\]

\[
\xrightarrow{e_2 - e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4 + e_2} \bar{1}\bar{4}\bar{2}5\bar{3} \xrightarrow{e_5 + e_2} \bar{1}\bar{4}\bar{5}\bar{2}\bar{3} \xrightarrow{e_4 + e_1} 4\bar{1}\bar{5}\bar{2}\bar{3} \xrightarrow{e_5 + e_1} 45\bar{1}\bar{2}\bar{3},
\]

for which we read off

\[
a = 201012103412312 \in \text{Red}(w^{(3,2)}).
\]
An example: $\text{Red}(w^{(d,r)}) \rightarrow \text{SYT}(Z(d, r))|_\lambda$

Kraśkiewicz’s insertion of $a = 201012103412312$ gives

$P^{(0)} = \emptyset$

$P^{(3)} = \begin{array}{ccc}
2 & 0 & 1 \\
\end{array}$

$P^{(4)} = \begin{array}{ccc}
2 & 1 & 0 \\
0 & \end{array}$

$P^{(5)} = \begin{array}{ccc}
2 & 1 & 0 & 1 \\
0 & \end{array}$

$P^{(15)} = \begin{array}{cccccc}
4 & 3 & 0 & 1 & 2 & 3 & 4 \\
3 & 0 & 1 & 2 & 3 & \\
0 & 1 & 2 & \end{array}$

$Q^{(0)} = \emptyset$

$Q^{(3)} = \begin{array}{ccc}
1 & 2 & 3 \\
\end{array}$

$Q^{(4)} = \begin{array}{ccc}
1 & 2 & 3 \\
4 & \end{array}$

$Q^{(5)} = \begin{array}{ccc}
1 & 2 & 3 & 5 \\
4 & \end{array}$

$Q^{(15)} = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 & 9 & 10 \\
4 & 7 & 11 & 12 & 13 & \\
8 & 14 & 15 & \end{array}$
Delete the largest entries from $Q(a)$ until $|\lambda|$ to get

$$T^+ = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 & 9 & 10 \\
4 & 7 & 11 & 12 & 13 \\
8 & 14 & 15 \\
\end{array}, \quad T = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 & 9 \\
4 & 7 \\
8 \\
\end{array}.$$  

We now completed the bijection

Example: the bijection $BS(\lambda) \rightarrow SYT(\lambda)$

$$B = \begin{array}{cccccc}
6 & 3 & 4 & 1 & 5 & 9 \\
7 & 8 \\
2 \\
\end{array}, \quad T = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 & 9 \\
4 & 7 \\
8 \\
\end{array}.$$
There are other notions of “balanced tableaux” in the literature, including

- “standard $w$-tableau” by Kraskiewicz,
- “balanced labeling” by Fomin-Greene-Reiner-Shimozono,
- and its type $B$ analogue by Hamaker.

Our definition focuses on hook length formula.
The above definitions focus on Rothe diagram.
The difference is analogous to dominant permutations v.s. Grassmannian permutations.
Some remarks

A shifted shape $\lambda$ can also be thought of as an order ideal in the principal order filter of the (co)minuscule node in the type $B_n$ root poset:

We have $\text{SYT}(Z(d, r)) = \# \text{Red}(45\bar{3}6\bar{2}7\bar{1})$.
But showing $\# \text{Red}(45\bar{1}2\bar{3}) = \# \text{Red}(45\bar{3}6\bar{2}7\bar{1})$ is far from trivial!
Some remarks

What about other root systems?
Unfortunately, \( \# \text{Red}(w_0(D_4)) = 2316 \) and the number of linear extensions of the root poset is \( e(\Phi(D_r)^+) = 2400 \).
These two quantities also fail to be equal in \( F_4 \).

**Conjecture (Stanley 1984)**

*For any Coxeter group \( W \) and \( J \subset S \),*

\[
\# \text{Red}(w_0^J) \leq e(\Phi_J^+).
\]
Thanks!

We thank Alex Yong and Jianping Pan for helpful conversations.

Thank you for listening!