Derived Weil Representation and Relative Langlands Duality

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Abstract

The Weil representation is a particularly significant linear representation of the metaplectic group, used in the study of theta correspondence. In this paper, I introduce a derived category version of the Weil representation in the local field case. For the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$, I will give a coherent description of this category, in the philosophy of relative Langlands duality.

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1 Introduction

1.1 Weil representations

The Weil representation is a special representation of symplectic group. The finite field case is defined as follows: let V be a symplectic vector space over the finite field $k = \mathbb{F}_q$ with odd characteristic, and $\mathrm{Heis}(V)$ be the Heisenberg group defined by the symplectic form:

$$1 \to k \to \mathrm{Heis}(V) \to V \to 1.$$

For a character $\psi\colon \to \mathbb{C}^\times$, we can define an irreducible representation $H_{V,\psi}$ of $\mathrm{Heis}(V)$ with central character ψ. It is a subspace of functions on the set V and its dimension is $q^{\frac{1}{2}\dim V}$. This can be extend to a projective representation ω_ψ of $\mathrm{Sp}(V)$, called the Weil representation. In general, it can be descent to a representation of the double cover $\widetilde{\mathrm{Sp}}(V)$ of the symplectic group. The case in local field $k((t))$ is defined similarly using residue.

A dual pair (G, H) is the subgroup $G \times H \to \mathrm{Sp}(V)$ such that they are the centralizer of each other. Examples are $(\mathrm{Sp}(V_1), \mathrm{O}(V_2))$ where $V = V_1 \otimes V_2$, and $(\mathrm{GL}(L_1), \mathrm{GL}(L_2))$ where $V = \mathrm{Hom}(L_1, L_2) \oplus \mathrm{Hom}(L_2, L_1)$. By restricting Weil representation to this subgroup, we obtain $\mathrm{Weil}_{G,H}$ as a representation of $G \times H$. Associated to it, we can define theta functions and construct theta lifts by using it as an integral kernel.

By choosing a Lagrangian $L \subset V$, the Weil representation can be identified with L^2-functions on L or V/L. Thus it has a natural categorification $D(L)$. In [14], the action of $D(\mathrm{Sp}(V))$ is constructed via the functor

$$D(\mathrm{Sp}(V)) \to \mathrm{End}(D(L)) \simeq D(L \times L) \simeq D(V)$$

giving by a sheaf in $D(\mathrm{Sp}(V) \times V)$.

In the local field case, one geometric model of Weil representation is constructed in [19].

When studying Weil representations, we would expect more compatibilities such as the commutativity of these two actions. By mimicking the lattice model of the Weil representation, I could define the derived Weil category with the action of Hecke categories of $G \times H$ at the same time.

Theorem 1. *Let F be a local field and $\mathcal{O}$ its ring of integers. For a variety X, let $X_\mathcal{O}$ be its arc space and X_F be its loop space.*

Let $\mathrm{Weil}_{G,H}$ be the category of $G_\mathcal{O} \times H_\mathcal{O}$-equivariant $(V_\mathcal{O}, \psi)$-equivariant sheaves on V_F. Let $\mathrm{Sat}_G = D_{G_\mathcal{O}}(G_F/G_\mathcal{O})$ be the derived Satake category. Then we have the action of $\mathrm{Sat}_{G\times H} \simeq \mathrm{Sat}_G \otimes \mathrm{Sat}_H$ on $\mathrm{Weil}_{G,H}$. Hence the actions of Sat_G and Sat_H commute in the strongest sense.

In [21], Lysenko constructed the functor from the heart of derived Satake category to the semisimplification of the heart of the Weil category

$$\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G^\vee) \to (\mathrm{Weil}_{G,H}^\heartsuit)^{\mathrm{ss}},$$

and showed that this is an equivalence in the case of $(\mathrm{GL}_n, \mathrm{GL}_m)$-case and conjectured it is also true in the $(\mathrm{Sp}_{2m}, \mathrm{SO}_{2n})$-cases. We will show this conjecture is true in section 3.

Under derived Satake equivalence [3], we can construct the functor

$$D_{G_\mathcal{O}}(\mathrm{Gr}_G) \simeq \mathrm{QCoh}^{G^\vee}_{\mathrm{perf}}(\mathfrak{g}^{\vee *}[2]) \to \mathrm{Weil}_{G,H}.$$

However, this functor is not an isomorphism in general as the left hand side does not have any information of H. Hence a natural question is to give a coherent description of $\mathrm{Weil}_{G,H}$ in terms of $G^\vee$ and $H^\vee$.

Consider the following cases:

- $G = \mathrm{GL}_n, H = \mathrm{GL}_m, n < m$. Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{m-n}$ and further embedded into $\mathfrak{gl}_m = \mathfrak{h}^\vee$;
- $G = \mathrm{SO}_{2n}, H = \mathrm{Sp}_{2m}, n \le m$. Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{so}_{2m-2n+1}$ and further embedded into $\mathfrak{so}_{2m+1} = \mathfrak{h}^\vee$;
- $G = \mathrm{Sp}_{2n}, H = \mathrm{SO}_{2m}, n < m$. Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{so}_{2m-2n-1}$ and further embedded into $\mathfrak{so}_{2m} = \mathfrak{h}^\vee$.

Let S be the Slodowy slice $f + \mathfrak{z}_{\mathfrak{h}^\vee}(e)$ corresponding to the $\mathfrak{sl}_2$-triple (e, h, f) inside $\mathfrak{h}^\vee$ and hence inside $\mathfrak{h}^{\vee *}$ using the canonical isomorphism $\mathfrak{h}^{\vee *} \simeq \mathfrak{h}^\vee$. S carries the action of $G^\vee$ because it acts trivially on the $\mathfrak{sl}_2$-triple. Besides, S carries the grading defined by $t^2 \exp(th)$ commuting with $G^\vee$-action. Let $S^{/\!\!/}$ be the dg-scheme with this cohomological grading.

Conjecture 1. *We have the equivalence of categories*

$$\mathrm{Weil}_{G,H} \simeq \mathrm{QCoh}^{G^\vee}_{\mathrm{perf}}(S^{/\!\!/}),$$

and the equivalence is compatible with the action of derived Hecke categories of G and H on both side.

Remark. The action of $D_{G_\mathcal{O}}(\mathrm{Gr}_G) \simeq \mathrm{QCoh}^{G^\vee}_{\mathrm{perf}}(\mathfrak{g}^{\vee *}[2])$ comes from the stack map $S^{/\!\!/}/G^\vee \to \mathfrak{g}^{\vee *}[2]/G^\vee$. For the action of $\mathrm{Sat}_H \simeq \mathrm{QCoh}^{H^\vee}_{\mathrm{perf}}(\mathfrak{h}^{\vee *}[2])$, it is first mapped to $\mathrm{QCoh}^{G^\vee \times \mathbb{G}_m}_{\mathrm{perf}}(\mathfrak{h}^{\vee *}[2])$, which is equivalent to $\mathrm{QCoh}^{G^\vee \times \mathbb{G}_m}_{\mathrm{perf}}(\mathfrak{h}^{\vee * /\!\!/})$. This category acts on $\mathrm{QCoh}^{G^\vee}_{\mathrm{perf}}(S^{/\!\!/})$ via the stack map $S^{/\!\!/}/G^\vee \to \mathfrak{h}^{\vee *}[2]/G^\vee$.

In this paper, the first case is proved:

Theorem 2. *In the case of $G = \mathrm{GL}_n, H = \mathrm{GL}_m, n < m$, the categories are equivalent. If the functor in section 5 of [21] is symmetric monoidal, then the above statement about Hecke action is true.*

For the case $G = H = \mathrm{GL}_n$, the space S in the equivalence is $\mathfrak{g}^{\vee *} \oplus \mathrm{std} \oplus \mathrm{std}^*$ and this result is claimed by Tsao-Hsien Chen and Jonathan Wang.

1.2 Relative Langlands duality

In [17], Gross and Prasad proposed the problem of restricting representations of SO_n to SO_{n-1}. For irreducible representations π_1 of SO_n and π_2 of SO_{n-1}, to find the multiplicity of trivial representation in $\pi_1 \boxtimes \pi_2$ as a representation of SO_{n-1} requires to calculate the matrix coefficients $\int_{\mathrm{SO}_{n-1}} \langle \pi_1 \boxtimes \pi_2(g) v, v^\vee \rangle \mathrm{d}g$. In [18], authors proved that when v is spherical, this is equal to

$$\Delta_{\mathrm{SO}_n} \frac{L(\frac{1}{2}, \pi_1 \boxtimes \pi_2, \mathrm{std})}{L(0, \pi_1 \boxtimes \pi_2, \mathrm{ad})},$$

where Δ_{SO_n} is a constant. std is the standard representation of Langlands dual group of $\mathrm{SO}_{n-1} \times \mathrm{SO}_n$, and ad is the adjoint representation.

Sakellaridis and Venkatesh [23] conjectured a generalized result regarding a group G and its spherical variety X, which are $\mathrm{SO}_{n-1} \times \mathrm{SO}_n$ and $\mathrm{SO}_{n-1} \setminus \mathrm{SO}_{n-1} \times \mathrm{SO}_n$ in the previous discussion. In [22], Sakellaridis gave the description of $C_c^\infty(X_F)^{G_\mathcal{O}}$ under this framework. The categorical version of this conjecture, proposed by Ben-Zvi, Sakellaridis and Venkatesh in [10], is as follows: the category $D_{G_\mathcal{O}}(X_F)$ is equivalent to $\mathrm{QCoh}^{G_X^\vee}_{\mathrm{perf}}(V_X^{/\!\!/})$ for some group $G_X^\vee \to G^\vee$ and its representation V_X with a compatible grading.

This categorical equivalence for $(G, X) = (\mathrm{GL}_{n-1} \times \mathrm{GL}_n, \mathrm{GL}_{n-1} \setminus \mathrm{GL}_{n-1} \times \mathrm{GL}_n)$ is proved in [6], and the case for $(G, X) = (\mathrm{SO}_{n-1} \times \mathrm{SO}_n, \mathrm{SO}_{n-1} \setminus \mathrm{SO}_{n-1} \times \mathrm{SO}_n)$ is proved in [9].

For GGP problem of Bessel case, i.e., SO_n and SO_m when $m - n$ is odd, the Jacobi group $J = \mathrm{SO}_n \ltimes U^{\mathrm{SO}}_{m,n} \subset \mathrm{SO}_n \times \mathrm{SO}_m$ is used. In the case $G = \mathrm{SO}_n \times \mathrm{SO}_m, X = J \setminus G$, it is expected that $G_X^\vee = G^\vee$

and V_X is the standard representation. The case for $G = \mathrm{GL}_m \times \mathrm{GL}_n$, $X = \mathrm{GL}_n \ltimes U_{m,n}^{\mathrm{GL}} \backslash \mathrm{GL}_m$ is proved in [24].

In the framework of [10], relative Langlands duality is between the pairs $(G, M = T^*X)$ and $(G^\vee, M^\vee = V_X \times^{G_X^\vee} G^\vee)$. Our result verifies $(G^{\vee\vee}, M^{\vee\vee}) = (G, X)$ in the Bessel period case and general linear group case. In fact, one can verify that $T^*(\mathrm{SO}_m/_f U_{m,n}) = \mathrm{SO}_m \times^{U_{m,n}} (f + U_{m,n}^{\perp}) = \mathrm{SO}_m \times S$.

Note that the construction of dual space in [10] is not self-dual a priori. For example, it is not clear if $M^\vee$ is hypersphrical for a general M.

1.3 Connection with Coulomb branches

In [8], the authors give a mathematical definition of Coulomb branch $\mathcal{M}_{G,N}$ for a group G and its representation N and showed that it only depends on the symplectic representation $T^*N = N \oplus N^*$.

Recently, [5] gives the construction for any symplectic representation. The method is by the geometric Weil representation. For the metaplectic group $\widetilde{\mathrm{Sp}}(V)$, consider its $\widetilde{\mathrm{Sp}}(V)_{\mathcal{O}}$-equivariant Weil representation category $\mathrm{Weil}_{G,H}$ and the special object IC_0. Then we obtain an algebraic object as inner Hom of IC_0 in the derived Satake category of $\widetilde{\mathrm{Sp}}(V)$.

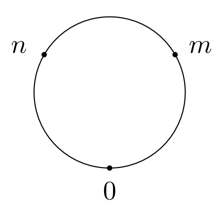
In the case of dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$ or $(\mathrm{SO}_{2n}, \mathrm{Sp}_{2m})$, the anomaly condition in [5] is satisfied. The !-pullback gives a genuine object in the derived Satake category of $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2m}$. So we can take global section to get an algebra and the Coulomb branch of group $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2m}$ and its representation $\mathrm{std} \otimes \mathrm{std}$.

Furthermore, the !-pullback of an inner Hom is still an inner Hom, the above construction is exactly considering the inner End of $\mathrm{IC}_0 \in \mathrm{Weil}_{G,H}$. From the equivalence of categories $\mathrm{Weil}_{G,H} \simeq \mathrm{QCoh}_{\mathrm{perf}}(S^{/\!\!/}/G^\vee)$, which identifies IC_0 and the structure sheaf of $S^{/\!\!/}/G^\vee$. Then the inner Hom of $\mathcal{O}_{S^{/\!\!/}/G^\vee}$ in $\mathrm{QCoh}(\mathfrak{g}^{\vee *}[2]/G \times \mathfrak{h}^{\vee *}[2]/H)$ is just the pushforward of $\mathcal{O}_{S^{/\!\!/}/G^\vee}$.

By [3], taking equivariant cohomology as $H^*_{G_{\mathcal{O}} \times H_{\mathcal{O}}}(\mathrm{pt})$-module refers to the pullback along Kostant section $\Sigma_{\mathfrak{g}^\vee} \times \Sigma_{\mathfrak{h}^\vee} \to \mathfrak{g}^{\vee *}[2]/G^\vee \times \mathfrak{h}^{\vee *}[2]/H^\vee$. Hence the coulomb branch in this case is

$$S/G^\vee \underset{\mathfrak{g}^{\vee *}[2]/G^\vee \times \mathfrak{h}^{\vee *}[2]/H^\vee}{\times} (\Sigma_{\mathfrak{g}^\vee} \times \Sigma_{\mathfrak{h}^\vee}).$$

In [16], the authors showed that the Coulomb branch associated with a quiver of affine type A with Cherkis bow varieties. If we apply this result to the following quiver:



we get

$$((\mathrm{GL}_m \times \Sigma_{\mathrm{GL}_m}) \times (\mathrm{GL}_m \times S) \times (\mathrm{GL}_n \times \Sigma_{\mathrm{GL}_n}))/\!\!/\!\!/(\mathrm{GL}_m \times \mathrm{GL}_n),$$

where $/\!\!/\!\!/$ means the Hamiltonian quotient. This is exactly what is stated above in the case of $(G, H) = (\mathrm{GL}_n, \mathrm{GL}_m)$.

2 Definition of the categories

2.1 Notations

Let k be an algebraically closed field used in the definition of geometric object. Let $\Lambda = \overline{\mathbb{Q}_\ell}$ or $\mathbb{C}$ be the field of the coefficient of sheaves. $\psi\colon k \to \Lambda^\times$ is a non-trivial character. Then we get the Artin-Schreier sheaf $\mathcal{L}_\psi \in D(\mathbb{A}^1)$. In the case $k = \Lambda = \mathbb{C}$, this is the exponential D-module.

$F = k((t))$ is the field of Laurent series, and $\mathcal{O} = k[[t]]$ is the ring of integers in F. ψ naturally extends to a character of F via residue: $\psi \colon F \xrightarrow{\text{res}} k \xrightarrow{\psi} \Lambda^\times$.

When V is a symplectic vector space, use $\omega \colon V \times V \to k$ to denote the symplectic pairing. It naturally extends to a symplectic pairing on V_F:

$$V_F \times V_F \to F \xrightarrow{\text{res}} k,$$

which also gives a pairing on $t^{-r}V_{\mathcal{O}}/t^r V_{\mathcal{O}}$. By abuse of notation, we still use ω to denote them.

For an algebraic group G, define $\mathrm{Gr}_G = G_F/G_{\mathcal{O}}$ be the affine Grassmannian of G and define $\mathrm{Sat}_G = D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ be the derived Satake category.

If needed, we assume our categories are $(\infty, 1)$-categories. By saying derived category, we mean stable $(\infty, 1)$-categories. For a derived category $\mathcal{C}$ with certain t-structure, we use $\mathcal{C}^\heartsuit$ to denote the heart of this t-structure.

2.2 Schrödinger model

In the general linear group case, the vector space V has a polarization $V = T^*L$ such that $L = \mathrm{Hom}(V_1, V_2)$ is a representation of $G \times H$. In this case, the Weil representation can be identified with $G_{\mathcal{O}} \times H_{\mathcal{O}}$-equivariant sheaves on L_F. More concretely, it is defined as a colimit of categories of the diagram:

$$\cdots \to D_{G_{2r} \times H_{2r}}(t^{-r}L_{\mathcal{O}}/t^r L_{\mathcal{O}}) \to D_{G_{2r+2} \times H_{2r+2}}(t^{-r-1}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}}) \to \cdots .$$

The arrows are given by $i_*p^\dagger = i_*p^*[\dim L]$, where $p \colon t^{-r}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}} \to t^{-r}L_{\mathcal{O}}/t^r L_{\mathcal{O}}$ is the projection and $i \colon t^{-r}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}} \to t^{-r-1}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}}$ is the inclusion. The degree is chosen such that the middle perverse t-structure is preserved.

2.3 Lattice model

When the case V is possibly not canonically split, the above construction lacks the equivariance structure. We propose another approach through the so-called lattice model. We first explain our construction through the finite case.

2.3.1 Finite case

Pick any Lagrangian $L \subset V$, we can think of L as a group acting on V via addition. Then we have a relative character on L: $L \times V \xrightarrow{\psi \circ \omega} \Lambda^\times$ and corresponding sheaf $\omega^*\mathcal{L}_\psi$. Call a sheaf $\mathcal{F}$ is (L, ψ)-equivariant if we have an isomorphism

$$\mathrm{act}^*\mathcal{F} \cong \mathrm{proj}^*\mathcal{F} \otimes \omega^*\mathcal{L}_\psi.$$

Hence we can form the category $D(V/(L, \psi))$ of (L, ψ)-equivariant sheaves on V.

2.3.2 Local case

Consider the $G_{\mathcal{O}} \times H_{\mathcal{O}}$-stable Lagrangian $V_{\mathcal{O}} \subset V_F$. To mimic the finite case, we want a category $D(V_F/(V_{\mathcal{O}}, \psi))$. As the colimit of finite cases, we define this category as the colimit of the following diagram:

$$\cdots \to D((t^{-r}V_{\mathcal{O}}/t^r V_{\mathcal{O}})/(V_{\mathcal{O}}/t^r V_{\mathcal{O}}, \psi)) \xrightarrow{i_*p^\dagger} D((t^{-r-1}V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}})/(V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}}, \psi)) \to \cdots .$$

Even G_r can act on the space $t^{-r}V_{\mathcal{O}}/V_{\mathcal{O}}$, it cannot act on $(V_{\mathcal{O}}/t^rV_{\mathcal{O}},\psi)$-equivariant sheaves on $t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}}$. Rather, we only have the action of G_{2r}. Hence the unramified Weil representation $D_{G_{\mathcal{O}}\times H_{\mathcal{O}}}(V_F/(V_{\mathcal{O}},\psi))$ is the colimit of the following diagram:

$$\cdots \to D_{G_{2r}\times H_{2r}}((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}},\psi)) \to$$
$$\to D_{G_{2r+2}\times H_{2r+2}}((t^{-r-1}V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}})/(V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}},\psi)) \to \cdots .$$

We will define the Hecke action in the next section.

2.4 Fourier transform

While the lattice model is defined without the assumption of V having a polarization, we want to show this construction is equivalent to the Schrödinger model in polarizable case.

By the colimit description of the category, it suffices to show $D(t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}})$ is equivalent to $D((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}},\psi))$. By taking Fourier transform, we know the latter is equivalent to $D((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}},\psi))$. Hence it suffices to show the following statement:

Proposition 1. *If a particular splitting of the short exact sequence $0\to L\to V\to V/L\to 0$ is chosen, we get a non-canonical equivalence of categories*

$$D(V/(L,\psi)) \cong D(V/L).$$

If the splitting preserves G-action, we have $D_G(V/(L,\psi))\cong D_G(V/L)$.

Proof. Consider the space $L\times V/L$. It carries an L-action by $L\times L\times V/L\to L\times V/L$ by $(l_1,l_2,v+L)\mapsto(l_1+l_2,v+L)$. From the map $L\times L\times V/L\to\mathbb{A}^1,(l_1,l_2,v+L)\mapsto\omega(l_1,v)$, we can define (L,ψ)-equivariant sheaves on $L\times V/L$.

Then we have the canonical equivalence $D(V/L)\simeq D((L\times V/L)/L)\simeq D((L\times V/L)/(L,\psi))$, where the second is given by $\mathcal{F}\mapsto\mathcal{F}\otimes\mathcal{L}_\psi$. This comes from $\mathcal{L}_\psi$ is (L,ψ)-equivariant, as $\omega(l_1+l_2,v)=\omega(l_1,v)+\omega(l_2,v)$.

For a given section $V/L\to V$, we get a non-canonical isomorphism $V\cong L\times V/L$. This isomorphism makes the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[d, equal] & L\times V \arrow[l] \arrow[r, shift left, "\mathrm{act}"] \arrow[r, shift right, "\mathrm{proj}"'] \arrow[d, "\cong"] & V \arrow[d, "\cong"] \\
\mathbb{A}^1 & L\times L\times V/L \arrow[l] \arrow[r, shift left, "\mathrm{act}"] \arrow[r, shift right, "\mathrm{proj}"'] & L\times V/L
\end{tikzcd}$$

This gives the equivalence $D(V/(L,\psi))\cong D((L\times V/L)/(L,\psi))$.

If the G-action preserves the isomorphism $V\cong L\times V/L$, the above equivalences preserves G-actions. □

3 Irreducible objects

3.1 Cotangent space

Here we compute $T^*(V/(L,\psi))$. The character ω induces a map $\mathbb{A}^1\times V\to\mathrm{Lie}(L)^*\simeq L^*$ given by $V\xrightarrow{\omega}V^*\to L^*$. The moment map of L-action $T^*V\to L^*$ is given by $(v,v^*)\mapsto(l\mapsto\langle l,v^*\rangle)$. Its fiber at $1\in\mathbb{A}^1$ is

$$T^*V\times_{L^*\times V}(1\times V)=\{(v,v^*):\omega(v)|_L=v^*|_L\}=\{(v,v^*):v-\omega^{-1}(v^*)\in L\}.$$

Here the last equation uses the fact that L is a Lagrangian, i.e.,

$$0 \to L \to V \simeq V^* \to L^* \to 0$$

is an exact sequence. Hence we have

$$T^*(V/(L,\psi)) = (T^*V \times_{L^* \times V} (1 \times V))/L \simeq V.$$

Similarly, we should expect $T^*(V_F/(V_\mathcal{O},\psi)) \simeq V_F$. In fact, we see the singular support of sheaves in $D(V_F/(V_\mathcal{O},\psi))$ lies in the colimit of the sets

$$\cdots \to \{L \subset t^{-r}V_\mathcal{O}/t^r V_\mathcal{O} \text{ is Lagrangian}\} \xrightarrow{p^* i_*} \{L \subset t^{-r-1}V_\mathcal{O}/t^{r+1}V_\mathcal{O} \text{ is Lagrangian}\} \to \cdots,$$

which is Lagrangians in V_F that contains some $t^N V_\mathcal{O}$.

Then we consider the behavior of $G_\mathcal{O}$-action on sheaves to its singular support.

Proposition 2. *The moment map of the $G_\mathcal{O}$-action is given by $V_F \to \mathfrak{g}^*_\mathcal{O}, v \mapsto (g \mapsto \omega(v, gv))$.*

Proof. First, for the finite case, if a group G acts on the symplectic space (V,ω) and fixes the Lagrangian L, we show the moment map of G-action on $V/(L,\psi)$ is by $V \to \mathfrak{g}^*, v \mapsto (g \mapsto \omega(v, gv))$.

The moment map of G-action on V is by $T^*V \to \mathfrak{g}, (v, v^*) \mapsto (gv, v^*)$. It restricts to a map from $T^*V \times_{L^* \times V} (1 \times V)$. The isomorphism $T^*V \times_{L^* \times V} (1 \times V) \simeq V$ is given by $(v, v^*) \mapsto \frac{1}{2}(v + \omega^{-1}(v^*))$ or $v \mapsto \{(v+l, \omega(v-l))\}/L$. Hence the image of V under the moment map is $g \mapsto \omega(g(v+l), v-l) = \omega(gv, v)$.

Then, for the local case, we have moment maps $t^{-r}V_\mathcal{O}/t^r V_\mathcal{O} \to \mathfrak{g}^*_{2r}, v \mapsto (g \mapsto \omega(v, gv))$. It is clear they are compatible for different r. By taking colimit, we get the desired moment map $V_F \to \mathfrak{g}^*_\mathcal{O}$. □

3.2 Singular support

The above result is compatible with the singular support calculated using Schrödinger models.

3.3 Relevant orbits

If a $(V_\mathcal{O},\psi)$-equivariant sheaf on V_F is $G_\mathcal{O}$-equivariant, its singular support must be contained in the preimage of $0 \in \mathfrak{g}^*_\mathcal{O}$.

Any section $V_F/V_\mathcal{O} \to V_F$ induces a non-canonical equivalence $D(V_F/(V_\mathcal{O},\psi))$ with $D(V_F/V_\mathcal{O})$, which does not preserve $G_\mathcal{O}$-action. However, by singular support calculation, we can still determine when a $G_\mathcal{O}$-orbit on $V_F/V_\mathcal{O}$ that could occur as the support of an irreducible object in $D_{G_\mathcal{O}}(V_F/(V_\mathcal{O},\psi))$.

Proposition 3. *Let $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ and $n \leq m$. Consider the subset*

$$\{(v, v^*) : v^*v \in \mathfrak{gl}_n(\mathcal{O}), vv^* \in \mathfrak{gl}_m(\mathcal{O})\} \subset V(F) \times V^*(F)$$

and its image in $V(F)/V(\mathcal{O}) \times V^*(F)/V^*(\mathcal{O})$. Under suitable $\mathrm{GL}_n(\mathcal{O}) \times \mathrm{GL}_m(\mathcal{O})$-action, any element in the quotient can be conjugate to

$$\left(\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \ldots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(t^{-b_1}, \ldots, t^{-b_s}) \end{pmatrix} \right) \tag{1}$$

for $r + s \leq n, a_1 \geq \cdots \geq a_r \geq 1, b_s \geq \cdots \geq b_1 \geq 1$.

Proof. By row and column operators on an elements in $V(F)$, one can make it diagonal, i.e, of the form

$$\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \ldots, t^{-a_n}) \\ 0 \end{pmatrix}$$

for $a_1 \geq \cdots \geq a_n$. Let $r = \max\{i : a_r > 0\}$.

Write $v^* = (x_{ij})_{1\leq i\leq n, 1\leq j\leq m}$. The condition $v^*v \in \mathfrak{gl}_m(\mathcal{O})$ and $vv^* \in \mathfrak{gl}_n(\mathcal{O})$ is equivalent to $x_{ij} \in t^{\max\{a_i,a_j\}}\mathcal{O}$. Hence v^* is of the form

$$\begin{pmatrix} A_{r,r} & A_{r,m-r} \\ A_{n-r,r} & A_{n-r,m-r} \end{pmatrix}$$

where $A_{i,j} \in \mathrm{Mat}_{i,j}(F)$ and $A_{r,r}, A_{r,m-r}, A_{n-r,r}$ has coefficients in $t\mathcal{O}$.

Next, use $\mathrm{GL}_{n-r}(\mathcal{O}) \times \mathrm{GL}_{m-r}(\mathcal{O})$ to do row and column operators to make $A_{n-r,m-r}$ diagonal. Thus we get $v^* + V^*(\mathcal{O})$ is conjugate to $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(t^{-b_1}, \ldots, t^{-b_s}) \end{pmatrix} + V^*(\mathcal{O})$.

Since $v \in \begin{pmatrix} \mathrm{diag}(t^{-a_1}, \ldots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$ and matrices $\begin{pmatrix} 1 & \\ & \mathrm{GL}_{n-r} \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ & \mathrm{GL}_{m-r} \end{pmatrix}$ fix this set, we know $v + V(\mathcal{O})$ is conjugate to $\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \ldots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$. $\square$

Corollary 1. *Let $n \leq m$. The irreducible elements in $D_{\mathrm{GL}_{n\mathcal{O}}\times \mathrm{GL}_{m\mathcal{O}}}((T^*V)_F/(T^*V)_{\mathcal{O}})$ is indexed by $X_\bullet(\mathrm{GL}_n)$.*

Proof. Just note that the element in (1) corresponds to $(a_1, \ldots, a_r, 0 \ldots, 0, -b_1, \ldots, -b_s)$ in $X_\bullet(\mathrm{GL}_n)$. $\square$

Proposition 4. *Let $V = \mathrm{Hom}(\mathbb{C}^{2n}, \mathbb{C}^{2m})$ and $n \leq m$. $\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ is equipped with standard symmetric inner product and $\mathbb{C}^{2m} = \mathbb{C}^m \oplus (\mathbb{C}^m)^*$ is equipped with standard anti-symmetric inner product. Consider the subset*

$$\{v \in V(F) : v^*v \in \mathfrak{so}_{2n}(\mathcal{O}), vv^* \in \mathfrak{sp}_{2m}(\mathcal{O})\}$$

and its image in $V(F)/V(\mathcal{O})$. Under suitable $\mathrm{O}_{2n}(\mathcal{O}) \times \mathrm{Sp}_{2m}$-action, any element in the quotient can be conjugate to

$$\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \ldots, t^{-a_r}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{diag}(t^{-b_1}, \ldots, t^{-b_s}) \end{pmatrix}$$

for $r + s \leq n, a_1 \geq \cdots \geq a_r \geq 1$, $b_s \geq \cdots \geq b_1 \geq 1$.

Proof. Write

$$v = (v_1, v_2, v_3, v_4) \in \mathrm{Hom}(F^n, F^m) \oplus \mathrm{Hom}(F^n, (F^m)^*) \oplus \mathrm{Hom}((F^n)^*, F^m) \oplus \mathrm{Hom}((F^n)^*, (F^m)^*),$$

and

$$v^* = (-v_4^{\mathrm{t}}, -v_2^{\mathrm{t}}, v_3^{\mathrm{t}}, v_1^{\mathrm{t}}) \in \mathrm{Hom}(F^m, F^n) \oplus \mathrm{Hom}(F^m, (F^n)^*) \oplus \mathrm{Hom}((F^m)^*, F^n) \oplus \mathrm{Hom}((F^m)^*, (F^n)^*).$$

Then the condition of $vv^* \in \mathfrak{sp}_{2m}(\mathcal{O})$ is equivalent to $v_1v_4^{\mathrm{t}} + v_3v_2^{\mathrm{t}}, v_1v_3^{\mathrm{t}} + v_3v_1^{\mathrm{t}}, v_2v_4^{\mathrm{t}} + v_4v_2^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O})$. The condition of $v^*v \in \mathfrak{so}_{2n}(\mathcal{O})$ is equivalent to $v_3^{\mathrm{t}}v_2 - v_4^{\mathrm{t}}v_1, v_3^{\mathrm{t}}v_4 - v_4^{\mathrm{t}}v_3, v_1^{\mathrm{t}}v_2 - v_2^{\mathrm{t}}v_1 \in \mathfrak{gl}_n(\mathcal{O})$.

Use elements in $\mathrm{GL}_n(\mathcal{O}), \mathrm{GL}_m(\mathcal{O})$ and permutations $(\mathbb{Z}/2\mathbb{Z})^n \ltimes \mathfrak{S}_n, (\mathbb{Z}/2\mathbb{Z})^m \ltimes \mathfrak{S}_m$, we can make v_1 diagonal and $v_t((v_1)_{jj}) \leq v_t((v_2)_{ij})$, $v_t((v_1)_{ii}) \leq v_t((v_3)_{ij})$.

In particular, write $v_1 = \begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_n}) \\ 0 \end{pmatrix}$ for $a_1 \geq \dots \geq a_n$. Let $r = \max\{i : a_r > 0\}$. Write v_2 as follows

$$\begin{pmatrix} t^{-a_1}x_{11} & \cdots & t^{-a_n}x_{1n} \\ \vdots & & \vdots \\ t^{-a_1}x_{m1} & \cdots & t^{-a_n}x_{mn} \end{pmatrix},$$

where $x_{ij} \in \mathcal{O}$. Then the condition $v_1^{\mathrm{t}} v_2 - v_2^{\mathrm{t}} v_1 \in \mathfrak{gl}_n(\mathcal{O})$ gives $x_{ij} - x_{ji} \in t^{a_i + a_j}\mathcal{O}, 1 \leq i, j \leq n$.

Take $y_{ij} = y_{ji} = x_{ji}$ for $i \leq r, i \leq j$ and $y_{ij} = 0$ for $i, j > r$. This gives an element Y in $\mathrm{Sym}^2 \mathcal{O}^m \subset \mathrm{Sp}_{2m}(\mathcal{O})$. Take the action, we get $x'_{ij} = x_{ij} - x_{ji}$ and $x'_{ji} = 0$ for $i \leq r, i \leq j$. Thus $(v'_2)_{ij} = t^{-a_j}(x_{ij} - x_{ji}) \in t^{a_i}\mathcal{O} \subset \mathcal{O}$ and $(v'_2)_{ji} = 0$ for $i \leq r, i \leq j$. When $i, j > r$, we have $(v'_2)_{ij} = (v_2)_{ij} = t^{-a_j}x_{ij} \in t^{-a_j}\mathcal{O} \subset \mathcal{O}$. In conclusion, we have $v'_2 \in \mathrm{Hom}(\mathcal{O}^n, (\mathcal{O}^m)^*)$.

Similarly, write

$$v_3 = \begin{pmatrix} t^{-a_1}x_{11} & \cdots & t^{-a_1}x_{1m} \\ \vdots & & \vdots \\ t^{-a_n}x_{n1} & \cdots & t^{-a_n}x_{nn} \\ x_{n+1,1} & \cdots & x_{n+1,n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix},$$

where $x_{ij} \in \mathcal{O}$ for $1 \leq i, j \leq n$. The condition $v_1 v_3^{\mathrm{t}} + v_3 v_1^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O})$ gives $x_{ij} + x_{ji} \in t^{a_i + a_j}\mathcal{O}$ for $1 \leq i, j \leq n$ and $x_{ij} \in t^{a_j}\mathcal{O}$ for $i > n$.

If $a_n \leq 0$, from our construction of v_1, we know $x_{ij} \in \mathcal{O}$ for $i > n$. Otherwise, we have $a_1 \geq \dots \geq a_n \geq 1$, then $x_{ij} \in t^{a_j}\mathcal{O} \subset \mathcal{O}$ for $i > n$. Anyway, we have $x_{ij} \in \mathcal{O}$ for $i > n$.

For the remaining, use exactly the same method as before to use an element in $\Lambda^2\mathcal{O}^n \subset \mathrm{SO}_{2n}(\mathcal{O})$ to make $v_3 \in \mathrm{Hom}((\mathcal{O}^n)^*, \mathcal{O}^m)$.

Now $v_3 v_2^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O}), v_3^{\mathrm{t}} v_2 \in \mathfrak{gl}_n(\mathcal{O})$, we get $v_1 v_4^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O}), v_4^{\mathrm{t}} v_1 \in \mathfrak{gl}_n(\mathcal{O})$. Using the result in Proposition 3, we can make v_4 into a diagonal matrix. □

Corollary 2. *Let $n \leq m$. The irreducible elements in $D_{\mathrm{O}_{2n} \times \mathrm{Sp}_{2m}}(V_F/(V_{\mathcal{O}}, \psi))$ is indexed by $X_\bullet(\mathrm{O}_{2n})$.*

Proof. As $r + s \leq n$, we can further use permutations in Weyl group to make $v + V(\mathcal{O})$ is conjugate to $\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$ for $r \leq n$. Thus it corresponds to $(a_1, \dots, a_r, 0 \dots, 0) \in X_\bullet(\mathrm{O}_{2n})$. □

4 Deequivariantization

4.1 Hecke action on the lattice model

For a group homomorphism $G \to \widetilde{\mathrm{Sp}}(V)$, we want to define the action of $D(G)$ on $D(V/(L, \psi))$, we need a kernel sheaf on $G \times V$. This is done in [15] and also [19]. Let $\widetilde{\mathcal{L}}(V)$ be the space of all Lagrangians on V, [19] constructed a sheaf $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ on $\widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times V$ with properties. By the map $G \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V)$ given by $g \mapsto (gL, L)$, we obtain a sheaf $\mathcal{F}_G$ on $G \times V$. Thus we can define the action by

$$\mathcal{S} * \mathcal{F} = \mathrm{act}_!(\mathrm{pr}_2^* \mathcal{F}_G \otimes \mathrm{pr}_{23}^* \mathcal{S} \otimes \mathrm{pr}_{13}^* \mathcal{F} \otimes \mathcal{L}_\psi),$$

Here $\mathrm{act}\colon G \times V \times V \to V$ is given by $(g, v_1, v_2) \mapsto gv_1 + v_2$; pr are corresponding projections; $\mathcal{L}_\psi$ is the sheaf on $G \times V \times V$ given by the pullback of Artin-Schreier sheaf through $G \times V \times V \to \mathbb{A}^1, (g, v_1, v_2) \mapsto \omega(gv_1, v_2)$.

The properties of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ ensures this action gives a genuine module structure.

For the unit, take $\mathcal{S} = \delta_1 \in D(G)$. From the property $\Delta^*\mathcal{F}_{\widetilde{\mathcal{L}}(V)} = \mathcal{F}_\Delta$, we know $\mathcal{F}_G|_1 = \Lambda_L$ and thus the convolution product with an (L,ψ)-equivariant sheaf is just identity.

Proposition 5. *The associativity holds. I.e., we have $\mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) \simeq (\mathcal{S}_1 * \mathcal{S}_2) * \mathcal{F}$.*

Proof. For clarity, we use $(g_1, g_2v_1 + v_2, v_3)$ to denote the map $G \times G \times V \times V \times V \to G \times V \times V$ given by $(g_1, g_2, v_1, v_2, v_3) \mapsto (g_1, g_2v_1 + v_2, v_3)$ and similarly for other maps. Then we have

$$\begin{aligned}\mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) =&(g_1(g_2v_1 + v_2) + v_3)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F})\otimes\\ &\otimes (g_2, v_2)^*\mathcal{F}_G \otimes (g_1, v_3)^*\mathcal{F}_G \otimes \omega(g_2v_1, v_2)^*\mathcal{L}_\psi \otimes \omega(g_1(g_2v_1 + v_2), v_3)^*\mathcal{L}_\psi).\end{aligned}$$

From the convolution property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$, we have the following isomorphism in $\widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times V$:

$$\mathrm{add}_!(\mathrm{pr}_{15}^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)} \otimes \mathrm{pr}_{34}^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)} \otimes \mathcal{L}_\psi) \simeq \mathrm{pr}_2^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)}.$$

Take the pullback by the map $G \times G \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V), (g_1, g_2) \mapsto (g_1g_2L, g_1L, L)$, we see

$$\mathrm{add}_!((g_1, v_1)^*\mathcal{F}_G \otimes (g_2, g_1^{-1}v_2)^*\mathcal{F}_G \otimes \mathcal{L}_\psi) \simeq \mathrm{mult}^* \mathcal{F}_G.$$

Here, we used the fact that $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ is G-equivariant. By change of variables, we see

$$(g_1, g_2, v_1 + g_1v_2)_!((g_1, v_1)^*\mathcal{F}_G \otimes (g_2, v_2)^*\mathcal{F}_G \otimes \omega(v_1, g_1v_2)^*\mathcal{L}_\psi) \simeq \mathrm{mult}^* \mathcal{F}_G.$$

Hence we can simplify, by letting $u = g_1v_2 + v_3$,

$$\begin{aligned}\mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) =&(g_1g_2v_1 + g_1v_2 + v_3)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F})\otimes\\ &\otimes (g_2, v_2)^*\mathcal{F}_G \otimes (g_1, v_3)^*\mathcal{F}_G \otimes \omega(g_1g_2v_1, g_1v_2 + v_3)^*\mathcal{L}_\psi \otimes \omega(g_1v_2, v_3)^*\mathcal{L}_\psi)\\ =&(g_1g_2v_1 + u)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F}) \otimes (g_1g_2, u)^*\mathcal{F}_G \otimes \omega(g_1g_2v_1, u)^*\mathcal{L}_\psi)\\ =&(gv_1 + u)_!((g, v_1)^*((\mathcal{S}_1 * \mathcal{S}_2) \boxtimes \mathcal{F}) \otimes (g, u)^*\mathcal{F}_G \otimes \omega(gv_1, u)^*\mathcal{L}_\psi).\end{aligned}$$

The right hand side is exactly $(\mathcal{S}_1 * \mathcal{S}_2) * \mathcal{F}$. $\square$

The image of an (L,ψ)-equivariant sheaf is still an (L,ψ)-equivariant sheaf comes from the act_{lr}-equivariant property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$.

If a subgroup $H \subset G$ fixes (L,ψ), we get the map $G/H \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V)$, using it, we can define the action of $D(H\backslash G/H)$ on $D_H(V/(L,\psi))$ similarly:

$$\mathcal{S} * \mathcal{F} = \mathrm{act}_!(\mathrm{pr}_2^* \mathcal{F}_G \otimes \mathrm{pr}_3^*(\mathcal{S}\widetilde{\boxtimes}\mathcal{F}) \otimes \mathcal{L}_\psi),$$

Here $\mathrm{act}\colon H\backslash((G \times^H V) \times V) \to H\backslash V$ is given by $(g, v_1, v_2) \mapsto gv_1 + v_2$. Since H fixes L, $\mathcal{F}_G$ descends to a sheaf $\mathcal{F}_{G/H}$ on $G/H \times V$. The act_G-equivariant property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ ensures $\mathcal{F}_{G/H}$ is H-equivariant under the action of $h \cdot (gH, v) = (hgH, hv)$. In conclusion, the action $\mathcal{S} * \mathcal{F}$ is well-defined. The proof of properties such as associativity is identical as above.

Proposition 6. *Take a subspace $W \subset L$ and subgroup $H \subset G$ that fixes W. Then H acts on the symplectic space $W^\perp/W$. We have the compatibility of both actions:*

$$\begin{array}{ccccc} D(H) & \otimes & D((W^\perp/W)/(L/W,\psi)) & \xrightarrow{\mathrm{act}} & D((W^\perp/W)/(L/W,\psi)) \\ \downarrow & & \downarrow & & \downarrow \\ D(G) & \otimes & D(V/(L,\psi)) & \xrightarrow{\mathrm{act}} & D(V/(L,\psi)) \end{array}$$

The compatibility of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ under taking a subquotient $W^{\perp}/W$ of a Lagrangian $W \subset V$ ensures the actions

$$\mathrm{Sat}_{G_n} \otimes D_{G_{\mathcal{O}}}((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}},\psi)) \to D_{G_{\mathcal{O}}}((t^{-r-n}V_{\mathcal{O}}/t^{r+n}V_{\mathcal{O}})/(V_{\mathcal{O}}/t^{r+n}V_{\mathcal{O}},\psi))$$

are compatible. In conclusion, we have the action of Sat_G on $D_{G_{\mathcal{O}}}(V_F/(V_{\mathcal{O}},\psi))$.

For our cases, $G \times H \to \mathrm{Sp}(V)$ has a lift to $\widetilde{\mathrm{Sp}}(V)$, we obtain a $D_{G_{\mathcal{O}}\times H_{\mathcal{O}}}(\mathrm{Gr}_{G\times H})$-action on $D_{G_{\mathcal{O}}\times H_{\mathcal{O}}}(V_F/(V_{\mathcal{O}},\psi))$.

4.2 Through deequivariantized Hecke category

Let $\mathcal{O}(S) = \mathrm{Hom}(\delta_V, \delta_V \underset{G}{*} \mathcal{O}(G^\vee))$. From [3], we have the isomorphism $\mathcal{O}(\mathfrak{g}^{\vee *}) = \mathrm{Sym}(\mathfrak{g}^\vee[-2]) = \mathrm{Hom}(\delta_G, \delta_G \underset{G}{*} \mathcal{O}(G^\vee))$ and similarly $\mathcal{O}(\mathfrak{h}^{\vee *}) = \mathrm{Sym}(\mathfrak{h}^\vee[-2]) = \mathrm{Hom}(\delta_H, \delta_H \underset{H}{*} \mathcal{O}(H^\vee))$. We define the following maps:

$$\mathrm{Sym}(\mathfrak{g}^\vee[-2]) \to \mathcal{O}(S) \quad \text{and} \quad \mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) \to \mathcal{O}(S).$$

From the shearing on $\mathfrak{h}^\vee$, we have the map $\mathfrak{g}^\vee[-2] \to \mathfrak{h}^{\vee /\!\!/}$. We will show the following diagram commute:

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) \arrow[r] & \mathrm{Hom}(\delta_V, \delta_V \underset{G}{*} \mathrm{Res}(\mathcal{O}(H^\vee))) \arrow[d] \\
\mathrm{Sym}(\mathfrak{g}^\vee[-2]) \arrow[u] \arrow[r] & \mathcal{O}(S) = \mathrm{Hom}(\delta_V, \delta_V \underset{G}{*} \mathcal{O}(G^\vee))
\end{tikzcd}$$

The first map $\mathrm{Hom}(\delta_G, \delta_G \underset{G}{*} \mathcal{O}(G^\vee)) \to \mathrm{Hom}(\delta_V, \delta_V \underset{G}{*} \mathcal{O}(G^\vee))$ is just defined via the action of Satake category on the category of Weil representation.

Lemma 1. $\mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) \simeq \bigoplus_{W \in \mathrm{Irr}\, H^\vee} \mathrm{Hom}(\delta_G, \mathrm{IC}_{W^*}) \otimes \mathrm{gr}(W)$.

Proof. From [3], we have $\mathrm{Sym}^i \mathfrak{h}^\vee \simeq \bigoplus_{W \in \mathrm{Irr}\, H^\vee} \mathrm{Ext}^{2i}(\delta_H, \mathrm{IC}_{W^*}) \otimes W$ as $H^\vee$ representations. Thus we can apply the grading of elements in the Cartan subgroup:

$$\mathrm{Sym}^i \mathrm{gr}(\mathfrak{h}^\vee) \simeq \bigoplus_{W \in \mathrm{Irr}\, H^\vee} \mathrm{Ext}^{2i}(\delta_H, \mathrm{IC}_{W^*}) \otimes \mathrm{gr}(W),$$

hence

$$\mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) = \mathrm{Sym}(\mathrm{gr}(\mathfrak{h}^\vee)[-2]) \simeq \bigoplus_{W \in \mathrm{Irr}\, H^\vee} \mathrm{Hom}(\delta_H, \mathrm{IC}_{W^*}) \otimes \mathrm{gr}(W). \qquad \square$$

The generators of this algebra is $\mathfrak{h}^{\vee /\!\!/} = \bigoplus_{\mathrm{Irr}\, H^\vee \ni W \subset \mathfrak{h}^\vee} \mathrm{Ext}^2(\delta_H, \mathrm{IC}_{W^*}) \otimes \mathrm{gr}(W)[-2]$.

We have maps from the Hecke action:

$$\mathrm{Ext}^{2i}(\delta_H, \mathrm{IC}_{W'^*}) \otimes \mathrm{gr}(W') \to \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{H}{*} W'^*) \otimes \mathrm{gr}(W') \simeq \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} \mathrm{gRes}(W'^*)) \otimes \mathrm{gr}(W').$$

Let $\mathrm{gRes}(W'^*) = \bigoplus_{W \in \mathrm{Irr}\, G^\vee} W \otimes M_W$, where M_W is a graded vector space associated to the multiplicity of W. Then $\mathrm{gr}(W') = \bigoplus_{W \in \mathrm{Irr}\, G^\vee} W^* \otimes M_W^*$.

Hence we get the direct summand

$$\bigoplus_{W \in \mathrm{Irr}\, G^\vee} \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} W \otimes M_W) \otimes W^* \otimes M_W^* \subset \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} \mathrm{gRes}(W'^*)) \otimes \mathrm{gr}(W').$$

Write $M_W = \oplus_{k\in\mathbb{Z}} M_{W,k}[k]$ and $M_W^* = \oplus_{k\in\mathbb{Z}} M_{W,k}^*[-k]$. Thus the first term has the direct summand

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} W \otimes M_{W,k}[k]) \otimes W^* \otimes M_{W,k}[-k] = \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^{2i+k}(\delta_V, \delta_V \underset{G}{*} W \otimes M_{W,k}) \otimes W^* \otimes M_{W,k}[-k].$$

Taking traces of each M_W, we obtain a map to $\mathrm{Ext}^{2i+k}(\delta_V, \delta_V \underset{G}{*} W) \otimes W^*[-k]$.

In conclusion, we get the map

$$\mathrm{Ext}^{2i}(\delta_H, \mathrm{IC}_{W'^*}) \otimes \mathrm{gr}(W') \to \bigoplus_{k\in\mathbb{Z}} \bigoplus_{W\in\mathrm{Irr}\, G^\vee} \mathrm{Ext}^{2i+k}(\delta_V, \delta_V \underset{G}{*} W) \otimes W^*[-k] = \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} \mathcal{O}(G^\vee)),$$

and hence

$$\mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) = \bigoplus_{W'\in\mathrm{Irr}\, H^\vee} \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^{2i}(\delta_H, \mathrm{IC}_{W'^*}) \otimes \mathrm{gr}(W')[-2i] \to \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^{2i}(\delta_V, \delta_V \underset{G}{*} \mathcal{O}(G^\vee))[-2i] = \mathcal{O}(S).$$

4.3 Algebraic map

Under the assumption that gRes is monoidal, we have this map is an algebraic homomorphism.

On the other hand, we have a map $\mathfrak{h}^{\vee /\!\!/} \hookrightarrow \mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) \to \mathcal{O}(S)$. By showing $\mathcal{O}(S)$ is commutative, we obtain another map $\mathrm{Sym}(\mathfrak{h}^{\vee /\!\!/}) \to \mathcal{O}(S)$. But without the assumption, it is not known if this map coincide with the map defined before.

5 Examples

5.1 The action by standard representations

Here I give an explicit calculation of $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n$ in the $\mathrm{GL}_n \times \mathrm{GL}_m$ case.

Note that $\mathrm{Gr}_{\mathrm{GL}_n, e_1} = \mathbb{P}^{n-1}$, we have $\mathrm{IC}_{\mathrm{std}} = \mathbb{C}_{\mathbb{P}^{n-1}}$. Thus $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n$ is the pushforward of the constant sheaf on $V_{\mathcal{O}} \widetilde{\times} \mathrm{Gr}_{\mathrm{GL}_n, e_1}$ to V_F.

Since $\mathrm{Gr}_{\mathrm{GL}_n, e_1}$ parameterize lattices Λ such that $\mathcal{O}^n \subset \Lambda \subset (t^{-1}\mathcal{O})^n$ and $\dim \Lambda/\mathcal{O}^n = 1$, by definition $V_{\mathcal{O}} \widetilde{\times} \mathrm{Gr}_{\mathrm{GL}_n, e_1}$ parameterize such a lattice Λ and m vectors in this lattice, and the map $V_{\mathcal{O}} \widetilde{\times} \mathrm{Gr}_{\mathrm{GL}_n, e_1} \to V_F$ forgets this lattice.

Hence the image lies in $t^{-1}V_{\mathcal{O}}$. For any element in $V_{\mathcal{O}}$, its preimage is the whole $\mathbb{P}^{n-1}$. For any element in the image and not in $V_{\mathcal{O}}$, the preimage is just one point. Thus $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n$ can be viewed as a sheaf on $t^{-1}V_{\mathcal{O}}/V_{\mathcal{O}} = V$. In this viewpoint, the support is the elements in V whose rank is less or equal to 1. the stalk at rank 1 is $\mathbb{C}$, and the stalk at 0 is $H^*(\mathbb{P}^{n-1})[m+n-1]$.

Besides, we can calculate the intersection complex directly. A rank 1 matrix can be written as the product of a non-zero row vector and a non-zero column vector. Thus the open part is $(\mathbb{C}^n\backslash\{0\} \times \mathbb{C}^m\backslash\{0\})/\mathbb{G}_m$, or the $\mathbb{C}^*$ bundle $\mathcal{O}(-1,-1)$ on $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$. Then the whole space is the affine cone of this line bundle. Now the stalk of the intersection complex at 0 is

$$(\mathrm{IC}_{e_1})_0 = \tau_{\le -1}(H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})/c_1(\mathcal{O}(-1,-1))[m+n-1]).$$

Here, τ is the truncation functor related to the classical t-structure, and quotient means the taking the cone of the map

$$c_1(\mathcal{O}(-1,-1)) : H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}) \to H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})[-2].$$

When $n < m$, this turns out to be isomorphic to $H^*(\mathbb{P}^{n-1})[m+n-1]$ and also $(\tau_{\leq 2n-2}H^*(\mathbb{P}^{m-1}))[m+n-1]$. Thus we see $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n \simeq \mathrm{IC}_{e_1}$.

Similarly, $\delta_V \underset{\mathrm{GL}_m}{*} \mathrm{std}_m$ supports on the same set and the stalk at 0 is $H^*(\mathbb{P}^{m-1})[m+n-1]$. The decomposition theorem says that this complex is a direct sum of simple objects. Besides IC_{e_1}, the remaining support at 0 and the stalk is $(\tau_{\geq 2n}H^*(\mathbb{P}^{m-1}))[m+n-1]$. Hence is $\mathbb{C}_0[m-n-1] \oplus \mathbb{C}_0[m-n-3] \cdots \oplus \mathbb{C}_0[-m+n+1]$.

This calculation verifies the result in [21] that $\delta_V \underset{\mathrm{GL}_m}{*} \mathrm{std}_m \simeq \delta_V \underset{\mathrm{GL}_n}{*} \mathrm{gRes}(\mathrm{std}_m)$.

From this viewpoint, it is clear that the action of $H^2(\mathbb{P}^{n-1}) = \mathrm{Ext}^2(\mathrm{IC}_{\mathrm{std}_n})$ on $(\delta_V * \mathrm{IC}_{\mathrm{std}_n})_0$ coincide with the action of $H^2(\mathbb{P}^{m-1}) = \mathrm{Ext}^2(\mathrm{IC}_{\mathrm{std}_m})$ on $(\delta_V * \mathrm{IC}_{\mathrm{std}_n})_0$.

In fact, we can calculate $\mathrm{End}(\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n)$ directly. $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n = \tau_{\leq -1}j_*\mathbb{C}[m+n-1]$ fits into an exact triangle:

$$\tau_{\leq -1}j_*\mathbb{C}[m+n-1] \to j_*\mathbb{C}[m+n-1] \to \tau_{\geq 0}j_*\mathbb{C}[m+n-1].$$

Hence an exact triangle

$$\begin{aligned}&\mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \tau_{\leq -1}j_*\mathbb{C}[m+n-1]) \to \\ &\to \mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], j_*\mathbb{C}[m+n-1]) \to \mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \tau_{\geq 0}j_*\mathbb{C}[m+n-1]).\end{aligned}$$

We can calculate

$$\begin{aligned}&\mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \tau_{\leq -1}j_*\mathbb{C}[m+n-1])\\ =&\mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], j_*\mathbb{C}[m+n-1])\\ =&\mathrm{Hom}(j^*\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \mathbb{C}[m+n-1])\\ =&\mathrm{Hom}(\mathbb{C}[m+n-1], \mathbb{C}[m+n-1])\\ =&H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})/c_1(\mathcal{O}(-1,-1)),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \tau_{\geq 0}j_*\mathbb{C}[m+n-1])\\ =&\mathrm{Hom}(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], i_*H^*(\mathbb{P}^{n-1})[n-m])\\ =&\mathrm{Hom}(H^*(\mathbb{P}^{n-1})[m+n-1], H^*(\mathbb{P}^{n-1})[n-m]).\end{aligned}$$

When $n=1$, $\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n = \mathbb{C}[m]$, it is clear the endomorphism is $\mathbb{C}$ of degree 0. When $m > n \geq 2$, the minimum degree of the complex $\mathrm{Hom}(H^*(\mathbb{P}^{n-1})[m+n-1], H^*(\mathbb{P}^{n-1})[n-m])$ is $2m-1-(2n-2) \geq 3$. Thus we have

$$\mathrm{Ext}^2(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], \tau_{\leq -1}j_*\mathbb{C}[m+n-1]) \simeq \mathrm{Ext}^2(\tau_{\leq -1}j_*\mathbb{C}[m+n-1], j_*\mathbb{C}[m+n-1]).$$

By tracking the action, the map from $H^*(\mathbb{P}^{n-1}) = \mathrm{End}(\mathrm{IC}_{\mathrm{std}_n})$ and $H^*(\mathbb{P}^{m-1}) = \mathrm{End}(\mathrm{IC}_{\mathrm{std}_m})$ to $H^*(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})/c_1(\mathcal{O}(-1,-1))$ is the canonical map. Thus the images of $H^2(\mathbb{P}^{n-1})$ and $H^2(\mathbb{P}^{m-1})$ are the same. Furthermore, the image to $\mathrm{Ext}^2(\delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n, \delta_V \underset{\mathrm{GL}_n}{*} \mathrm{std}_n)$ is the same.

5.2 The case when $n = 1$

6 Localization

6.1 Compatibility of two actions

6.2 Pass through the Slodowy slice

Consider the case $(G, H) = (\mathrm{GL}_n, \mathrm{GL}_m)$.

6.3 Linear algebra

We calculate the fiber of the map $S_f \to \mathfrak{g}^{\vee *} /\!\!/ G^\vee \times \mathfrak{h}^{\vee *} /\!\!/ H^\vee$ up to codimension one.

Elements in S_f look like

$$\begin{pmatrix} x & & & & & v \\ v^* & a_1 & a_2 & a_3 & \cdots & a_{m-n} \\ & c_1 & a_1 & a_2 & \cdots & a_{m-n-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & c_{m-n-2} & a_1 & a_2 \\ & & & & c_{m-n-1} & a_1 \end{pmatrix}, x \in \mathfrak{gl}_n, v \in \mathrm{std}_n, v^* \in \mathrm{std}_n^*,$$

where c_i are positive constants. Its image is given by characteristic polynomial of x and this whole matrix. The later is calculated by

$$\chi_x(\lambda)(\lambda^{m-n} - d_1 a_1 \lambda^{m-n-1} + \cdots + (-1)^{m-n} d_{m-n} a_{m-n} + d_{m-n+1} v^*(\lambda I - x)^{-1} v).$$

Here d_i are positive constants.

Proposition 7. *The fiber at a generic point of $\mathfrak{g}^{\vee *} /\!\!/ G^\vee \times \mathfrak{h}^{\vee *} /\!\!/ H^\vee$ is isomorphic to $G^\vee$.*

Proof. Given two polynomials $f(\lambda), g(\lambda)$ of degree n and m. If the discriminant of f and the resultant of f and g are non-zero, we will show the fiber is GL_n.

Write $g = qf + r$ such that $\deg r < n$. Then we know $q(\lambda) = \lambda^{m-n} - d_1 a_1 \lambda^{m-n-1} + \cdots + (-1)^{m-n} d_{m-n} a_{m-n}$, which shows that a_i are fixed.

Since x has characteristic polynomial $\chi_x(\lambda) = f(\lambda)$, x is conjugated to a diagonal matrix by an element in GL_n. Write $x = g\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n) g^{-1}$. Then $v^*(\lambda I - x)^{-1} v = \sum (v^* g^{-1})_i (gv)_i \frac{1}{\lambda - \lambda_i}$. Hence we have

$$(v^* g)_i (g^{-1} v)_i = e_i := \frac{r(\lambda_i)}{d_{m-n+1} \prod_{j \neq i} (\lambda_i - \lambda_j)}.$$

By taking an action of a diagonal matrix, x is unchanged, and we can make $(g^{-1}v)_i = 1$. In conclusion, $x = g\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n) g^{-1}, v = g(1, \ldots, 1)^{\mathrm{t}}, v^* = (e_1, \ldots, e_n) g^{-1}$ gives all the possible fibers and they are different. $\square$

Proposition 8. *The fiber at the hyperplane given by resultant is isomorphic to* $(\mathrm{GL}_n \times \mathbb{A}^1)/\mathbb{G}_m$.

Proposition 9. *The fiber at the hyperplane given by root hyperplanes of $\mathfrak{gl}_n /\!\!/ \mathrm{GL}_n$ is isomorphic to* GL_n.

6.4 Localization

Proposition 10. *$\mathcal{O}(S)$ is normal.*

Proof. Choose a splitting $V_F/V_{\mathcal{O}} \to V_F$. We regard sheaves in $D(V_F/(V_{\mathcal{O}}, \psi))$ as sheaves in $D(V_F/V_{\mathcal{O}})$. Then we have

$$\mathcal{O}(S) = \bigoplus_{W \in \mathrm{Irr}\, G^\vee} \mathrm{Hom}(\delta_V, \delta_V \underset{G}{*} W) = \bigoplus_{W \in \mathrm{Irr}\, G^\vee} i_0^!(\delta_V \underset{G}{*} W).$$

As direct sums of costalks, $\mathcal{O}(S)$ is a free $H^*_{\mathrm{GL}_n}(\mathrm{pt}) \otimes H^*_{\mathrm{GL}_m}(\mathrm{pt})$-module. $\square$

We can use localization to calculate it by fixed points of torus actions.

For the generic point in $t \in \mathfrak{t}_n \times \mathfrak{t}_m$, the corresponding T-action on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ has fixed point $\{0\}$. Let $i_0 \colon \{0\} \to V_F$ be the embedding. Hence we have

$$\begin{aligned}\mathcal{O}(S) \otimes_{\mathbb{C}[\mathfrak{t}_n/\mathfrak{S}_n \times \mathfrak{t}_m/\mathfrak{S}_m]} \mathbb{C}(\mathfrak{t}_n/\mathfrak{S}_n \times \mathfrak{t}_m/\mathfrak{S}_m) &= \mathrm{Hom}(i_0^*\delta_V, i_0^*(\delta_V \underset{\mathrm{GL}_n}{*} \mathcal{O}(\mathrm{GL}_n))) \\ &= \mathrm{Hom}(\mathbb{C}, \mathbb{C} \otimes \mathcal{O}(\mathrm{GL}_n)^{/\!\!/}) = \mathcal{O}(\mathrm{GL}_n)^{/\!\!/}.\end{aligned}$$

For the point $t \in \mathfrak{t}_n \times \mathfrak{t}_m$ lies in the hyperplane given by the resultant, the corresponding T-action on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ has fixed point $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$, where $\mathbb{C} \subset \mathbb{C}^n$ and $\mathbb{C} \subset \mathbb{C}^m$ are the eigenspaces. Let i_2 be the embedding.

To calculate $i_2^*(\delta_V \underset{\mathrm{GL}_n}{*} \mathcal{O}(\mathrm{GL}_n))$, one can first pull back through $i_1 \colon \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}) \to V$. Then we have $i_1^*(\delta_V \underset{\mathrm{GL}_n}{*} \mathcal{O}(\mathrm{GL}_n)) \simeq \delta_{(\mathbb{C}^n)^*} \underset{\mathrm{GL}_1}{*} \mathrm{gRes}\, \mathcal{O}(\mathrm{GL}_n) = \oplus_{k \in \mathbb{Z}} \mathrm{IC}_k \otimes \mathcal{O}(\mathrm{GL}_n)^{/\!\!/}_k$.

By calculation, we have $\mathrm{Hom}(i_2^*\mathrm{IC}_0, i_2^*\mathrm{IC}_k) = \mathbb{C}[-|k|m]$, and hence

$$\mathcal{O}(S)_t = \mathrm{Hom}(i_2^*\delta_V, i_2^*(\delta_V \underset{\mathrm{GL}_n}{*} \mathcal{O}(\mathrm{GL}_n))) = \oplus_{k \in \mathbb{Z}} \mathcal{O}(\mathrm{GL}_n)^{/\!\!/}_k[-|k|m] = \mathcal{O}(\mathrm{GL}_n \times \mathbb{A}^1/\mathbb{G}_m)^{/\!\!/}.$$

Corollary 3. *$\mathcal{O}(S)$ is commutative.*

In conclusion, the algebraic map $\mathcal{O}(S_f^{/\!\!/}) \to \mathcal{O}(S)$ is isomorphism over $\mathrm{Spec}\, H^*_{\mathrm{GL}_n}(\mathrm{pt}) \otimes H^*_{\mathrm{GL}_m}(\mathrm{pt})$ up to a codimension 2 subspace. By localization theorem, these two algebras are isomorphic.

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