The Hodge structure on the fundamental group

Def Consider the category of local systems $\mathrm{Loc}(X)$ over X.
$\mathrm{Loc}(X) = \mathrm{Rep}(\pi_1(X))$
$F_x : \mathrm{Loc}(X) \to \mathrm{Vect}$ is the fiber functor at x.

Then the fundamental group at x, $\varpi_1(X,x)$ is $\underline{\mathrm{Aut}}^{\otimes}(F_x)$, a pro-algebraic group.

$\mathrm{Loc}^{nil}(X)$ is the subcategory generated by trivial objects

Def Consider the full subcategory $\mathrm{Loc}^{ss}(X)$ consists of all semi-simple local systems, and its fiber functor $F_x : \mathrm{Loc}^{ss}(X) \to \mathrm{Vect}$, then its automorphic group $\underline{\mathrm{Aut}}^{\otimes}(F_x)$ is the reductive quotient of $\varpi_1(X,x)$, denoted by $\varpi_1^{red}(X,x)$.

Prop $\varpi_1(X,x) = \varprojlim (G,\rho)$, where $\rho : \pi_1(X,x) \to G$ for all algebraic group G.

pro-reductive group $\to$ $\varpi_1^{red}(X,x) = \varprojlim (G,\rho)$, where G is a reductive group.

We can also define $\varpi_1^{nil}(X,x) = \varprojlim (G,\rho)$ for nilpotent group G.

The main topic of this section is to give a pure Hodge structure on $\varpi_1^{red}(X,x)$ and a mixed Hodge structure on $\varpi_1^{nil}(X,x)$, when X is a compact Kähler manifold.

Recall the equivalence of categories between $\mathrm{Loc}(X)$ and semi-stable Higgs bundles with vanishing chern classes.
We have a $\mathbb{G}_m$-action by $(E,\theta) \mapsto (E, t\theta)$.
Hence we obtain a $\mathbb{G}_m$-action on $\varpi_1(X,x)$.

Also $\mathrm{Loc}^{ss}(X) \simeq$ Harmonic bundles, so the $\mathbb{G}_m$-action on it induces a $\mathbb{G}_m$-action on $\varpi_1^{red}(X,x)$.

Def a pure non-abelian Hodge structure is an affine group G over $\mathbb{R}$ defined, a finite generated subgroup $\Gamma \subset G_{\mathbb{R}}$, and an action of $U(1) \subset \mathbb{C}^\times$ on $G_{\mathbb{R}}$ by homomorphisms of algebraic group such that

1. Γ is Zariski dense in $G_{\mathbb{C}}$,
2. $U(1) \times \Gamma \to G(\mathbb{C})^{an}$ is continuous.
3. $C = -1 \in U(1)$ acts by Cartan involution on $G_{\mathbb{R}}$.

In case G is pro-algebraic, this refers to pro-analytic topology.

3. means if σ is the complex conjugation, then $\tau = C\sigma = \sigma C$ gives a compact real form $G(\mathbb{C})^{an,\tau} \subset G(\mathbb{C})^{an}$.

Theorem 7 The action of $U(1) \subset \mathbb{C}^\times$ on $\varpi_1^{red}(X, x)$ gives a pure non-abelian Hodge structure. proof in the next talk

Results from section 4:

① If G has a pure Hodge structure, then the real form $G_{\mathbb{R}}$ is a group of Hodge type.

Proof: From the assumption, $U(1) \to \mathrm{Hom}(\Gamma, G) \to \mathrm{Aut}(G)$ is continuous, so it extends to an algebraic map $\mathbb{C}^\times \to \mathrm{Aut}(G)$.

② If $G \xrightarrow{\rho} H$ is a surjective group hom such that $\Gamma \to H$ is a rigid representation of Γ, then $U(1)$ fixes $\ker(\rho)$, so that $U(1)$ acts on H, making $\mathrm{im}(\Gamma) \to H$ a pure Hodge structure.

Proof. the continuity of $U(1) \times \Gamma \to H$ implies $\rho \circ t$ is near ρ, thus there exists $h(t) \in H$ such that

$$\rho(t, \gamma) = h_t\, \rho(\gamma)\, h_t^{-1}, \text{ hence } \rho(t, g) = h_t\, \rho(g)\, h_t^{-1}.$$

If $g \in \ker(\rho)$, then $t \cdot g \in \ker(\rho)$. The Hodge structure comes from G.

A real Hodge representation of weight w is a representation $\rho_{\mathbb{R}}: G_{\mathbb{R}} \to GL(V_{\mathbb{R}})$, such that $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, and $\overline{V^{p,q}} = V^{q,p}$.
and if we let $U(1)$ acts on V by $t \cdot v = t^p v$ for $v \in V^{p,q}$,
then $\rho: G \to GL(V_{\mathbb{C}})$ is compatible with $U(1)$-action.

A polarization is a bilinear form $S(u,v)$ defined over $\mathbb{R}$ such that
$\langle u,v \rangle = i^w S(u,\bar{v})$ is Hermitian, $S(u,v) = S(v,u)$ symmetric
$G_{\mathbb{R}}$ preserve the form $\langle u,v \rangle$.
$(u,u) = (-1)^p \langle u,u \rangle = i^{q-p} S(u,\bar{u})$ is positive definite for $u \in V^{p,q}$.

③ $\rho: G \to GL(V)$ has a structure of Hodge representation iff
$\ker(\rho)$ is fixed by $U(1)$. then a polarization always exists.
if $\rho: G_{\mathbb{R}} \to GL(V_{\mathbb{R}})$ such that irreducible and $\ker(\rho)$ is fixed by $U(1)$,
then it has a real Hodge rep structure and a real polarization exists.
In particular, representations of algebraic group of Hodge type has a Hodge str.

④ In the case of Hodge structure on $\varpi_1^{red}(X,x)$,
there's a one-to one correspondence between complex variations of Hodge structure and Hodge representations of $\varpi_1^{red}(X,x)$.

Proof. $U(1)$ acts on V, compatible w/ G-action gives an isomorphism between (E,θ) and $(E,t\theta)$.

Similar statements for real Hodge rep, polarizations, and real polarizations also holds. It seems to be trivial.
(The structure compatible w/ G-action means that it can extend to the whole local system.)

Next is the case for $\varpi_1^{nil}(X, x)$. suffice to consider its Lie algebra $\mathcal{L}$

Morgan-Hain: a Hodge filtration F^p decreasing by neg numbers
such that $[F^p, F^q] \subset F^{p+q}$.
a weight filtration W_k by derived series
$W_0 = L$, $W_k = [L, W_{k+1}]$.
The complex conjugate defined by the real form $L_{\mathbb{R}}$
determines the $\mathbb{R}$-mixed Hodge structure.

Result in this paper: Theorem 8
The $\mathbb{C}^\times$ action on L induces the Hodge filtration:
If $H^r \subset L$ is the subspace of weight r by $\mathbb{C}^\times$-action,
then $F^p = \bigoplus_{r \geq p} H^r$.
Note: $\mathbb{R}$-structure is not split.

Similar results like ④ also holds for $\varpi_1^{nil}(X, x)$:
unipotent variation of Hodge structure is equivalent to
mixed Hodge representations for $\varpi_1^{nil}(X, x)$.

Final result about variation of Hodge structures:

Hain-Zucker: as x varies, Lie $\varpi_1^{nil}(X, x)$ fits into
a pro-unipotent variation of mixed Hodge structure.

Result in this paper:
$K(X, x)$ = subgroup of $\varpi_1^{red}(X, x)_{\mathbb{R}}$ that is fixed by $U(1)$-action,
$\widetilde{X}$ = pointed universal cover $= \{(y, p : x \to y)\}$.

Theorem 9: There's a $\pi_1(X, x)$-equivariant map
$$\Phi : \widetilde{X} \to \varpi_1^{red}(X, x)_{\mathbb{R}} / K(X, x)$$
such that pullback of $\varpi_1^{red}(X, y)$ along p is $\varpi_1^{red}(X, x)$ conj by $\Phi(y, p)$.

The case of Bun_G

A G-torsor is a map $X \to BG$, or equivalently a monoidal functor $\mathrm{Rep}(G) \to$ vector bundles $\hookrightarrow \mathrm{Coh}(X)$.

analytic version: $\mathrm{Rep}(G) \to C^\infty$ vector bundles $\overset{\text{algebraic version}}{\hookrightarrow} \mathrm{Shv}(X^{an})$

Locally constant version: $\mathrm{Rep}(G) \to \mathrm{Loc}(X) \hookrightarrow \mathrm{Coh}(X_{dR})$.

The G-torsor $X \times_{BG} pt$ is given by $\underline{\mathrm{Isom}}^{\otimes}(\mathrm{Rep}(G), \omega_X)$.

an algebraic object in the category $\mathrm{IndVect}(X)$, etc.

$= \int_V V^* \otimes P(V)$, where P is the functor

$= P(\mathcal{O}(G))$, $\mathcal{O}(G) \in \mathrm{IndRep}(G)$

Now, if we consider the category $\mathrm{Loc}(X) \simeq \mathrm{Higgs}(X)$, and map $\mathrm{Funct}^{\otimes}(\mathrm{Rep}(G), \mathrm{Loc}(X)) \to \mathrm{Funct}^{\otimes}(\mathrm{Rep}(G), \mathrm{Vect}(X))$ the fiber is how to lift a functor $\mathrm{Rep}(G) \to \mathrm{Vect}(X)$ to a functor $\mathrm{Rep}(G) \to \mathrm{Higgs}(X)$.

That is, to give every image $V \in \mathrm{Rep}(G)$ a Higgs field θ_V, which is compatible w/ the tensor structure.

Then $\{\theta_V\}$ lies in $\underline{\mathrm{End}}^{\otimes}(\mathrm{Rep}(G), \omega_X) \otimes \Omega^1 = (\mathfrak{g} \times_G P) \otimes \Omega^1$ for the torsor P.

$\theta_V \wedge \theta_V = 0$ for all $V \Leftrightarrow [\theta, \theta] = 0$.

Hence $M_{Dol}(G) \simeq M_{DR}(G)$.

Real group cases

υ - involution real group is a complex group w/ conjugation.

what is Bun_G when G is real group?

(G, υ) - a real group, $(\mathrm{Vect}(X), \upsilon)$ - tannakian category,

$\upsilon = C \Rightarrow \nu = \mathrm{id}$

$\upsilon = \sigma$ or $\tau \Rightarrow \nu = \mathrm{conj}$

Define $\mathrm{Bun}_{(G,\upsilon)} = \mathrm{Funct}^{\otimes}(\mathrm{Rep}(G,\upsilon), (\mathrm{Vect}(X), \upsilon), \text{good})$

(family of υ-structure : call it good if $\omega_x \simeq \omega_G$ for all x.)

an isom exists

Lemma: a υ-structure $\Leftrightarrow$ an involution of the principal bundle P.

good $\Leftrightarrow$ has fixed points in at least one P_x.

$\Rightarrow$ P^{υ} form a G^{υ}-principal bundle &

$P = P^{\upsilon} \times_{G^{\upsilon}} G$.

Tannakian category over $\mathbb{R}$?

$\mathrm{Rep}_{\mathbb{R}}(G_{\mathbb{R}})$ is the same thing as $\mathrm{Rep}_{\mathbb{C}}(G_{\mathbb{C}})$ with semi-linear tensor isomorphism $X \mapsto \sigma X$ such that $\sigma^2 = \mathrm{id}$ & $\sigma^3 = \sigma$.

the benefit is it can be used for Cartan inv (C, σ, τ)

Pf $F : \mathrm{Rep}(G) \to \mathrm{Vect}(X)$,

$\upsilon : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ & $\mathrm{Vect}(X) \to \mathrm{Vect}(X)$,

such that $F\upsilon = \upsilon F$.

a point is an isom

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, "\ell"] \arrow[d, "\text{forget}"'] & \mathrm{Rep}(G) \arrow[d, "F"] \\ \mathrm{Vect} & \mathrm{Vect}(X) \arrow[l, "\omega_x"'] \end{tikzcd}$$

its image under ν is

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, "\upsilon"] \arrow[d] & \mathrm{Rep}(G) \arrow[r, "\ell"] & \mathrm{Rep}(G) \arrow[d, "F"] \\ \mathrm{Vect} & \mathrm{Vect} \arrow[l, "\nu"'] & \mathrm{Vect}(X) \arrow[l] \end{tikzcd}$$

To show one point can give all points, we do it locally.

If $(\omega_x, \upsilon) \simeq (\omega_G, \upsilon)$ for some point, then P_x has fixed points, they are defined smoothly, so we get G^υ - principal bundle.

The Tannakian category $\mathcal{E} = \mathcal{E}_{Dol} = \mathcal{E}_{DR}$ has υ structure for $\upsilon = \sigma$ or $\upsilon = C$.

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we can also consider the maps $\varpi_1(X, x) \to G$ or $\varpi_1^{red}(X, x) \to G$ that is compatible w/ υ, so we can study more str.

- $\upsilon = \tau$ and G^τ is a compact real form. so we can get $\varpi_1^{red}(X, x) \to G$, so that the image of $Rep(G) \in \mathcal{E}^{ss}$. then we have Harmonic structure on these vector bundles.

- $\upsilon = \sigma$ so we get $\varpi_1(X, x)^\sigma \to G^\sigma$, LHS is the real algebraic closure. So it is the same thing as representations $\pi_1(X, x) \to G^\sigma$. In this case, the principal bundle has locally constant structure.
- $\upsilon = C$, so we get an algebraic structure.

What about the Higgs structure?

C gives the involution of the Higgs bundles
$\Upsilon : P \times_G CV \xrightarrow{\sim} C(P \times_G V) = P \times_G V$ with $-\theta$.

It preserves the Higgs field $\theta = (\varphi \in G,\ \theta_\varphi \in \mathfrak{g} \otimes \Omega^1)$,
which means $\Upsilon\, \theta_{CV} = -\theta_V \Upsilon \Rightarrow C\theta_\varphi = -\theta_{C\varphi}$.

Hence when $\varphi \in G^C$, we have $\theta_\varphi \in \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is Cartan decomp.

In other word, θ is a section of $P^C \times_{G^C} \mathfrak{p} \otimes \Omega^1$.

Finally is the case $v = (C, \sigma, \tau)$.

then we get a $G^{C,\sigma,\tau}$ - principal bundle such that the corresponding G^C, G^σ, G^τ - bundle has corresponding structures.

Call a (G, C) or (G, σ) torsor is reductive if the image in $\mathcal{E}^{ss}$.

Theorem the category of (G, C, σ, τ)-torsor is equiv to reductive (G, C) - torsor or reductive (G, σ) - torsor.

Hence, for a reductive G^σ - torsor, we can get a G^c - torsor P^c with a Higgs field $\theta \in P^c \times_{G^c} \mathfrak{p} \otimes \Omega^1$.