

The number of edges in k -quasi-planar graphs*

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Abstract

A graph drawn in the plane is called k -quasi-planar if it does not contain k pairwise crossing edges. It has been conjectured for a long time that for every fixed k , the maximum number of edges of a k -quasi-planar graph with n vertices is $O(n)$. The best known upper bound is $n(\log n)^{O(\log k)}$. In the present note, we improve this bound to $(n \log n)2^{\alpha^{c_k}(n)}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for k -quasi-planar graphs in which every edge is drawn as an x -monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^{c_k^6} n \log n$.

1 Introduction

A *topological graph* is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is *simple* if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called *geometric*.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see [2, 3, 4, 6, 10, 12, 16, 21, 23]). It follows from Euler's Polyhedral Formula that every topological graph on n vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called k -quasi-planar if it can be drawn

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as a topological graph with no k pairwise crossing edges. A graph is 2-quasi-planar if and only if it is planar. According to an old conjecture (see Problem 1 in Section 9.6 of [5]), for any fixed $k \geq 2$ there exists a constant c_k such that every k -quasi-planar graph on n vertices has at most $c_k n$ edges. Agarwal, Aronov, Pach, Pollack, and Sharir [4] were the first to prove this conjecture for *simple* 3-quasi-planar graphs. Later, Pach, Radoičić, and Tóth [17] generalized the result to *all* 3-quasi-planar graphs. Ackerman [1] proved the conjecture for $k = 4$.

For larger values of k , first Pach, Shahrokhi, and Szegedy [18] showed that every simple k -quasi-planar graph on n vertices has at most $c_k n (\log n)^{2k-4}$ edges. For $k \geq 3$ and for all (not necessarily simple) k -quasi-planar graphs, Pach, Radoičić, and Tóth [17] established the upper bound $c_k n (\log n)^{4k-12}$. Plugging into these proofs the above mentioned result of Ackerman [1], for $k \geq 4$, we obtain the slightly better bounds $c_k n (\log n)^{2k-8}$ and $c_k n (\log n)^{4k-16}$, respectively. For large values of k , the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a k -quasi-planar graph on n vertices is $n(\log n)^{O(\log k)}$.

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound $O(n \log n)$. He also extended this result to *simple* topological graphs whose edges are drawn as x -monotone curves [23].

The aim of this paper is to improve the best known bound, $n(\log n)^{O(\log k)}$, on the number of edges of a k -quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as x -monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$.

Theorem 1.1. *Let $G = (V, E)$ be a k -quasi-planar simple topological graph with n vertices. Then $|E(G)| \leq (n \log n) 2^{\alpha(n)^{c_k}}$, where $\alpha(n)$ denotes the inverse of the Ackermann function and c_k is a constant that depends only on k .*

Recall that the *Ackermann* (more precisely, the *Ackermann-Péter*) function $A(n)$ is defined as follows. Let $A_1(n) = 2n$, and $A_k(n) = A_{k-1}(A_k(n-1))$ for $k = 2, 3, \dots$. In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of n two's. Now let $A(n) = A_n(n)$, and let $\alpha(n)$ be defined as $\alpha(n) = \min\{k \geq 1 : A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

Theorem 1.2. *Let $G = (V, E)$ be a k -quasi-planar (not necessarily simple) topological graph with n vertices, whose edges are drawn as x -monotone curves. Then $|E(G)| \leq 2^{ck^6} n \log n$, where c is an absolute constant.*

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport-Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr's result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on k from double exponential to single exponential.

2 Generalized Davenport-Schinzel Sequences

The sequence $u = a_1, a_2, \dots, a_m$ is called l -regular if any l consecutive terms are pairwise different. For integers $l, t \geq 2$, the sequence

$$S = s_1, s_2, \dots, s_{lt}$$

of length lt is said to be of *type up*(l, t) if the first l terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \dots = s_{i+(t-1)l}$$

for every $i, 1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c, a, b, c,$$

is a *type up*(3, 4) sequence or, in short, an *up*(3, 4) sequence. We need the following theorem of Klazar [13] on generalized Davenport-Schinzel sequences.

Theorem 2.1 (Klazar). *For $l \geq 2$ and $t \geq 3$, the length of any l -regular sequence over an n -element alphabet that does not contain a subsequence of type *up*(l, t) has length at most*

$$n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)^{lt}}.$$

For $l \geq 2$, the sequence

$$S = s_1, s_2, \dots, s_{3l-2}$$

of length $3l - 2$ is said to be of *type up-down-up*(l) if the first l terms are pairwise different and

$$s_i = s_{2l-i} = s_{(2l-2)+i}$$

for every $i, 1 \leq i \leq l$. For example,

$$a, b, c, d, c, b, a, b, c, d,$$

is an *up-down-up*(4) sequence. Valtr and Klazar [14] showed that any l -regular sequence over an n -element alphabet, which contains no subsequence of type *up-down-up*(l), has length at most $2^{lc} n$ for some constant c . This has been improved by Pettie [20], who proved the following.

Lemma 2.2 (Pettie). *For $l \geq 2$, the length of any l -regular sequence over an n -element alphabet, which contains no subsequence of type *up-down-up*(l), has length at most $2^{O(l^2)} n$.*

For more results on generalized Davenport-Schinzel sequences, see [15, 20, 19].

3 On intersection graphs of curves

In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection C of curves, no two of which intersect many times, contains a large subcollection C' such that in the partition of C' into its connected components C_1, \dots, C_t in the intersection graph of C , each component C_i has a vertex connected to all other $|C_i| - 1$ vertices.

For a graph $G = (V, E)$, a subset V_0 of the vertex set is said to be a *separator* if there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq \frac{2}{3}|V|$ such that no edge connects a vertex in V_1 to a vertex in V_2 . We need the following separator lemma for intersection graphs of curves, established in [9].

Lemma 3.1 (Fox–Pach). *There is an absolute constant c_1 such that every collection C of curves with x intersection points has a separator of size at most $c_1\sqrt{x}$.*

Call a collection C of curves in the plane *decomposable* if there is a partition $C = C_1 \cup \dots \cup C_t$ such that each C_i contains a curve which intersects all other curves in C_i , and for $i \neq j$, the curves in C_i are disjoint from the curves in C_j . The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is include here, for completeness.

Lemma 3.2. *There is an absolute constant $c > 0$ such that every collection C of $m \geq 2$ curves such that each pair of them intersect in at most t points has a decomposable subcollection of size at least $\frac{cm}{t \log m}$.*

Proof of Lemma 3.2 We prove the following stronger statement. There is an absolute constant $c > 0$ such that every collection C of $m \geq 2$ curves whose intersection graph has at least x edges, and each pair of curves intersect in at most t points, has a decomposable subcollection of size at least $\frac{cm}{t \log m} + \frac{x}{m}$. Let $c = \frac{1}{576c_1^2}$, where $c_1 \geq 1$ is the constant in Lemma 3.1. The proof is by induction on m , noting that all collections of curves with at most three elements are decomposable. Define $d = d(m, x, t) := \frac{cm}{t \log m} + \frac{x}{m}$.

Let Δ denote the maximum degree of the intersection graph of C . We have $\Delta < d - 1$. Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in C that intersect it, is decomposable and its size is at least d , and we are done. Also, $\Delta \geq 2\frac{x}{m}$, since $2\frac{x}{m}$ is the average degree of the vertices in the intersection graph of C . Hence, if $\Delta \geq 2\frac{cm}{t \log m}$, then the desired inequality holds. Thus, we may assume $\Delta < 2\frac{cm}{t \log m}$.

Applying Lemma 3.1 to the intersection graph of C , we obtain that there is a separator $V_0 \subset C$ with $|V_0| \leq c_1\sqrt{tx}$, where c_1 is the absolute constant in Lemma 3.1. That is, there is a partition $C = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2|V|/3$ such that no curve in V_1 intersects any curve in V_2 . For $i = 1, 2$, let $m_i = |V_i|$ and x_i denote the number of pairs of curves in V_i that intersect, so that

$$x_1 + x_2 \geq x - \Delta|V_0| \geq x - 2\frac{cm}{t \log m}c_1\sqrt{tx}. \quad (1)$$

As no curve in V_1 intersects any curve in V_2 , the union of a decomposable subcollection of V_1 and a decomposable subcollection of V_2 is decomposable. Thus, by the induction hypothesis, C contains decomposable subcollection of size at least

$$\begin{aligned} d(m_1, x_1, t) + d(m_2, x_2, t) &= \frac{cm_1}{t \log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{t \log m_2} + \frac{x_2}{m_2} \\ &\geq \frac{c(m_1 + m_2)}{t \log(2m/3)} + \frac{(x_1 + x_2)}{2m/3}. \end{aligned}$$

We split the rest of the proof into two cases.

Case 1. $x \geq t^{-1} \left(12c_1c\frac{m}{\log m}\right)^2$. In this case, by (1), we have $x_1 + x_2 \geq \frac{5}{6}x$ and hence there is a decomposable subcollection of size at least

$$\begin{aligned} d(m_1, x_1, t) + d(m_2, x_2, t) &\geq \frac{c(m_1 + m_2)}{t \log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{t \log m} \\ &\geq d + \frac{x}{4m} - \frac{c_1c\sqrt{tx}}{t \log m} > d, \end{aligned}$$

completing the analysis.

Case 2. $x < t^{-1} \left(12c_1 c_{\frac{m}{\log m}} \right)^2$. There is a decomposable subcollection of size at least

$$\begin{aligned} d(m_1, x_1, t) + d(m_2, x_2, t) &\geq \frac{c(m_1 + m_2)}{t \log(2m/3)} \geq \frac{c}{t} \left(m - c_1 \sqrt{tx} \right) \left(\frac{1}{\log m} + \frac{1}{2 \log^2 m} \right) \\ &\geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{2 \log^2 m} - \frac{2c_1 \sqrt{tx}}{\log m} \right) \geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{4 \log^2 m} \right) \\ &\geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{4 \log^2 m} \right) \geq \frac{cm}{t \log m} + \frac{x}{m} = d, \end{aligned}$$

where we used $c = \frac{1}{576c_1^2}$. □

4 Simple Topological Graphs

In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

Theorem 4.1. *Let $G = (V, E)$ be a k -quasi-planar simple topological graph with n vertices. Suppose that G has an edge that crosses every other edge. Then we have $|E| \leq n \cdot 2^{\alpha(n)^{c'_k}}$, where $\alpha(n)$ denotes the inverse Ackermann function and c'_k is a constant that depends only on k .*

Proof of Theorem 4.1. Let $k \geq 5$ and $c'_k = 40 \cdot 2^{k^2+2k}$. To simplify the presentation, we do not make any attempt to optimize the value of c'_k . Label the vertices of G from 1 to n , i.e., let $V = \{1, 2, \dots, n\}$. Let $e = uv$ be the edge that crosses every other edge in G . Note that $d(u) = d(v) = 1$.

Let E' denote the set of edges that cross e . Suppose without loss of generality that no two of elements of E' cross e at the same point. Let $e_1, e_2, \dots, e_{|E'|}$ denote the edges in E' listed in the order of their intersection points with e from u to v . We create two sequences of vertices $S_1 = p_1, p_2, \dots, p_{|E'|}$ and $S_2 = q_1, q_2, \dots, q_{|E'|} \subset V$, as follows. For each $e_i \in E'$, as we move along edge e from u to v and arrive at the intersection point with e_i , we turn left and move along edge e_i until we reach its endpoint u_i . Then we set $p_i = u_i$. Likewise, as we move along edge e from u to v and arrive at edge e_i , we turn right and move along edge e_i until we reach its other endpoint w_i . Then we set $q_i = w_i$. Thus, S_1 and S_2 are sequences of length $|E'|$ over the alphabet $\{1, 2, \dots, n\}$. See Figure 1 for a small example.

We need two lemmas. The first one is due to Valtr [23].

Lemma 4.2 (Valtr). *For $l \geq 1$, at least one of the sequences S_1, S_2 defined above contains an l -regular subsequence of length at least $|E'|/(4l)$.* □

Since each edge in E' crosses e exactly once, the proof of Lemma 4.2 can be copied almost *verbatim* from the proof of Lemma 4 in [23] and is left to the reader as an exercise.

For the rest of this section, we set $l = 2^{k^2+k}$ and $t = 2^k$.

Lemma 4.3. *Neither of the sequences S_1 and S_2 has a subsequence of type $up(l, t)$.*

Proof. By symmetry, it suffices to show that S_1 does not contain a subsequence of type $up(l, t)$. The argument is by contradiction. We will prove by induction on k that the existence of such a

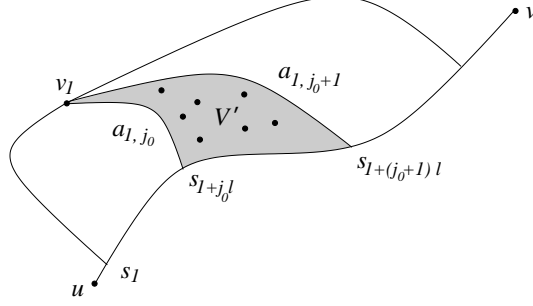


Figure 2: Vertices of V' lie in the region enclosed by $a_{1,j_0}, a_{1,j_0+1}, e$.

$$a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.$$

Since $a_{i,j_0+j_1}, a_{1,j_0+1}$, and e pairwise intersect at precisely one point, γ is well defined. We define points $x = a_{i,j_0+j_1} \cap a_{1,j_0+1}$ and $y = a_{1,j_0+1} \cap e$, and orient γ in the direction from x to y along γ .

In view of the fact that a_{i,j_0+j_1} intersects a_{1,j_0+1} , the vertex v_i must lie to the right of γ . Moreover, since the arc from x to y along a_{1,j_0+1} is a subset of γ , the points corresponding to the subsequence

$$S' = \{s_q \in S \mid 2 + (j_0 + 1)l \leq q \leq (i - 1) + (j_0 + j_1)l\}$$

must lie to the left of γ . Hence, γ separates vertex v_i and the points of S' . Therefore, using again that G is simple, each arc from A must cross a_{1,j_0+1} (these arcs cannot cross a_{i,j_0+j_1}). See Figure 4.

By the same argument, if the arc a_{i,j_0-j_1} crosses a_{1,j_0} for $j_1 \geq 1$, then the arcs emanating from v_i ,

$$a_{i,j_0-1}, a_{i,j_0-2}, \dots, a_{i,j_0-j_1+1}$$

must also cross a_{1,j_0} . Since $a_{i,j_0+t/2} = a_{i,j_0-t/2}$, we have the following observation.

Observation 4.4. *For half of the vertices $v_i \in V'$, the arcs emanating from v_i satisfy*

1. $a_{i,j_0+1}, a_{i,j_0+2}, \dots, a_{i,j_0+t/2}$ all cross a_{1,j_0+1} , or
2. $a_{i,j_0-1}, a_{i,j_0-2}, \dots, a_{i,j_0-t/2}$ all cross a_{1,j_0} .

Since $t/2 = 2^{k-1}$ and

$$\frac{|V'|}{2} \geq \frac{l-1}{2t} = \frac{2^{k^2+k}-1}{2 \cdot 2^k} \geq 2^{(k-1)^2+(k-1)},$$

by Observation 4.4, we obtain a $up(2^{(k-1)^2+(k-1)}, 2^{k-1})$ sequence such that the corresponding arcs all cross either a_{1,j_0} or a_{1,j_0+1} . By the induction hypothesis, it follows that there exist k pairwise crossing edges. \square

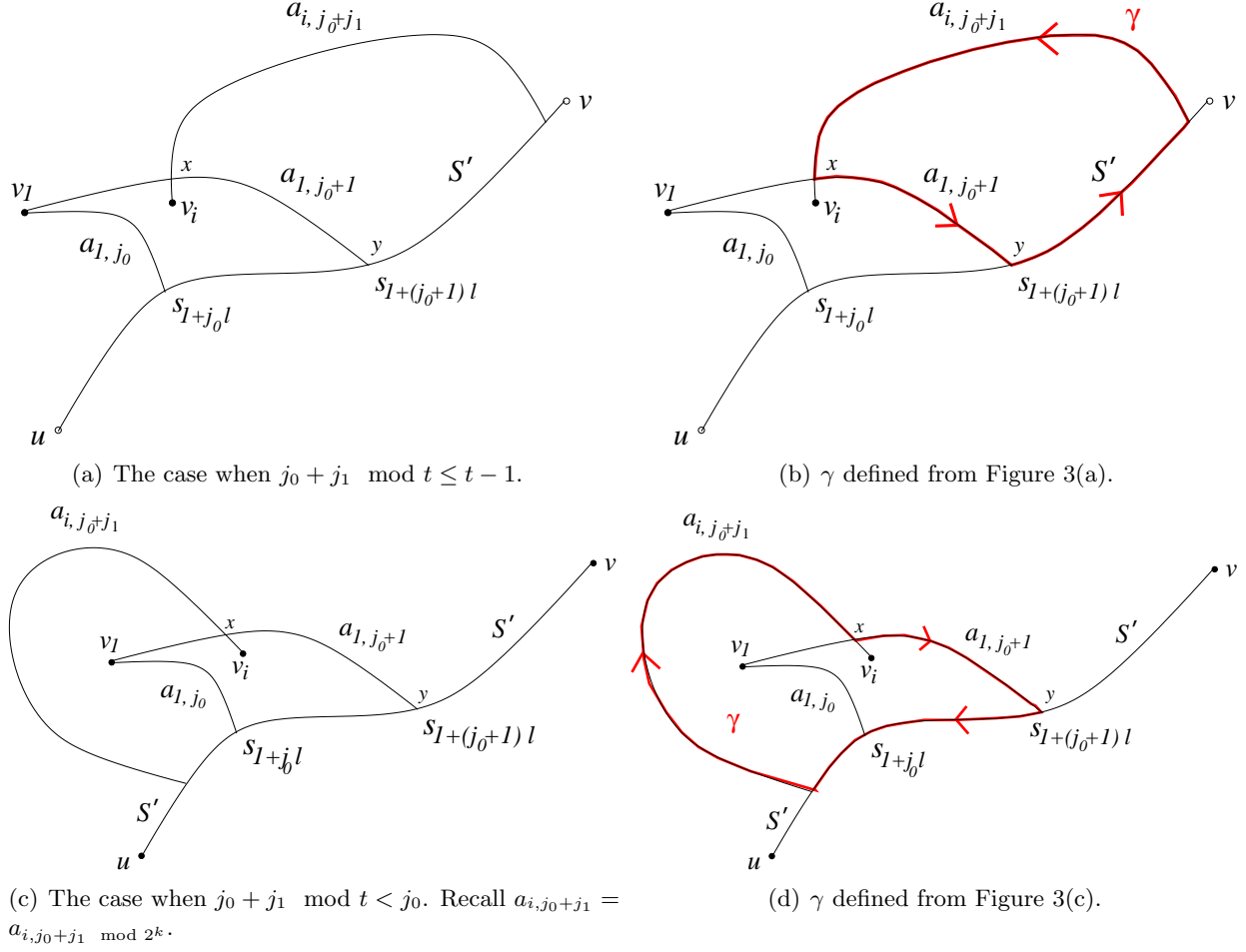


Figure 3: Defining γ and its orientation.

Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, S_1 contains an l -regular subsequence of length $|E'|/(4l)$. By Theorem 2.1 and Lemma 4.3, this subsequence has length at most

$$n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)lt}.$$

Therefore, we have

$$\frac{|E'|}{4 \cdot l} \leq n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)lt},$$

which implies

$$|E'| \leq 4 \cdot n \cdot l^2 2^{(lt-3)} \cdot (10l)^{10\alpha(n)lt}.$$

Since $c'_k = 40 \cdot lt = 40 \cdot 2^{k^2+2k}$, $\alpha(n) \geq 2$ and $k \geq 5$, we have

$$|E| = |E'| + 1 \leq n \cdot 2^{\alpha_{c'_k}(n)},$$

which completes the proof of Lemma 4.1. \square

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $G = (V, E)$ be a k -quasi-planar simple topological graph on n vertices. By Lemma 3.2, there is a subset $E' \subset E$ such that $|E'| \geq c|E|/\log|E|$, where c is an absolute constant and E' is decomposable. Hence, there is a partition

$$E' = E_1 \cup E_2 \cup \dots \cup E_t$$

such that each E_i has an edge e_i that intersects every other edge in E_i , and for $i \neq j$, the edges in E_i are disjoint from the edges in E_j . Let V_i denote the set of vertices that are the endpoints of the edges in E_i , and let $n_i = |V_i|$. By Lemma 4.1, we have

$$|E_i| \leq n_i 2^{\alpha_{c_k}'(n_i)} + 2n_i,$$

where the $2n_i$ term accounts for the edges that share a vertex with e_i . Hence,

$$\frac{c|E|}{\log|E|} \leq \sum_{i=1}^t n_i 2^{\alpha_{c_k}'(n_i)} + 2n_i \leq n 2^{\alpha_{c_k}'(n)} + 2n,$$

Therefore, we obtain

$$|E| \leq (n \log n) 2^{\alpha_{c_k}(n)},$$

for a sufficiently large constant c_k . \square

5 x -Monotone Topological Graphs

The aim of this section is to prove Theorem 1.2.

Proof of Theorem 1.2. For $k \geq 2$, let $g_k(n)$ be the maximum number of edges in a k -quasi-planar topological graph whose edges are drawn as x -monotone curves. We will prove by induction on n that

$$g_k(n) \leq 2^{ck^6} n \log n$$

where c is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let $G = (V, E)$ be a k -quasi-planar topological graph whose edges are drawn as x -monotone curves, and let the vertices be labeled $1, 2, \dots, n$. Let L be a vertical line that partitions the vertices into two parts, V_1 and V_2 , such that $|V_1| = \lfloor n/2 \rfloor$ vertices lie to the left of L , and $|V_2| = \lceil n/2 \rceil$ vertices lie to the right of L . Furthermore, let E_1 denote the set of edges induced by V_1 , let E_2 denote the set of edges induced by V_2 , and let E' be the set of edges that intersect L . Clearly, we have

$$|E_1| \leq g_k(\lfloor n/2 \rfloor) \quad \text{and} \quad |E_2| \leq g_k(\lceil n/2 \rceil).$$

It suffices that show that

$$|E'| \leq 2^{ck^6/2} n, \tag{2}$$

since this would imply

$$g_k(n) \leq g_k(\lfloor n/2 \rfloor) + g_k(\lceil n/2 \rceil) + 2^{ck^6/2} n \leq 2^{ck^6} n \log n.$$

In the rest of the proof, we only consider the edges belonging to E' . For each vertex $v_i \in V_1$, consider the graph G_i whose vertices are the edges with v_i as a left endpoint, and two vertices in G_i are adjacent if the corresponding edges cross at some point to the left of L . Since G_i is an *incomparability graph* (see [7], [11]) and does not contain a clique of size k , G_i contains an independent set of size $|E(G_i)|/(k-1)$. We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to v_i . After repeating this process for all vertices in V_1 , we are left with at least $|E'|/(k-1)$ edges.

Now we continue this process on the other side. For each vertex $v_j \in V_2$, consider the graph G_j whose vertices are the edges with v_j as a right endpoint, and two vertices in G_j are adjacent if the corresponding edges cross at some point to the right of L . Since G_j is an incomparability graph and does not contain a clique of size k , G_j contains an independent set of size $|E(G_j)|/(k-1)$. We keep all edges that corresponds to this independent set, and discard all other edges incident to v_j . After repeating this process for all vertices in V_2 , we are left with at least $|E'|/(k-1)^2$ edges.

We order the remaining edges e_1, e_2, \dots, e_m in the order in which they intersect L from bottom to top. (We assume without loss of generality that any two intersection points are distinct.) Define two sequences, $S_1 = p_1, p_2, \dots, p_m$ and $S_2 = q_1, q_2, \dots, q_m$, such that p_i denotes the left endpoint of edge e_i and q_i denotes the right endpoint of e_i . We need the following lemma.

Lemma 5.1. *Neither of the sequences S_1 and S_2 has subsequence of type up-down-up($k^3 + 2$).*

Proof. By symmetry, it suffices to show that S_1 does not have a subsequence of type up-down-up($k^3 + 2$). Suppose for contradiction that S_1 does contain such a subsequence. Then there is a sequence

$$S = s_1, s_2, \dots, s_{2(k^3+2)-2}$$

such that the integers s_1, \dots, s_{k^3+2} are pairwise distinct and

$$s_i = s_{2(k^3+2)-i} = s_{2(k^3+2)-2+i}$$

for $i = 1, 2, \dots, k^3 + 2$.

For each $i \in \{1, 2, \dots, k^3 + 2\}$, let $v_i \in V_1$ denote the label (vertex) of s_i and let x_i denote the x -coordinate of the vertex v_i . Moreover, let a_i be the arc emanating from vertex v_i to the point on L that corresponds to $s_{2(k^3+2)-i}$. Let $A = \{a_2, a_3, \dots, a_{k^3+1}\}$. Note that the arcs in A are enumerated downwards with respect to their intersection points with L , and they correspond to the elements of the “middle” section of the up-down-up sequence. We define two partial orders on A as follows.

$$a_i \prec_1 a_j \quad \text{if } i < j, \quad x_i < x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect,}$$

$$a_i \prec_2 a_j \quad \text{if } i < j, \quad x_i > x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect.}$$

Clearly, \prec_1 and \prec_2 are partial orders. If two arcs are not comparable by either \prec_1 or \prec_2 , then they must cross. Since G does not contain k pairwise crossing edges, by Dilworth’s theorem, there exist k arcs $\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ such that they are pairwise comparable by either \prec_1 or \prec_2 . Now the proof falls into two cases.

Case 1. Suppose that $a_{i_1} \prec_1 a_{i_2} \prec_1 \dots \prec_1 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ to the points corresponding to $s_{2(k^3+2)-2+i_1}, s_{2(k^3+2)-2+i_2}, \dots, s_{2(k^3+2)-2+i_k}$ are pairwise crossing. See Figure 4.

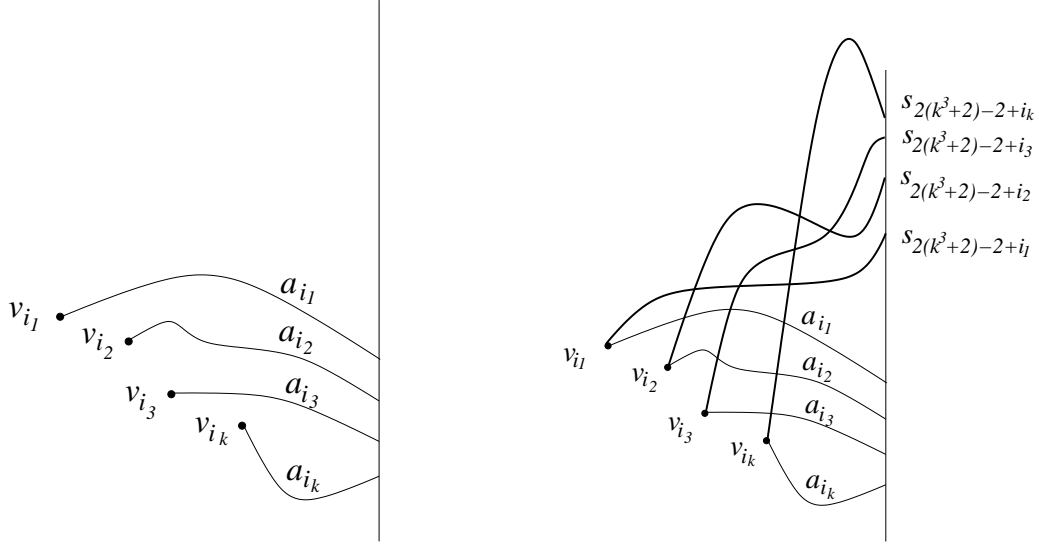


Figure 4: Case 1.

Case 2. Suppose that $a_{i_1} \prec_2 a_{i_2} \prec_2 \dots \prec_2 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ to the points corresponding to $s_{i_1}, s_{i_2}, \dots, s_{i_k}$ are pairwise crossing. See Figure 5. □

We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, S_1 , say, contains a $(k^3 + 2)$ -regular subsequence of length

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2}.$$

By Lemmas 2.2 and 5.1, this subsequence has length at most $2^{c'k^6}n$, where c' is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2} \leq 2^{c'k^6}n,$$

which implies that

$$|E'| \leq 4k^5 2^{c'k^6}n \leq 2^{ck^6/2}n$$

for a sufficiently large absolute constant c . □

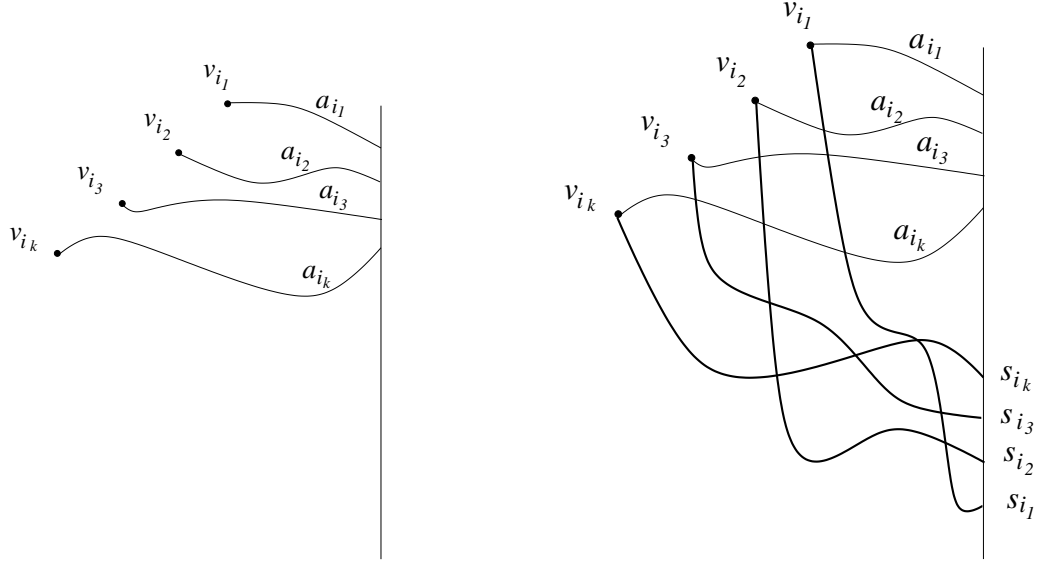


Figure 5: Case 2.

References

- [1] E. Ackerman, On the Maximum Number of Edges in Topological Graphs with No Four Pairwise Crossing Edges. *Discrete Comput. Geom.* **41** (2009), 365–375.
- [2] E. Ackerman, J. Fox, J. Pach, A. Suk, On Grids in Topological Graphs. *In Proceedings of the 25th Annual ACM Symposium on Computational Geometry*, 2009, 403–412.
- [3] E. Ackerman, G. Tardos, On the Maximum Number of Edges in Quasi-Planar Graphs. *J. Comb. Theory Ser. A* **114** (2007), 563–571.
- [4] P.K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir, Quasi-Planar Graphs Have a Linear Number of Edges. *Combinatorica* **17** (1997), 1–9.
- [5] P. Brass, W. Moser, and J. Pach, “Research Problems in Discrete Geometry.” Springer-Verlag, Berlin, 2005.
- [6] V. Capoleas and J. Pach, A Turán-Type Theorem on Chords of a Convex Polygon. *J. Combinatorial Theory, Series B* **56** (1992), 9–15.
- [7] R.P. Dilworth, A Decomposition Theorem for Partially Ordered Sets. *Annals of Math.* **51** (1950), 161–166.
- [8] J. Fox and J. Pach, Erdős–Hajnal-Type Results on Intersection Patterns of Geometric Objects. In “Horizons of Combinatorics” (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer (2008), 79–103.
- [9] J. Fox and J. Pach, Separator Theorems and Turán-Type Results for Planar Intersection Graphs. *Advances in Math.* **219** (2008), 1070–1080.

- [10] J. Fox and J. Pach, Coloring K_k -Free Intersection Graphs of Geometric Objects in the Plane. In *Proceedings of the 24th Annual ACM Symposium on Computational Geometry*, 2008, 346–354.
- [11] J. Fox, J. Pach, and Cs. Tóth, Intersection Patterns of Curves. *Journal of the London Mathematical Society* **83** (2011), 389–406.
- [12] R. Fulek and A. Suk, Disjoint Crossing Families, In “Thirty Essays in Geometric Graph Theory” (J. Pach, ed), Springer, Berlin, 2012, to appear. See also arXiv:1004.2850, 2010.
- [13] M. Klazar, A General Upper Bound in Extremal Theory of Sequences. *Commentationes Mathematicae Universitatis Carolinae* **33** (1992), 737–746.
- [14] M. Klazar and P. Valtr, Generalized Davenport-Schinzel Sequences. *Combinatorica* **14** (1994), 463–476.
- [15] G. Nivasch, Improved Bounds and New Techniques for Davenport–Schinzel Sequences and Their Generalizations. *J. ACM* **57** (2010), 1–44.
- [16] J. Pach, R. Pinchasi, M. Sharir, and G. Tóth, Topological Graphs with No Large Grids. *Graphs & Comb.* **21** (2005), 355–364.
- [17] J. Pach, R. Radoičić, and G. Tóth, Relaxing Planarity for Topological Graphs. *Lecture Notes in Comput. Sci.* **2866** (2003), Springer, 221–232.
- [18] J. Pach, F. Shahrokhi, and M. Szegedy, Applications of the Crossing Number. *J. Graph Theory* **22** (1996), 239–243.
- [19] S. Pettie, Generalized Davenport-Schinzel Sequences and Their 0-1 Matrix Counterparts. *J. Comb. Theory Ser. A* **118** (2011) 1863–1895.
- [20] S. Pettie, On the Structure and Composition of Forbidden Sequences, with Geometric Applications. In *Proceedings of the 27th Annual ACM Symposium on Computational Geometry*, 2011, 370–379.
- [21] G. Tardos and G. Tóth, Crossing Stars in Topological Graphs. *SIAM J. Discret. Math.* **21** (2007), 737–749.
- [22] P. Valtr, On Geometric Graphs with No k Pairwise Parallel Edges. *Discrete Comput. Geom.* **19** (1997), 461–469.
- [23] P. Valtr, Graph Drawings with No k Pairwise Crossing Edges. In *Proceedings of the 5th International Symposium on Graph Drawing*, 1997, 205–218.