

Induced Ramsey-type theorems

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Abstract

We present a unified approach to proving Ramsey-type theorems for graphs with a forbidden induced subgraph which can be used to extend and improve the earlier results of Rödl, Erdős-Hajnal, Prömel-Rödl, Nikiforov, Chung-Graham, and Łuczak-Rödl. The proofs are based on a simple lemma (generalizing one by Graham, Rödl, and Ruciński) that can be used as a replacement for Szemerédi's regularity lemma, thereby giving much better bounds. The same approach can be also used to show that pseudo-random graphs have strong induced Ramsey properties. This leads to explicit constructions for upper bounds on various induced Ramsey numbers.

1 Background and Introduction

Ramsey theory refers to a large body of deep results in mathematics concerning partitions of large structures. Its underlying philosophy is captured succinctly by the statement that “In a large system, complete disorder is impossible.” This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to graph theory and combinatorics but also to logic, analysis, number theory, and geometry. Since the publication of the seminal paper of Ramsey [43] in 1930, this subject has grown with increasing vitality, and is currently among the most active areas in combinatorics.

For a graph H , the *Ramsey number* $r(H)$ is the least positive integer n such that every two-coloring of the edges of the complete graph K_n on n vertices contains a monochromatic copy of H . Ramsey's theorem states that $r(H)$ exists for every graph H . A classical result of Erdős and Szekeres [26], which is a quantitative version of Ramsey's theorem, implies that $r(K_k) \leq 2^{2k}$ for every positive integer k . Erdős [19] showed using probabilistic arguments that $r(K_k) > 2^{k/2}$ for $k > 2$. Over the last sixty years, there has been several improvements on the lower and upper bounds of $r(K_k)$, the most recent by Conlon [15]. However, despite efforts by various researchers, the constant factors in the exponents of these bounds remain the same.

A subset of vertices of a graph is *homogeneous* if it is either an independent set (empty subgraph) or a clique (complete subgraph). For a graph G , denote by $\text{hom}(G)$ the size of the largest homogeneous subset of vertices of G . A restatement of the Erdős-Szekeres result is that every graph G on n vertices satisfies $\text{hom}(G) \geq \frac{1}{2} \log n$, while the Erdős result says that for each $n \geq 2$ there is a graph G on n

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vertices with $\text{hom}(G) \leq 2 \log n$. (Here, and throughout the paper, all logarithms are base 2.) The only known proofs of the existence of *Ramsey graphs*, i.e., graphs for which $\text{hom}(G) = O(\log n)$, come from various models of random graphs with edge density bounded away from 0 and 1. This supports the belief that any graph with small $\text{hom}(G)$ looks ‘random’ in one sense or another. There are now several results which show that Ramsey graphs have random-like properties.

A graph H is an *induced subgraph* of a graph G if $V(H) \subset V(G)$ and two vertices of H are adjacent if and only if they are adjacent in G . A graph is *k-universal* if it contains all graphs on at most k vertices as induced subgraphs. A basic property of large random graphs is that they almost surely are *k-universal*. There is a general belief that graphs which are not *k-universal* are highly structured. In particular, they should contain a homogeneous subset which is much larger than that guaranteed by the Erdős-Szekeres bound for general graphs.

In the early 1970’s, an important generalization of Ramsey’s theorem, known as the Induced Ramsey Theorem, was discovered independently by Deuber [16], Erdős, Hajnal, and Posa [25], and Rödl [44]. It states that for every graph H there is a graph G such that in every 2-edge-coloring of G there is an induced copy of H whose edges are monochromatic. The least positive integer n for which there is an n -vertex graph with this property is called the *induced Ramsey number* $r_{\text{ind}}(H)$. All of the early proofs of the Induced Ramsey Theorem give enormous upper bounds on $r_{\text{ind}}(H)$. It is still a major open problem to prove good bounds on induced Ramsey numbers. Ideally, we would like to understand conditions for a graph G to have the property that in every two-coloring of the edges of G , there is an induced copy of graph H that is monochromatic.

In this paper, we present a unified approach to proving Ramsey-type theorems for graphs with a forbidden induced subgraph which can be used to extend and improve results of various researchers. The same approach is also used to prove new bounds on induced Ramsey numbers. In the few subsequent sections we present in full detail our theorems and compare them with previously obtained results.

1.1 Ramsey properties of H -free graphs

As we already mentioned, there are several results (see, e.g., [27, 47, 2, 10]) which indicate that Ramsey graphs, graphs G with relatively small $\text{hom}(G)$, have random-like properties. The first advance in this area was made by Erdős and Szemerédi [27], who showed that the Erdős-Szekeres bound $\text{hom}(G) \geq \frac{1}{2} \log n$ can be improved for graphs which are very sparse or very dense. The edge density of a graph G is the fraction of pairs of distinct vertices of G that are edges. The Erdős-Szemerédi theorem states that there is an absolute positive constant c such that $\text{hom}(G) \geq \frac{c \log n}{\epsilon \log \frac{1}{\epsilon}}$ for every graph G on n vertices with edge density $\epsilon \in (0, 1/2)$. This result shows that the Erdős-Szekeres bound can be significantly improved for graphs that contain a large subset of vertices that is very sparse or very dense.

Rödl [45] proved that if a graph is not *k-universal* with k fixed, then it contains a linear-sized induced subgraph that is very sparse or very dense. A graph is called *H-free* if it does not contain H as an induced subgraph. More precisely, Rödl’s theorem says that for each graph H and $\epsilon \in (0, 1/2)$, there is a positive constant $\delta(\epsilon, H)$ such that every H -free graph on n vertices contains an induced subgraph on at least $\delta(\epsilon, H)n$ vertices with edge density either at most ϵ or at least $1 - \epsilon$. Together with the theorem of Erdős and Szemerédi, it shows that the Erdős-Szekeres bound can be improved

by any constant factor for any family of graphs that have a forbidden induced subgraph.

Rödl's proof uses Szemerédi's regularity lemma [48], a powerful tool in graph theory, which was introduced by Szemerédi in his celebrated proof of the Erdős-Turán conjecture on long arithmetic progressions in dense subsets of the integers. The regularity lemma roughly says that every large graph can be partitioned into a small number of parts such that the bipartite subgraph between almost every pair of parts is random-like. To properly state the regularity lemma requires some terminology. The edge density $d(X, Y)$ between two subsets of vertices of a graph G is the fraction of pairs $(x, y) \in X \times Y$ that are edges of G , i.e., $d(X, Y) = \frac{e(X, Y)}{|X||Y|}$, where $e(X, Y)$ is the number of edges with one endpoint in X and the other in Y . A pair (X, Y) of vertex sets is called ϵ -regular if for every $X' \subset X$ and $Y' \subset Y$ with $|X'| > \epsilon|X|$ and $|Y'| > \epsilon|Y|$ we have $|d(X', Y') - d(X, Y)| < \epsilon$. A partition $V = \bigcup_{i=1}^k V_i$ is called *equitable* if $||V_i| - |V_j|| \leq 1$ for all i, j .

Szemerédi's regularity lemma [48] states that for each $\epsilon > 0$, there is a positive integer $M(\epsilon)$ such that the vertices of any graph G can be equitably partitioned $V(G) = \bigcup_{i=1}^k V_i$ into k subsets with $\epsilon^{-1} \leq k \leq M(\epsilon)$ satisfying that all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ -regular. For more background on the regularity lemma, see the excellent survey by Komlós and Simonovits [37].

In the regularity lemma, $M(\epsilon)$ can be taken to be a tower of 2's of height proportional to ϵ^{-5} . On the other hand, Gowers [31] proved a lower bound on $M(\epsilon)$ which is a tower of 2's of height proportional to $\epsilon^{-\frac{1}{16}}$. His result demonstrates that $M(\epsilon)$ is inherently large as a function of ϵ^{-1} . Unfortunately, this implies that the bounds obtained by applications of the regularity lemma are often quite poor. In particular, this is a weakness of the bound on $\delta(\epsilon, H)$ given by Rödl's proof of his theorem. It is therefore desirable to find a new proof of Rödl's theorem that does not use the regularity lemma. The following theorem does just that, giving a much better bound on $\delta(\epsilon, H)$. Its proof works as well in a multicolor setting (see concluding remarks).

Theorem 1.1 *There is a constant c such that for each $\epsilon \in (0, 1/2)$ and graph H on $k \geq 2$ vertices, every H -free graph on n vertices contains an induced subgraph on at least $2^{-ck(\log \frac{1}{\epsilon})^2} n$ vertices with edge density either at most ϵ or at least $1 - \epsilon$.*

Nikiforov [41] recently strengthened Rödl's theorem by proving that for each $\epsilon > 0$ and graph H of order k , there are positive constants $\kappa = \kappa(\epsilon, H)$ and $C = C(\epsilon, H)$ such that for every graph $G = (V, E)$ that contains at most $\kappa|V|^k$ induced copies of H , there is an equitable partition $V = \bigcup_{i=1}^C V_i$ of the vertex set such that the edge density in each V_i ($i \geq 1$) is at most ϵ or at least $1 - \epsilon$. Using the same technique as the proof of Theorem 1.1, we give a new proof of this result without using the regularity lemma, thereby solving the main open problem posed in [41].

Erdős and Hajnal [23] gave a significant improvement on the Erdős-Szekeres bound on the size of the largest homogeneous set in H -free graphs. They proved that for every graph H there is a positive constant $c(H)$ such that $\text{hom}(G) \geq 2^{c(H)\sqrt{\log n}}$ for all H -free graphs G on n vertices. Erdős and Hajnal further conjectured that every such G contains a complete or empty subgraph of order $n^{c(H)}$. This beautiful problem has received increasing attention by various researchers, and was also featured by Gowers [32] in his list of problems at the turn of the century. For various partial results on the Erdős-Hajnal conjecture see, e.g., [4, 24, 29, 3, 28, 39, 12] and their references.

Recall that a graph is k -universal if it contains all graphs on at most k vertices as induced subgraphs. Note that the Erdős-Hajnal bound, in particular, implies that, for every fixed k , sufficiently large

Ramsey graphs are k -universal. This was extended further by Prömel and Rödl [42], who obtained an asymptotically best possible result. They proved that if $\text{hom}(G) \leq c_1 \log n$ then G is $c_2 \log n$ -universal for some constant c_2 which depends on c_1 .

Let $\text{hom}(n, k)$ be the largest positive integer such that every graph G on n vertices is k -universal or satisfies $\text{hom}(G) \geq \text{hom}(n, k)$. The Erdős-Hajnal theorem and the Prömel-Rödl theorem both say that $\text{hom}(n, k)$ is large for fixed or slowly growing k . Indeed, from the first theorem it follows that for fixed k there is $c(k) > 0$ such that $\text{hom}(n, k) \geq 2^{c(k)\sqrt{\log n}}$, while the second theorem says that for each c_1 there is $c_2 > 0$ such that $\text{hom}(n, c_2 \log n) \geq c_1 \log n$. One would naturally like to have a general lower bound on $\text{hom}(n, k)$ that implies both the Erdős-Hajnal and Prömel-Rödl results. This is done in the following theorem.

Theorem 1.2 *There are positive constants c_3 and c_4 such that for all n, k , every graph on n vertices is k -universal or satisfies $\text{hom}(G) \geq c_3 2^{c_4 \sqrt{\frac{\log n}{k}}} \log n$.*

Theorem 1.1 can be also used to answer a question of Chung and Graham [13], which was motivated by the study of quasirandom graphs. Given a fixed graph H , it is well known that a typical graph on n vertices contains many induced copies of H as n becomes large. Therefore if a large graph G contains no induced copy of H , its edge distribution should deviate from “typical” in a rather strong way. This intuition was made rigorous in [13], where the authors proved that if a graph G on n vertices is not k -universal, then there is a subset S of $\lfloor \frac{n}{2} \rfloor$ vertices of G such that $|e(S) - \frac{1}{16}n^2| > 2^{-2(k^2+27)}n^2$. For positive integers k and n , let $D(k, n)$ denote the largest integer such that every graph G on n vertices that is not k -universal contains a subset S of vertices of size $\lfloor \frac{n}{2} \rfloor$ with $|e(S) - \frac{1}{16}n^2| > D(k, n)$. Chung and Graham asked whether their lower bound on $D(k, n)$ can be substantially improved, e.g., replaced by $c^{-k}n^2$. Using Theorem 1.1 this can be easily done as follows.

A lemma of Erdős, Goldberg, Pach, and Spencer [22] implies that if a graph on n vertices has a subset R that deviates by D edges from having edge density $1/2$, then there is a subset S of size $\lfloor n/2 \rfloor$ that deviates by at least a constant times D edges from having edge density $1/2$. By Theorem 1.1 with $\epsilon = 1/4$, there is a positive constant C such that every graph on n vertices that is not k -universal has a subset R of size at least $C^{-k}n$ with edge density at most $1/4$ or at least $3/4$. This R deviates from having edge density $1/2$ by at least

$$\frac{1}{4} \binom{|R|}{2} \geq \frac{1}{16} |R|^2 \geq \frac{1}{16} C^{-2k} n^2$$

edges. Thus, the above mentioned lemma from [22] implies that there is an absolute constant c such that every graph G on n vertices which is not k -universal contains a subset S of size $\lfloor n/2 \rfloor$ with $|e(S) - \frac{n^2}{16}| > c^{-k}n^2$. Chung and Graham also ask for non-trivial upper bounds on $D(k, n)$. In this direction, we show that there are K_k -free graphs on n vertices for which $|e(S) - \frac{1}{16}n^2| = O(2^{-k/4}n^2)$ holds for every subset S of $\lfloor \frac{n}{2} \rfloor$ vertices of G . Together with the lower bound it determines the asymptotic behavior of $D(k, n)$ and shows that there are constants $c_1, c_2 > 1$ such that $c_1^{-k}n^2 < D(k, n) < c_2^{-k}n^2$ holds for all positive integers k and n . This completely answers the questions of Chung and Graham.

Moreover, we can obtain a more precise result about the relation between the number of induced copies of a fixed graph H in a large graph G and the edge distribution of G . In their celebrated paper,

Chung, Graham, and Wilson [14] introduced a large collection of equivalent graph properties shared by almost all graphs which are called *quasirandom*. For a graph $G = (V, E)$ on n vertices, two of these properties are

\mathbf{P}_1 : For each subset $S \subset V$,

$$e(S) = \frac{1}{4}|S|^2 + o(n^2).$$

\mathbf{P}_2 : For every fixed graph H with k vertices, the number of labeled induced copies of H in G is

$$(1 + o(1))n^k 2^{-\binom{k}{2}}.$$

So one can ask naturally, by how much does a graph deviate from \mathbf{P}_1 assuming a deviation from \mathbf{P}_2 ? The following theorem answers this question.

Theorem 1.3 *Let H be a graph with k vertices and $G = (V, E)$ be a graph with n vertices and at most $(1 - \epsilon)2^{-\binom{k}{2}}n^k$ labeled induced copies of H . Then there is a subset $S \subset V$ with $|S| = \lfloor n/2 \rfloor$ and $|e(S) - \frac{n^2}{16}| \geq \epsilon c^{-k}n^2$, where $c > 1$ is an absolute constant.*

The proof of Theorem 1.3 can be easily adjusted if we replace the “at most” with “at least” and the $(1 - \epsilon)$ factor by $(1 + \epsilon)$. Note that this theorem answers the original question of Chung and Graham in a very strong sense.

1.2 Induced Ramsey numbers

Recall that the induced Ramsey number $r_{\text{ind}}(H)$ is the minimum n for which there is a graph G with n vertices such that for every 2-edge-coloring of G , one can find an induced copy of H in G whose edges are monochromatic. One of the fundamental results in graph Ramsey theory (see chapter 9.3 of [17]), the Induced Ramsey Theorem, says that $r_{\text{ind}}(H)$ exists for every graph H . Rödl [45] noted that a relatively simple proof of the theorem follows from a simple application of his result discussed in the previous section. However, all of the early proofs of the Induced Ramsey Theorem give poor upper bounds on $r_{\text{ind}}(H)$.

Since these early proofs, there has been a considerable amount of research on induced Ramsey numbers. Erdős [21] conjectured that there is a constant c such that every graph H on k vertices satisfies $r_{\text{ind}}(H) \leq 2^{ck}$. Erdős and Hajnal [20] proved that $r_{\text{ind}}(H) \leq 2^{2^{k^{1+o(1)}}}$ holds for every graph H on k vertices. Kohayakawa, Prömel, and Rödl [36] improved this bound substantially and showed that if a graph H has k vertices and chromatic number χ , then $r_{\text{ind}}(H) \leq k^{ck(\log \chi)}$, where c is a universal constant. In particular, their result implies an upper bound of $2^{ck(\log k)^2}$ on the induced Ramsey number of any graph on k vertices. In their proof, the graph G which gives this bound is randomly constructed using projective planes.

There are several known results that provide upper bounds on induced Ramsey numbers for sparse graphs. For example, Beck [8] studied the case when H is a tree; Haxell, Kohayakawa, and Łuczak [35] proved that the cycle of length k has induced Ramsey number linear in k ; and, settling a conjecture of Trotter, Łuczak and Rödl [40] showed that the induced Ramsey number of a graph with bounded degree is at most polynomial in the number of its vertices. More precisely, they proved that for every integer d , there is a constant c_d such that every graph H on k vertices and maximum degree at most

d satisfies $r_{\text{ind}}(H) \leq k^{c_d}$. Their proof, which also uses random graphs, gives an upper bound on c_d that is a tower of 2's of height proportional to d^2 .

As noted by Schaefer and Shah [46], all known proofs of the Induced Ramsey Theorem either rely on taking G to be an appropriately chosen random graph or give a poor upper bound on $r_{\text{ind}}(H)$. However, often in combinatorics, explicit constructions are desirable in addition to existence proofs given by the probabilistic method. For example, one of the most famous such problems was posed by Erdős [5], who asked for the explicit construction of a graph on n vertices without a complete or empty subgraph of order $c \log n$. Over the years, this intriguing problem and its bipartite variant has drawn a lot of attention by various researches (see, e.g., [30, 1, 6, 9, 7]), but, despite these efforts, it is still open. Similarly, one would like to have an explicit construction for the Induced Ramsey Theorem. We obtain such a construction using pseudo-random graphs.

The *random graph* $G(n, p)$ is the probability space of all labeled graphs on n vertices, where every edge appears randomly and independently with probability p . An important property of $G(n, p)$ is that, with high probability, between any two large subsets of vertices A and B , the edge density $d(A, B) = \frac{e(A, B)}{|A||B|}$ is approximately p . This observation is one of the motivations for the following useful definition. A graph $G = (V, E)$ is (p, λ) -*pseudo-random* if the following inequality holds for all subsets $A, B \subset V$:

$$|d(A, B) - p| \leq \frac{\lambda}{\sqrt{|A||B|}}.$$

It is easy to show that if $p < 0.99$, then with high probability, the random graph $G(n, p)$ is (p, λ) -pseudo-random with $\lambda = O(\sqrt{pn})$. Moreover, there are also many explicit constructions of pseudo-random graphs which can be obtained using the following fact. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of a graph G . An (n, d, λ) -*graph* is a d -regular graph on n vertices with $\lambda = \max_{i \geq 2} |\lambda_i|$. It was proved by Alon (see, e.g., [5], [38]) that every (n, d, λ) -graph is in fact $(\frac{d}{n}, \lambda)$ -pseudo-random. Therefore to construct good pseudo-random graphs we need regular graphs with $\lambda \ll d$. For more details on pseudo-random graphs, including many constructions, we refer the interested reader to the recent survey [38].

A graph is d -*degenerate* if every subgraph of it has a vertex of degree at most d . The degeneracy number of a graph H is the smallest d such that H is d -degenerate. This quantity, which is always bounded by the maximum degree of the graph, is a natural measure of its sparseness. In particular, in a d -degenerate graph every subset X spans at most $d|X|$ edges. The chromatic number $\chi(H)$ of graph H is the minimum number of colors needed to color the vertices of H such that adjacent vertices get different colors. Using a greedy coloring, it is easy to show that d -degenerate graphs have chromatic number at most $d + 1$. The following theorem, which is special case of a more general result which we prove in Section 4, shows that any sufficiently pseudo-random graph of appropriate density has strong induced Ramsey properties.

Theorem 1.4 *There is an absolute constant c such that for all integers $k, d, \chi \geq 2$, every $(\frac{1}{k}, n^{0.9})$ -pseudo-random graph G on $n \geq k^{cd \log \chi}$ vertices satisfies that every d -degenerate graph on k vertices with chromatic number at most χ occurs as an induced monochromatic copy in all 2-edge-colorings of G . Moreover, all of these induced monochromatic copies can be found in the same color.*

This theorem implies that, with high probability, $G(n, p)$ with $p = 1/k$ and $n \geq k^{cd \log \chi}$ satisfies that every d -degenerate graph on k vertices with chromatic number at most χ occurs as an induced

monochromatic copy in all 2-edge-colorings of $G(n, p)$. It gives the first polynomial upper bound on the induced Ramsey numbers of d -degenerate graphs. In particular, for bounded degree graphs this is a significant improvement of the above mentioned Łuczak-Rödl result. It shows that the exponent of the polynomial in their theorem can be taken to be $O(d \log d)$, instead of the previous bound of a tower of 2's of height proportional to d^2 .

Corollary 1.5 *There is an absolute constant c such that every d -degenerate graph H on k vertices with chromatic number $\chi \geq 2$ has induced Ramsey number $r_{\text{ind}}(H) \leq k^{cd \log \chi}$.*

A significant additional benefit of Theorem 1.4 is that it leads to explicit constructions for induced Ramsey numbers. One such example can be obtained from a construction of Delsarte and Goethals and also of Turyn (see [38]). Let r be a prime power and let G be a graph whose vertices are the elements of the two dimensional vector space over the finite field \mathbb{F}_r , so G has r^2 vertices. Partition the $r + 1$ lines through the origin of the space into two sets P and N , where $|P| = t$. Two vertices x and y of the graph G are adjacent if $x - y$ is parallel to a line in P . This graph is known to be $t(r - 1)$ -regular with eigenvalues, besides the largest one, being either $-t$ or $r - t$. Taking $t = \frac{r^2}{k(r-1)}$, we obtain an (n, d, λ) -graph with $n = r^2$, $d = n/k$, and $\lambda = r - t \leq r \leq n^{1/2}$. This gives a (p, λ) -pseudo-random graph with $p = d/n = 1/k$ and $\lambda \leq n^{1/2}$ which satisfies the assertion of Theorem 1.4.

Another well-known explicit construction is the Paley graph P_n . Let n be a prime power which is congruent to 1 modulo 4 so that -1 is a square in the finite field \mathbb{F}_n . The *Paley graph* P_n has vertex set \mathbb{F}_n and distinct elements $x, y \in \mathbb{F}_n$ are adjacent if $x - y$ is a square. It is well known and not difficult to prove that the Paley graph P_n is $(1/2, \lambda)$ -pseudo-random with $\lambda = \sqrt{n}$. This can be used together with the generalization of Theorem 1.4, which we discuss in Section 5, to prove the following result.

Corollary 1.6 *There is an absolute constant c such that for prime $n \geq 2^{ck \log^2 k}$, every graph on k vertices occurs as an induced monochromatic copy in all 2-edge-colorings of the Paley graph P_n .*

This explicit construction matches the best known upper bound on induced Ramsey numbers of graphs on k vertices obtained by Kohayakawa, Prömel, and Rödl [36]. Similarly, we can prove that there is a constant c such that, with high probability, $G(n, 1/2)$ with $n \geq 2^{ck \log^2 k}$ satisfies that every graph on k vertices occurs as an induced monochromatic copy in all 2-edge-colorings of G .

Very little is known about lower bounds for induced Ramsey numbers beyond the fact that an induced Ramsey number is at least its corresponding Ramsey number. A well-known conjecture of Burr and Erdős [11] from 1973 states that for each positive integer d there is a constant $c(d)$ such that the Ramsey number $r(H)$ is at most $c(d)k$ for every d -degenerate graph H on k vertices. As mentioned earlier, Haxell et al. [35] proved that the induced Ramsey number for the cycle on k vertices is linear in k . This implies that the induced Ramsey number for the path on k vertices is also linear in k . Also, using a star with $2k - 1$ edges, it is trivial to see that the induced Ramsey number of a star with k edges is $2k$. It is natural to ask whether the Burr-Erdős conjecture extends to induced Ramsey numbers. The following result shows that this fails already for trees, which are 1-degenerate graphs.

Theorem 1.7 *For every $c > 0$ and sufficiently large integer k there is a tree T on k vertices such that $r_{\text{ind}}(T) \geq ck$.*

The tree T in the above theorem can be taken to be any sufficiently large tree that contains a matching of linear size and a star of linear size as subgraphs. It is interesting that the induced Ramsey number for a path on k vertices or a star on k vertices is linear in k , but the induced Ramsey number for a tree which contains both a path on k vertices and a star on k vertices is superlinear in k .

Organization of the paper. In the next section we give short proofs of Theorem 1.1 and Theorem 1.2 which illustrate our methods. Section 3 contains the key lemma that is used as a replacement for Szemerédi's regularity lemma in the proofs of several results. We answer questions of Chung-Graham and Nikiforov on the edge distribution in graphs with a forbidden induced subgraph in Section 4. In Section 5 we show that any sufficiently pseudo-random graph of appropriate density has strong induced Ramsey properties. Combined with known examples of pseudo-random graphs, this leads to explicit constructions which match and improve the best known estimates for induced Ramsey numbers. The proof of the result that there are trees whose induced Ramsey number is superlinear in the number of vertices is in Section 6. The last section of this paper contains some concluding remarks together with a discussion of a few conjectures and open problems. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

2 Ramsey-type results for H -free graphs

In this section, we prove Theorems 1.1 and 1.2. While we obtain more general results later in the paper, the purpose of this section is to illustrate on simple examples the main ideas and techniques that we will use in our proofs. Our theorems strengthen and generalize results from [45] and [42] and the proofs we present here are shorter and simpler than the original ones. We start with the proof of Theorem 1.1, which uses the following lemma of Erdős and Hajnal [23]. We prove a generalization of this lemma in Section 4.

Lemma 2.1 *For each $\epsilon \in (0, 1/2)$, graph H on k vertices, and H -free graph $G = (V, E)$ on $n \geq 2$ vertices, there are disjoint subsets A and B of V with $|A|, |B| \geq \epsilon^{k-1} \frac{n}{k}$ such that either every vertex in A has at most $\epsilon|B|$ neighbors in B , or every vertex in A has at least $(1 - \epsilon)|B|$ neighbors in B .*

Actually, the statement of the lemma in [23] is a bit weaker than that of Lemma 2.1 but it is easy to get the above statement by analyzing more carefully the proof of Erdős and Hajnal. Lemma 2.1 roughly says that every H -free graph contains two large disjoint vertex subsets such that the edge density between them is either very small or very large. However, to prove Theorem 1.1, we need to find a large induced subgraph with such edge density. Our next lemma shows how one can iterate the bipartite density result of Lemma 2.1 in order to establish the complete density result of Theorem 1.1.

For $\epsilon_1, \epsilon_2 \in (0, 1)$ and a graph H , define $\delta(\epsilon_1, \epsilon_2, H)$ to be the largest δ (which may be 0) such that for each H -free graph on n vertices, there is an induced subgraph on at least δn vertices with edge density at most ϵ_1 or at least $1 - \epsilon_2$. Notice that for $2 \leq n_0 \leq n_1$, the edge-density of a graph on n_1 vertices is the average of the edge-densities of the induced subgraphs on n_0 vertices. Therefore, from definition of δ , it follows that for every $2 \leq n_0 \leq \delta(\epsilon_1, \epsilon_2, H)n$ and H -free graph G on n vertices, G contains an induced subgraph on exactly n_0 vertices with edge density at most ϵ_1 or at least $1 - \epsilon_2$.

Recall that the edge-density $d(A)$ of a subset A of G equals $e(A)/\binom{|A|}{2}$, where $e(A)$ is the number of edges spanned by A .

Lemma 2.2 *Suppose $\epsilon_1, \epsilon_2 \in (0, 1)$ with $\epsilon_1 + \epsilon_2 < 1$ and H is a graph on $k \geq 2$ vertices. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. We have*

$$\delta(\epsilon_1, \epsilon_2, H) \geq (\epsilon/4)^k k^{-1} \min\left(\delta(3\epsilon_1/2, \epsilon_2, H), \delta(\epsilon_1, 3\epsilon_2/2, H)\right).$$

Proof. Let G be a H -free graph on $n \geq 2$ vertices. If $n < k$ then we may consider any two-vertex induced subgraph of G which has always density either 0 or 1. Therefore, for G of order less than k we can take $\delta = 2/k$, which is clearly larger than the right hand side of the inequality in the assertion of the lemma. Thus we can assume that $n \geq k$. Applying Lemma 2.1 to G with $\epsilon/4$ in place of ϵ , we find two subsets A and B with $|A|, |B| \geq (\epsilon/4)^{k-1} n/k$, such that either every vertex in A is adjacent to at most $\frac{\epsilon}{4}|B|$ vertices of B or every vertex of A is adjacent to at least $(1 - \frac{\epsilon}{4})|B|$ vertices of B .

First consider the case in which every vertex in A is adjacent to at most $\frac{\epsilon}{4}|B|$ vertices of B (the other case can be treated similarly) and let $G[A]$ be the subgraph of G induced by the set A . By definition of function δ , $G[A]$ contains a subset A' with

$$|A'| = \delta(3\epsilon_1/2, \epsilon_2, H) \left(\frac{\epsilon}{4}\right)^k \frac{n}{k} \leq \delta(3\epsilon_1/2, \epsilon_2, H) |A|,$$

such that the subgraph induced by A' has edge density at most $\frac{3}{2}\epsilon_1$ or at least $1 - \epsilon_2$. If A' has edge density at least $1 - \epsilon_2$ we are done, since $G[A']$ is an induced subgraph of G with at least $(\epsilon/4)^k k^{-1} \delta(3\epsilon_1/2, \epsilon_2, H) n$ vertices and edge density at least $1 - \epsilon_2$. So we may assume that the edge density in A' is at most $\frac{3}{2}\epsilon_1$.

Let $B_1 \subset B$ be those vertices of B that have at most $\frac{\epsilon}{2}|A'|$ neighbors in A' . Since $A' \subset A$, each vertex of A' has at most $\frac{\epsilon}{4}|B|$ neighbors in B and the number of edges $e(A', B) \leq \frac{\epsilon}{4}|A'||B|$. Therefore B_1 has at least $|B|/2$ vertices. Then, by definition of δ , B_1 contains a subset B' with

$$|B'| = \delta(3\epsilon_1/2, \epsilon_2, H) \left(\frac{\epsilon}{4}\right)^k \frac{n}{k} \leq \delta(3\epsilon_1/2, \epsilon_2, H) |B_1|,$$

such that the induced subgraph $G[B']$ has edge density at most $\frac{3}{2}\epsilon_1$ or at least $1 - \epsilon_2$. If it has edge density at least $1 - \epsilon_2$ we are done, so we may assume that the edge density $d(B')$ is at most $\frac{3}{2}\epsilon_1$.

Finally to complete the proof note that, since $|A'| = |B'|$, $|A' \cup B'| = 2|A'|$, $d(A'), d(B') \leq \frac{3}{2}\epsilon_1$, and $d(A', B') \leq \frac{\epsilon_1}{2}$, we have that

$$\begin{aligned} e(A' \cup B') &= e(A') + e(B') + e(A', B') \leq \frac{3}{2}\epsilon_1 \binom{|A'|}{2} + \frac{3}{2}\epsilon_1 \binom{|B'|}{2} + \frac{\epsilon_1}{2} |A'| |B'| \\ &= 2\epsilon_1 |A'|^2 - 3\epsilon_1 |A'|/2 \leq \epsilon_1 \binom{2|A'|}{2}. \end{aligned}$$

Therefore, $d(A' \cup B') \leq \epsilon_1$. □

From this lemma, the proof of our first result, that every H -free graph on n vertices contains a subset of at least $2^{-ck(\log \frac{1}{\epsilon})^2} n$ vertices with edge density either $\leq \epsilon$ or $\geq 1 - \epsilon$, follows in a few lines.

Proof of Theorem 1.1: Notice that if $\epsilon_1 + \epsilon_2 \geq 1$, then trivially $\delta(\epsilon_1, \epsilon_2, H) = 1$. In particular, if $\epsilon_1 \epsilon_2 \geq \frac{1}{4}$, then $\epsilon_1 + \epsilon_2 \geq 1$ and $\delta(\epsilon_1, \epsilon_2, H) = 1$. Therefore, by iterating Lemma 2.2 for $t = \log \frac{1}{\epsilon^2} / \log \frac{3}{2}$ iterations and using that $\epsilon \leq 1/2$, we obtain

$$\delta(\epsilon, \epsilon, H) \geq \left(\frac{\epsilon^k}{4^k k} \right)^t = 2^{-\frac{2}{\log 3/2} (k(\log 1/\epsilon)^2 + (2k + \log k) \log 1/\epsilon)} \geq 2^{-15k(\log 1/\epsilon)^2},$$

which, by definition of δ , completes the proof of the theorem. \square

Recall the Erdős-Szemerédi theorem, which states that there is an absolute constant c such that every graph G on n vertices with edge density $\epsilon \in (0, 1/2)$ has a homogeneous set of size at least $\frac{c \log n}{\epsilon \log \frac{1}{\epsilon}}$. Theorem 1.2 follows from a simple application of Theorem 1.1 and the Erdős-Szemerédi theorem.

Proof of Theorem 1.2: Let G be a graph on n vertices which is not k -universal, i.e., it is H -free for some fixed graph H on k vertices. Fix $\epsilon = 2^{-\frac{1}{5} \sqrt{\frac{\log n}{k}}}$ and apply Theorem 1.1 to G . It implies that G contains a subset $W \subset V(G)$ of size at least $2^{-15k(\log \frac{1}{\epsilon})^2} n = n^{2/5}$ such that the subgraph induced by W has edge density at most ϵ or at least $1 - \epsilon$. Applying the Erdős-Szemerédi theorem to the induced subgraph $G[W]$ or its complement and using that $\epsilon \log 1/\epsilon \leq 4\epsilon^{1/2}$ for all $\epsilon \leq 1$, we obtain a homogeneous subset $W' \subset W$ with

$$|W'| \geq \frac{c \log n^{2/5}}{\epsilon \log \frac{1}{\epsilon}} \geq \frac{c \log n}{10\epsilon^{1/2}} \geq \frac{c}{10} 2^{\frac{1}{10} \sqrt{\frac{\log n}{k}}} \log n,$$

which completes the proof of Theorem 1.2. \square

3 Key Lemma

In this section we present our key lemma. We use it as a replacement for Szemerédi's regularity lemma in the proofs of several Ramsey-type results, thereby giving much better estimates. A very special case of this statement was essentially proved in Lemma 2.2 in the previous section. Our key lemma generalizes the result of Graham, Rödl, and Ruciński [34] and has a simpler proof than the one in [34]. Roughly, our result says that if (G_1, \dots, G_r) is a sequence of graphs on the same vertex set V with the property that every large subset of V contains a pair of large disjoint sets with small edge density between them in at least one of the graphs G_i , then every large subset of V contains a large set with small edge density in one of the G_i . To formalize this concept, we need a couple of definitions.

For a graph $G = (V, E)$ and disjoint subsets $W_1, \dots, W_t \subset V$, the *density* $d_G(W_1, \dots, W_t)$ between the $t \geq 2$ vertex subsets W_1, \dots, W_t is defined by

$$d_G(W_1, \dots, W_t) = \frac{\sum_{i < j} e(W_i, W_j)}{\sum_{i < j} |W_i| |W_j|}.$$

If $|W_1| = \dots = |W_t|$, then

$$d_G(W_1, \dots, W_t) = \binom{t}{2}^{-1} \sum_{i < j} d_G(W_i, W_j).$$

Also, in this section if $t = 1$ we define the density to be zero.

Definition 3.1 For $\alpha, \rho, \epsilon \in [0, 1]$ and positive integer t , a sequence (G_1, \dots, G_r) of graphs on the same vertex set V is $(\alpha, \rho, \epsilon, t)$ -sparse if for all subsets $U \subset V$ with $|U| \geq \alpha|V|$, there are positive integers t_1, \dots, t_r such that $\prod_{i=1}^r t_i \geq t$ and for each $i \in [r] = \{1, \dots, r\}$ there are disjoint subsets $W_{i,1}, \dots, W_{i,t_i} \subset U$ with $|W_{i,1}| = \dots = |W_{i,t_i}| = \lceil \rho|U| \rceil$ and $d_{G_i}(W_{i,1}, \dots, W_{i,t_i}) \leq \epsilon$.

We call a graph $(\alpha, \rho, \epsilon, t)$ -sparse if the one-term sequence consisting of that graph is $(\alpha, \rho, \epsilon, t)$ -sparse. By averaging, if $\alpha' \geq \alpha$, $\rho' \leq \rho$, $\epsilon' \geq \epsilon$, $t' \leq t$, and (G_1, \dots, G_r) is $(\alpha, \rho, \epsilon, t)$ -sparse, then (G_1, \dots, G_r) is also $(\alpha', \rho', \epsilon', t')$ -sparse. The following is our main result in this section.

Lemma 3.2 If a sequence of graphs (G_1, \dots, G_r) with common vertex set V is $(\frac{1}{2}\alpha\rho, \rho', \epsilon, t)$ -sparse and $(\alpha, \rho, \epsilon/4, 2)$ -sparse, then (G_1, \dots, G_r) is also $(\alpha, \frac{1}{2}\rho\rho', \epsilon, 2t)$ -sparse.

Proof. Since (G_1, \dots, G_r) is $(\alpha, \rho, \epsilon/4, 2)$ -sparse, then for each $U \subset V$ with $|U| \geq \alpha|V|$, there is $i \in [r]$ and disjoint subsets $X, Y \subset U$ with $|X| = |Y| = \rho|U|$ and $d_{G_i}(X, Y) \leq \epsilon/4$. Let X_1 be the set of vertices in X that have at most $\frac{\epsilon}{2}|Y|$ neighbors in Y in graph G_i . Then $e_{G_i}(X \setminus X_1, Y) \geq \epsilon|X \setminus X_1||Y|/2$ and we also have $e_{G_i}(X, Y) \leq \epsilon|X||Y|/4$. Therefore $|X_1| \geq |X|/2 \geq \frac{1}{2}\rho|U|$ and by removing extra vertices we assume that $|X_1| = \frac{1}{2}\rho|U|$.

Since (G_1, \dots, G_r) is $(\frac{1}{2}\alpha\rho, \rho', \epsilon, t)$ -sparse, then there are positive integers t_1, \dots, t_r such that $\prod_{j=1}^r t_j \geq t$ and for each $j \in [r]$ there are disjoint subsets $X_{j,1}, \dots, X_{j,t_j} \subset X_1$ of size $|X_{j,1}| = \dots = |X_{j,t_j}| = \rho'|X_1|$ with density $d_{G_j}(X_{j,1}, \dots, X_{j,t_j}) \leq \epsilon$. Let Y_1 be the set of vertices in Y that have at most $\epsilon|X_{i,1} \cup \dots \cup X_{i,t_i}|$ neighbors in $X_{i,1} \cup \dots \cup X_{i,t_i}$ in graph G_i . Since every vertex of X_1 is adjacent to at most $\frac{\epsilon}{2}|Y|$ vertices of Y and since $X_{i,1} \cup \dots \cup X_{i,t_i} \subset X_1$ we have that $d_{G_i}(X_{i,1} \cup \dots \cup X_{i,t_i}, Y) \leq \epsilon/2$. On the other hand, $d_{G_i}(X_{i,1} \cup \dots \cup X_{i,t_i}, Y \setminus Y_1) \geq \epsilon$. Therefore $|Y_1| \geq |Y|/2$, so again we can assume that $|Y_1| = \frac{1}{2}\rho|U| = |X_1|$. Since (G_1, \dots, G_r) is $(\frac{1}{2}\alpha\rho, \rho', \epsilon, t)$ -sparse, then there are positive integers s_1, \dots, s_r such that $\prod_{j=1}^r s_j \geq t$ and for each $j \in [r]$ there are disjoint subsets $Y_{j,1}, \dots, Y_{j,s_j} \subset Y_1$ with $d_{G_j}(Y_{j,1}, \dots, Y_{j,s_j}) \leq \epsilon$ and $|Y_{j,1}| = \dots = |Y_{j,s_j}| = \rho'|Y_1|$.

By the above construction, the edge density between $X_{i,1} \cup \dots \cup X_{i,t_i}$ and $Y_{i,1} \cup \dots \cup Y_{i,s_i}$ is bounded from above by ϵ . We also have that both $d_{G_i}(X_{i,1}, \dots, X_{i,t_i})$ and $d_{G_i}(Y_{i,1}, \dots, Y_{i,s_i})$ are at most ϵ . Therefore $d_{G_i}(X_{i,1}, \dots, X_{i,t_i}, Y_{i,1}, \dots, Y_{i,s_i}) \leq \epsilon$, implying that (G_1, \dots, G_r) is $(\alpha, \frac{1}{2}\rho\rho', \epsilon, u)$ -sparse with $u = (t_i + s_i) \prod_{j \in [r] \setminus \{i\}} \max(t_j, s_j)$ for some i . By the arithmetic mean-geometric mean inequality, we have

$$t^2 \leq \prod_{j=1}^r t_j \prod_{j=1}^r s_j \leq t_i s_i \left(\prod_{j \in [r] \setminus \{i\}} \max(t_j, s_j) \right)^2 = \frac{t_i s_i}{(t_i + s_i)^2} u^2 \leq \frac{u^2}{4}.$$

Thus $u \geq 2t$. Altogether this shows that (G_1, \dots, G_r) is $(\alpha, \frac{1}{2}\rho\rho', \epsilon, 2t)$ -sparse, completing the proof. \square

Rather than using this lemma directly, in applications we usually need the following two corollaries. The first one is obtained by simply applying Lemma 3.2 $h - 1$ times.

Corollary 3.3 If (G_1, \dots, G_r) is $(\alpha, \rho, \epsilon/4, 2)$ -sparse and h is a positive integer, then (G_1, \dots, G_r) is also $\left(\left(\frac{2}{\rho}\right)^{h-1}\alpha, 2^{1-h}\rho^h, \epsilon, 2^h\right)$ -sparse.

If we use the last statement with $h = r \log \frac{1}{\epsilon}$ and $\alpha = (\frac{\rho}{2})^{h-1}$, then we get that there is an index $i \in [r]$ and disjoint subsets $W_1, \dots, W_t \subset V$ with $t \geq 2^{h/r} = \frac{1}{\epsilon}$, $|W_1| = \dots = |W_t| = 2^{1-h} \rho^h |V|$, and $d_{G_i}(W_1, \dots, W_t) \leq \epsilon$. Since $\binom{|W_1|}{2} \leq \frac{\epsilon}{t} \binom{t|W_1|}{2}$, even if every W_i has edge density one, still the edge density in the set $W_1 \cup \dots \cup W_t$ is at most 2ϵ . Therefore, (using $\epsilon/2$ instead of ϵ) we have the following corollary.

Corollary 3.4 *If (G_1, \dots, G_r) is $((\frac{\rho}{2})^{h-1}, \rho, \epsilon/8, 2)$ -sparse where $h = r \log \frac{2}{\epsilon}$, then there is $i \in [r]$ and an induced subgraph G' of G_i on $2\epsilon^{-1}2^{1-h}\rho^h|V|$ vertices that has edge density at most ϵ .*

The key lemma in the paper of Graham, Rödl, and Rucinski [34] on the Ramsey number of graphs (their Lemma 1) is essentially the $r = 1$ case of Corollary 3.4.

4 Edge distribution in H -free graphs

In this section, we obtain several results on the edge distribution of graphs with a forbidden induced subgraph which answer open questions by Nikiforov and Chung-Graham. We first prove a strengthening of Rödl's theorem (mentioned in the introduction) without using the regularity lemma. Then we present a proof of Theorem 1.3 on the dependence of error terms in quasirandom properties. We conclude this section with an upper bound on the maximum edge discrepancy in subgraphs of H -free graphs. To obtain these results we need the following generalization of Lemma 2.1.

Lemma 4.1 *Let H be a k -vertex graph and let G be a graph on $n \geq k^2$ vertices that contains less than $n^k(1 - \frac{k^2}{2n}) \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i}$ labeled induced copies of H , where $\epsilon_0 = 1$ and $\epsilon_i, \delta_i \in (0, 1)$ for all $1 \leq i \leq k-1$. Then there is an index $i \leq k-1$ and disjoint subsets A and B of G with $|A| \geq \frac{\delta_i n}{k(k-i)} \prod_{j < i} \epsilon_j$ and $|B| \geq \frac{n}{k} \prod_{j < i} \epsilon_j$ such that either every vertex of A is adjacent to at most $\epsilon_i |B|$ vertices of B or every vertex of A is adjacent to at least $(1 - \epsilon_i) |B|$ vertices of B .*

Proof. Let M denote the number of labeled induced copies of H in G , which by our assumption is at most

$$M < n^k \left(1 - \frac{k^2}{2n}\right) \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i}. \quad (1)$$

We may assume that the vertex set of H is $[k]$. Consider a random partition $V_1 \cup \dots \cup V_k$ of the vertices of G such that each V_i has cardinality n/k . Note that for any such partition there are $(n/k)^k$ ordered k -tuples of vertices of G with the property that the i -th vertex of the k -tuple is in V_i for all $i \in [k]$. On the other hand the total number of ordered k -tuples of vertices is $n(n-1) \dots (n-k+1)$ and each of these k -tuples has the above property with equal probability. This implies that for any given k -tuple the probability that its i -th vertex is in V_i for all $i \in [k]$ equals $\prod_{i=1}^k \frac{n/k}{n-i+1}$. In particular, by linearity of expectation, the expected number of labeled induced copies of H in G for which the image of every vertex $i \in [k]$ is in V_i is $M \cdot \prod_{i=1}^k \frac{n/k}{n-i+1}$. Using that $\prod (1 - x_i) \geq 1 - \sum x_i$ for any $0 \leq x_i \leq 1$ and that $n \geq k^2$, we obtain

$$\begin{aligned} \prod_{i=1}^k \frac{n/k}{n-i+1} &= k^{-k} \prod_{i=0}^{k-1} (1 - i/n)^{-1} \leq k^{-k} \left(1 - \sum_{i=0}^{k-1} i/n\right)^{-1} = k^{-k} \left(1 - \binom{k}{2}/n\right)^{-1} \\ &< \left(1 - \frac{k^2}{2n}\right)^{-1} k^{-k}. \end{aligned}$$

This, together with (1), shows that there is a partition $V_1 \cup \dots \cup V_k$ of G into sets of cardinality n/k such that the total number of labeled induced copies of H in G for which the image of every vertex $i \in [k]$ is in V_i is less than

$$M \left(1 - \frac{k^2}{2n}\right)^{-1} k^{-k} < k^{-k} n^k \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i}. \quad (2)$$

We use this estimate to construct sets A and B which satisfy the assertion of the lemma. For a vertex $v \in V$, the *neighborhood* $N(v)$ is the set of vertices of G that are adjacent to v . For $v \in V_i$ and a subset $S \subset V_j$ with $i \neq j$, let $\tilde{N}(v, S) = N(v) \cap S$ if (i, j) is an edge of H and $\tilde{N}(v, S) = S \setminus N(v)$ otherwise. We will try iteratively to build many induced copies of H . After i steps, we will have vertices v_1, \dots, v_i with $v_j \in V_j$ for $j \leq i$ and subsets $V_{i+1,i}, V_{i+2,i}, \dots, V_{k,i}$ such that

1. $V_{\ell,i}$ is a subset of V_ℓ of size $|V_{\ell,i}| \geq \frac{n}{k} \prod_{j=1}^i \epsilon_j$ for all $i+1 \leq \ell \leq k$,
2. for $1 \leq j < \ell \leq i$, (v_j, v_ℓ) is an edge of G if and only if (j, ℓ) is an edge of H ,
3. and if $j \leq i < \ell$ and $w \in V_{\ell,i}$, then (v_j, w) is an edge of G if and only if (j, ℓ) is an edge of H .

In the first step, we call a vertex $v \in V_1$ *good* if $|\tilde{N}(v, V_i)| \geq \epsilon_1 |V_i|$ for each $i > 1$. If less than a fraction $1 - \delta_1$ of the vertices in V_1 are good, then, by the pigeonhole principle, there is a subset $A \subset V_1$ with $|A| \geq \frac{\delta_1}{k-1} |V_1| = \frac{\delta_1}{k(k-1)} n$ and an index $j > 1$ such that $|\tilde{N}(v, V_j)| < \epsilon_1 |V_j|$ for each $v \in A$. Letting $B = V_j$, one can easily check that A and B satisfy the assertion of the lemma. Hence, we may assume that at least a fraction $1 - \delta_1$ of the vertices $v_1 \in V_1$ are good, choose any good v_1 and define $V_{i,1} = \tilde{N}(v_1, V_i)$ for $i > 1$, completing the first step.

Suppose that after step i the properties 1-3 are satisfied. Then, in step $i+1$, we again call a vertex $v \in V_{i+1,i}$ *good* if $|\tilde{N}(v, V_{j,i})| \geq \epsilon_{i+1} |V_{j,i}|$ for each $j > i+1$. If less than a fraction $1 - \delta_{i+1}$ of the vertices of $V_{i+1,i}$ are good, then, by the pigeonhole principle, there is a subset $A \subset V_{i+1,i}$ with $|A| \geq \frac{\delta_{i+1}}{k-i-1} |V_{i+1,i}|$ and index $j > i+1$ such that $|\tilde{N}(v, V_{j,i})| < \epsilon_{i+1} |V_{j,i}|$ for each $v \in A$. Letting $B = V_{j,i}$, one can check using property 1, that A and B satisfy the assertion of the lemma. Hence, we may assume that a fraction $1 - \delta_{i+1}$ of the vertices $v_{i+1} \in V_{i+1,i}$ are good, choose any good v_{i+1} and define $V_{j,i+1} = \tilde{N}(v_{i+1}, V_{j,i})$ for $j \geq i+2$, completing step $i+1$. Notice that after step $i+1$, we have $|V_{j,i+1}| \geq \epsilon_{i+1} |V_{j,i}|$ for $j > i+1$, which guarantees that property 1 is satisfied. The remaining properties (2 and 3) follow from our construction of sets $V_{j,i+1}$.

Thus if our process fails in one of the first $k-1$ steps we obtain the desired sets A and B . Suppose now that we successfully performed $k-1$ steps. Note that in step $i+1$, we had at least $(1 - \delta_{i+1}) |V_{i+1,i}| \geq \frac{n}{k} (1 - \delta_{i+1}) \prod_{j=1}^i \epsilon_j$ vertices to choose for vertex v_{i+1} . Also note that, by property

3, after step $k - 1$ we can choose any vertex in the set $V_{k,k-1}$ to be v_k . Moreover, by the property 2, every choice of the vertices v_1, \dots, v_k form a labeled induced copy of H . Altogether, this gives at least

$$\begin{aligned} |V_{k,k-1}| \cdot \prod_{i=1}^{k-1} \left(\frac{n}{k} (1 - \delta_i) \prod_{0 \leq j < i} \epsilon_j \right) &\geq \frac{n}{k} \prod_{j=1}^{k-1} \epsilon_j \cdot \prod_{i=1}^{k-1} \left(\frac{n}{k} (1 - \delta_i) \prod_{0 \leq j < i} \epsilon_j \right) \\ &= (n/k)^k \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i} \end{aligned}$$

labeled induced copies of H for which the image of every vertex $i \in [k]$ is in V_i . This contradicts (2) and completes the proof. \square

Notice that the number of induced copies of H in any induced subgraph of G is at most the number of induced copies of H in G . Let $\epsilon_i = \epsilon \leq 1/2$ and $\delta_i = \frac{1}{2}$ for $1 \leq i \leq k-1$ and let $\alpha \geq k^2/n$. Applying Lemma 4.1 with these ϵ_i, δ_i to subsets of G of size αn and using that $\frac{\epsilon^{i-1}}{k-i} \geq \epsilon^{k-1}, 1 - \frac{k^2}{2\alpha n} \geq 1/2$ we obtain the following corollary.

Corollary 4.2 *Let H be a graph with k vertices, $\alpha \geq k^2/n$, $\epsilon \leq 1/2$, and G be a graph with at most $2^{-k} \epsilon^{\binom{k}{2}} (\alpha n)^k$ induced copies of H . Then the pair (G, \bar{G}) is $(\alpha, \frac{\epsilon^{k-1}}{2k}, \epsilon, 2)$ -sparse.*

The next statement strengthens Theorem 1.1 by allowing for many induced copies of H . It follows from Corollary 3.4 with $r = 2, h = 2 \log(2/\epsilon), \rho = \frac{\epsilon^{k-1}}{2k}$, combined with the last statement in which we set $\alpha = (\rho/2)^{h-1}$.

Corollary 4.3 *There is a constant c such that for each $\epsilon \in (0, 1/2)$ and graph H on k vertices, every graph G on n vertices with less than $2^{-c(k \log \frac{1}{\epsilon})^2} n^k$ induced copies of H contains an induced subgraph of size at least $2^{-ck(\log \frac{1}{\epsilon})^2} n$ with edge density at most ϵ or at least $1 - \epsilon$.*

This result demonstrates that for each $\epsilon \in (0, 1/2)$ and graph H , there exist positive constants $\delta^* = \delta^*(\epsilon, H)$ and $\kappa^* = \kappa^*(\epsilon, H)$ such that every graph $G = (V, E)$ on n vertices with less than $\kappa^* n^k$ induced copies of H contains a subset $W \subset V$ of size at least $\delta^* n$ such that the edge density of W is at most ϵ or at least $1 - \epsilon$. Furthermore, there is a constant c such that we can take $\delta^*(\epsilon, H) = 2^{-ck(\log \frac{1}{\epsilon})^2}$ and $\kappa^*(\epsilon, H) = 2^{-c(k \log \frac{1}{\epsilon})^2}$. Applying Corollary 4.3 recursively one can obtain an equitable partition of G into a small number of subsets each with low or high density.

Theorem 4.4 *For each $\epsilon \in (0, 1/2)$ and graph H on k vertices, there are positive constants $\kappa = \kappa(\epsilon, H)$ and $C = C(\epsilon, H)$ such that every graph $G = (V, E)$ on n vertices with less than κn^k induced copies of H , there is an equitable partition $V = \bigcup_{i=1}^{\ell} V_i$ such that $\ell \leq C$ and the edge density in each V_i is at most ϵ or at least $1 - \epsilon$.*

This extension of Rödl's theorem was proved by Nikiforov [41] using the regularity lemma and therefore it had quite poor (tower like) dependence of κ and C on ϵ and k . Obtaining a proof without using the regularity lemma was the main open problem raised in [41].

Proof of Theorem 4.4. Let $\kappa(\epsilon, H) = (\frac{\epsilon}{4})^k \kappa^*(\frac{\epsilon}{4}, H)$ and $C(\epsilon, H) = 4/(\epsilon \delta^*(\frac{\epsilon}{4}, H))$, where κ^* and δ^* were defined above. Take a subset $W_1 \subset V$ of size $\delta^*(\frac{\epsilon}{4}, H) \frac{\epsilon}{4} n$ whose edge density is at most $\frac{\epsilon}{4}$ or at

least $1 - \frac{\epsilon}{4}$, and set $U_1 = V \setminus W_1$. For $j \geq 1$, if $|U_j| \geq \frac{\epsilon}{4}n$, then by definition of κ we have that the number of induced copies of H in U_j is at most (the number of such copies in G) $\kappa n^k = (\frac{\epsilon}{4})^k \kappa^* n^k \leq \kappa^* |U_j|^k$. Therefore by definition of κ^* and δ^* we can find a subset $W_{j+1} \subset U_j$ of size $\delta^* \frac{\epsilon}{4}n \leq \delta^* |U_j|$ whose edge density is at most $\frac{\epsilon}{4}$ or at least $1 - \frac{\epsilon}{4}$, and set $U_{j+1} = U_j \setminus W_{j+1}$.

Once this process stops, we have disjoint sets W_1, \dots, W_ℓ , each with the same cardinality, and a subset U_ℓ of cardinality at most $\frac{\epsilon}{4}n$. The number ℓ is at most

$$n/|W_1| \leq 4/(\epsilon \delta^*(\frac{\epsilon}{4}, H)).$$

Partition the set U_ℓ into ℓ parts T_1, \dots, T_ℓ of equal size and let $V_j = W_j \cup T_j$ for $1 \leq j \leq \ell$. Notice that $V = V_1 \cup \dots \cup V_\ell$ is an equitable partition of V . By definition, $|T_j| = |U_\ell|/\ell \leq \frac{\epsilon}{4}n/\ell$. On the other hand $|W_j| = (n - |U_\ell|)/\ell \geq (1 - \epsilon/4)n/\ell$. Since $1 - \epsilon/4 > 7/8$, this implies that

$$|T_j| \leq \frac{\epsilon}{4}n/\ell \leq \frac{\epsilon}{4}(1 - \epsilon/4)^{-1}|W_j| \leq \frac{2\epsilon}{7}|W_j|.$$

We next look at the edge density in V_j . If the edge density in W_j is at most $\epsilon/4$, then using the above bound on $|T_j|$, it is easy to check that the number of edges in V_j is at most

$$\binom{|T_j|}{2} + |T_j||W_j| + \frac{\epsilon}{4}\binom{|W_j|}{2} \leq \epsilon \binom{|W_j|}{2} \leq \epsilon \binom{|V_j|}{2}.$$

Hence, the edge density in each such V_j is at most ϵ . Similarly, if the edge density in W_j is at least $1 - \frac{\epsilon}{4}$, then the edge density in V_j is at least $1 - \epsilon$. This completes the proof. \square

We next use Lemma 4.1 to prove that there is a constant $c > 1$ such that every graph G on n vertices which contains at most $(1 - \epsilon)2^{-\binom{k}{2}}n^k$ labeled induced copies of some fixed k -vertex graph H has a subset S of size $|S| = \lfloor n/2 \rfloor$ with $|e(S) - \frac{n^2}{16}| \geq \epsilon c^{-k}n^2$.

Proof of Theorem 1.3. For $1 \leq i \leq k-1$, let $\epsilon_i = \frac{1}{2}(1 - 2^{i-k-2}\epsilon)$ and $\delta_i = 2^{i-k-2}\epsilon$. Notice that for all $i \leq k-1$

$$\prod_{j < i} \epsilon_j = 2^{-i+1} \prod_{j < i} (1 - 2^{j-k-2}\epsilon) \geq 2^{-i+1} \left(1 - \epsilon \sum_{j < k-1} 2^{j-k-2}\right) \geq 2^{-i+1}(1 - \epsilon/8) > 2^{-i} \quad (3)$$

and also that

$$\begin{aligned} \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i} &= 2^{-\binom{k}{2}} \prod_{i=1}^{k-1} (1 - 2^{i-k-2}\epsilon)^{k-i+1} = 2^{-\binom{k}{2}} \prod_{j=2}^k (1 - \epsilon 2^{-j-1})^j \\ &\geq 2^{-\binom{k}{2}} \left(1 - \epsilon \sum_{j=2}^k \frac{j}{2^{j+1}}\right) > \left(1 - \frac{3\epsilon}{4}\right) 2^{-\binom{k}{2}}. \end{aligned}$$

We may assume that $\epsilon \geq 2k^2/n$ since otherwise by choosing constant c large enough we get that $\epsilon c^{-k}n^2 < n/8$ and the conclusion of the theorem follows easily. Therefore

$$\left(1 - \frac{k^2}{2n}\right) \prod_{i=1}^{k-1} (1 - \delta_i) \epsilon_i^{k-i} \geq \left(1 - \frac{\epsilon}{4}\right) \left(1 - \frac{3\epsilon}{4}\right) 2^{-\binom{k}{2}} > (1 - \epsilon) 2^{-\binom{k}{2}},$$

and we can apply Lemma 4.1 with ϵ_i and δ_i as above to our graph G since it contains at most $(1 - \epsilon)2^{-\binom{k}{2}}n^k$ labeled induced copies of H . This lemma, together with (3), implies that there is an index $i \leq k - 1$ and disjoint subsets A and B with

$$|A| \geq \frac{\delta_i n}{k(k-i)} \prod_{j < i} \epsilon_j \geq k^{-2} 2^{-k-2} n,$$

$$|B| \geq \frac{n}{k} \prod_{j < i} \epsilon_j \geq 2^{-i} k^{-1} n,$$

and every element of A is adjacent to at most $\epsilon_i |B|$ elements of B or every element of A is adjacent to at least $(1 - \epsilon_i) |B|$ elements of B . In either case, we have

$$\left| e(A, B) - \frac{1}{2} |A| |B| \right| \geq \left(\frac{1}{2} - \epsilon_i \right) |A| |B| = 2^{i-k-3} \epsilon |A| |B| \geq k^{-3} 2^{-2k-5} \epsilon n^2.$$

Note that

$$e(A, B) - \frac{1}{2} |A| |B| = \left(e(A \cup B) - \frac{1}{2} \binom{|A \cup B|}{2} \right) - \left(e(A) - \frac{1}{2} \binom{|A|}{2} \right) - \left(e(B) - \frac{1}{2} \binom{|B|}{2} \right).$$

It follows from the triangle inequality that there is some subset of vertices $R \in \{A, B, A \cup B\}$ such that

$$\left| e(R) - \frac{1}{2} \binom{|R|}{2} \right| \geq \frac{1}{3} k^{-3} 2^{-2k-5} \epsilon n^2, \quad (4)$$

i.e., it deviates by at least $\epsilon k^{-3} 2^{-2k-5} n^2 / 3$ edges from having edge density $1/2$. To finish the proof we will use the lemma of Erdős et al. [22], mentioned in the introduction. This lemma says that if graph G on n vertices with edge density η has a subset that deviates by D edges from having edge density η , then it also has a subset of size $n/2$ that deviates by at least $D/5$ edges from having edge density η . Note that if the edge density of our graph G is either larger than $1/2 + \epsilon k^{-3} 2^{-2k-5} n^2 / 30$ or smaller than $1/2 - \epsilon k^{-3} 2^{-2k-5} n^2 / 30$ than by averaging over all subsets of size $n/2$ we will find subset S satisfying our assertion. Otherwise, if the edge density η of G satisfies $|\eta - 1/2| \leq \epsilon k^{-3} 2^{-2k-5} n^2 / 30$, then the subset R from (4) deviates by at least $\epsilon k^{-3} 2^{-2k-5} n^2 / 3 - \epsilon k^{-3} 2^{-2k-5} n^2 / 30 \geq \epsilon k^{-3} 2^{-2k} n^2 / 4$ edges from having edge density η . Then, by the lemma of Erdős et al., G has a subset S of cardinality $n/2$ that deviates by at least $\epsilon k^{-3} 2^{-2k-5} n^2 / 20$ edges from having edge density η . Such S satisfies

$$\left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| \geq \epsilon k^{-3} 2^{-2k-5} n^2 / 20 - \epsilon k^{-3} 2^{-2k-5} n^2 / 30 = \Omega \left(\epsilon k^{-3} 2^{-2k} n^2 \right).$$

Since $\left| \frac{|S|^2}{4} - \frac{1}{2} \binom{|S|}{2} \right| < n$, this completes the proof. \square

For positive integers k and n , recall that $D(k, n)$ denotes the largest integer such that every graph G on n vertices that is H -free for some k -vertex graph H contains a subset S of size $n/2$ with $|e(S) - \frac{1}{16} n^2| > D(k, n)$. We end this section by proving the upper bound on $D(k, n)$.

Proposition 4.5 *There is a constant $c > 0$ such that for all positive integers k and $n \geq 2^{k/2}$, there is a K_k -free graph G on n vertices such that for every subset S of $n/2$ vertices of G ,*

$$\left| e(S) - \frac{1}{16} n^2 \right| < c 2^{-k/4} n^2.$$

Proof. Consider the random graph $G(\ell, 1/2)$ with $\ell = 2^{k/2}$. For every subset of vertices X in this graph the number of edges in X is a binomially distributed random variable with expectation $\frac{|X|(|X|-1)}{4}$. Therefore by Chernoff's bound (see, e.g., Appendix A in [5]), the probability that it deviates from this value by t is at most $2e^{-t^2/|X|^2}$. Thus choosing $t = 1.5\ell^{3/2}$ we obtain that the probability that there is a subset of vertices X such that $|e(X) - \frac{|X|(|X|-1)}{4}| > t$ is at most $2^\ell \cdot 2e^{-t^2/\ell^2} \ll 1$. Also, the probability that this random graph contains K_k is at most $2^{-\binom{k}{2}} \binom{\ell}{k} \leq 2^{-\binom{k}{2}} \ell^k/k! = 2^{k/2}/k! \ll 1$. This implies that there is K_k -free graph Γ on ℓ vertices such that every subset X of Γ satisfies

$$\left|e(X) - \frac{1}{4}|X|^2\right| \leq 2\ell^{3/2}. \quad (5)$$

Let G be the graph obtained by replacing every vertex u of Γ with an independent set I_u , of size n/ℓ , and by replacing every edge (u, v) of Γ with a complete bipartite graph, whose partition classes are the independent sets I_u and I_v . Clearly, since Γ does not contain K_k , then neither does G . We claim that graph G satisfies the assertion of the proposition. Suppose for contradiction that there is a subset S of $n/2$ vertices of G satisfying

$$e(S) - \frac{1}{16}n^2 > 4\ell^{3/2}(n/\ell)^2 = 4\ell^{-1/2}n^2 = 2^{-k/4+2}n^2,$$

(the other case when $e(S) - n^2/16 < -4\ell^{-1/2}n^2$ can be treated similarly). For every vertex $u \in \Gamma$ let the size of $S \cap I_u$ be $a_u n/\ell$. By definition, $0 \leq a_u \leq 1$ and since S has size $n/2$ we have that $\sum_u a_u = \ell/2$. We also have that

$$e(S) = \sum_{(u,v) \in E(\Gamma)} a_u a_v \cdot (n/\ell)^2 > \frac{1}{16}n^2 + 4\ell^{3/2}(n/\ell)^2,$$

and therefore

$$\sum_{(u,v) \in E(\Gamma)} a_u a_v > \ell^2/16 + 4\ell^{3/2} = \frac{1}{4} \left(\sum_u a_u \right)^2 + 4\ell^{3/2}.$$

Consider a random subset Y of Γ obtained by choosing every vertex u randomly and independently with probability a_u . Since all choices were independent we have that

$$\mathbb{E}[|Y|^2] = \sum_u a_u + \sum_{u \neq v} a_u a_v \leq \left(\sum_u a_u \right)^2 + \ell/2.$$

We also have that the expected number of edges spanned by Y is $\mathbb{E}[e(Y)] = \sum_{(u,v) \in E(\Gamma)} a_u a_v$. Then, by the above discussion, $\mathbb{E}[e(Y) - |Y|^2/4] > 3\ell^{3/2}$. In particular, there is subset Y of Γ with this property, which contradicts (5). This shows that every subset S of $n/2$ vertices of G satisfies

$$\left|e(S) - \frac{1}{16}n^2\right| \leq 2^{-k/4+2}n^2$$

and completes the proof. \square

5 Induced Ramsey Numbers and Pseudorandom Graphs

The main result in this section is Theorem 5.4, which shows that any sufficiently pseudo-random graph of appropriate density has strong induced Ramsey properties. It generalizes Theorem 1.4 and Corollary 1.6 from the introduction. Combined with known examples of pseudo-random graphs, this theorem gives various explicit constructions which match and improve the best known estimates for induced Ramsey numbers.

The idea of the proof of Theorem 5.4 is rather simple. We have a sufficiently large, pseudo-random graph G that is not too sparse or dense. We also have d -degenerate graphs H_1 and H_2 each with vertex set $[k]$ and chromatic number at most q . We suppose for contradiction that there is a red-blue edge-coloring of G without an induced red copy of H_1 and without an induced blue copy of H_2 . We may view the red-blue coloring of G as a red-blue-green edge-coloring of the complete graph $K_{|G|}$, in which the edges of G have their original color, and the edges of the complement \bar{G} are colored green. The fact that in G there is no induced red copy of H_1 means that the red-blue-green coloring of $K_{|G|}$ does not contain a particular red-green coloring of the complete graph K_k . Then we prove, similar to Lemma 2.1 of Erdős and Hajnal, that any large subset of vertices of G contains two large disjoint subsets for which the edge density in color red between them is small. By using the key lemma from Section 3, we find k large disjoint vertex subsets V_1, \dots, V_k of G for which the edge density in color red is small between any pair (V_i, V_j) for which (i, j) an edge of H_2 .

Next we try to find an induced blue copy of H_2 with vertex i in V_i for all $i \in [k]$. Since the edge density between V_i and V_j in color red is sufficiently small for every edge (i, j) of H_2 , we can build an induced blue copy of H_2 one vertex at a time. At each step of this process we use pseudo-randomness of G to make sure that the existing possible subsets for not yet embedded vertices of H_2 are sufficiently large and that the density of red edges does not increase a lot between any pair of subsets corresponding to adjacent vertices of H_2 . This last part of the proof, embedding an induced blue copy of H_2 , is the most technically involved and handled by Lemma 5.5.

Recall that $[i] = \{1, \dots, i\}$ and that a graph is d -degenerate if every subgraph has a vertex of degree at most d . For an edge-coloring $\Psi : E(K_k) \rightarrow [r]$, we say that another edge-coloring $\Phi : E(K_n) \rightarrow [s]$ is Ψ -free if, for every subset W of size k of the complete graph K_n , the restriction of Φ to W is not isomorphic to Ψ . In the following lemma, we have a coloring Ψ of the edges of the complete graph K_k with colors 1 and 2 such that the graph of color 2 is d -degenerate. We also have a Ψ -free coloring Φ of the edges of the complete graph K_n such that between any two large subsets of vertices there are sufficiently many edges of color 1. With these assumptions, we show that there are two large subsets of K_n which in coloring Φ have few edges of color 2 between them. A graph G is $bi-(\epsilon, \delta)$ -dense if $d(A, B) > \epsilon$ holds for all disjoint subsets $A, B \subset V(G)$ with $|A|, |B| \geq \delta|V(G)|$.

Lemma 5.1 *Let d and k be positive integers and $\Psi : E(K_k) \rightarrow [2]$ be a 2-coloring of the edges of K_k such that the graph of color 2 is d -degenerate. Suppose that $q, \epsilon \in (0, 1)$ and $\Phi : E(K_n) \rightarrow [s]$ is a Ψ -free edge-coloring such that the graph of color 1 is $bi-(q, \epsilon^d q^k k^{-2})$ -dense. Then there are disjoint subsets A and B of K_n with $|A|, |B| \geq \epsilon^d q^k k^{-2} n$ such that every vertex of A is connected to at most $\epsilon|B|$ vertices in B by edges of color 2.*

Proof. Note that from definition, the vertices of every d -degenerate graph can be labeled $1, 2, \dots$

such that for every vertex ℓ the number of vertices $j < \ell$ adjacent to it is at most d . (Indeed, remove from the graph a vertex of minimum degree, place it in the end of the list and repeat this process in the remaining subgraph.) Therefore we may assume that the labeling $1, \dots, k$ of vertices of K_k has the property that for every $\ell \in [k]$ there are at most d vertices $j < \ell$ such that the color $\Psi(j, \ell) = 2$. Partition the vertex set of K_n into sets V_1, \dots, V_k each of size $\frac{n}{k}$. For $w \in V_i$ and a subset $S \subset V_j$ with $j \neq i$, let $N(w, S) = \{s \in S \mid \Phi(w, s) = \Psi(i, j)\}$. For $i < \ell$, let $D(\ell, i)$ denote the number of vertices $j \leq i$ such that the color $\Psi(j, \ell) = 2$. By the above assumption, $D(\ell, i) \leq d$ for $1 \leq i < \ell \leq k$.

We will try iteratively to build a copy of K_k with coloring Ψ . After i steps, we either find two disjoint subsets of vertices A, B which satisfy the assertion of the lemma or we will have vertices v_1, \dots, v_i and subsets $V_{i+1,i}, V_{i+2,i}, \dots, V_{k,i}$ such that

1. $V_{\ell,i}$ is a subset of V_ℓ of size $|V_{\ell,i}| \geq \epsilon^{D(\ell,i)} q^{i-D(\ell,i)} |V_\ell|$ for all $i+1 \leq \ell \leq k$,
2. $\Phi(v_j, v_\ell) = \Psi(j, \ell)$ for $1 \leq j < \ell \leq i$,
3. and if $j \leq i < \ell$ and $w \in V_{\ell,i}$, then $\Phi(v_j, w) = \Psi(j, \ell)$.

In the first step, we call a vertex $w \in V_1$ *good* if $|N(w, V_j)| \geq \epsilon |V_j|$ for all $j > 1$ with $\Psi(1, j) = 2$ and $|N(w, V_j)| \geq q |V_j|$ for all $j > 1$ with $\Psi(1, j) = 1$. If there is no good vertex in V_1 , then there is a subset $A \subset V_1$ with $|A| \geq \frac{1}{k-1} |V_1|$ and index $j > 1$ such that either $\Psi(1, j) = 1$ and every vertex $w \in A$ is connected to less than $q |V_j|$ vertices in V_j by edges of color 1 or $\Psi(1, j) = 2$ and every vertex $w \in A$ is connected to less than $\epsilon |V_j|$ vertices in V_j by edges of color 2. Letting $B = V_j$, we conclude that the first case is impossible since the graph of color 1 is $\text{bi-}(q, \epsilon^d q^k k^{-2})$ -dense, while in the second case we would be done, since A and B would satisfy the assertion of the lemma. Therefore, we may assume that there is a good vertex $v_1 \in V_1$, and we define $V_{i,1} = N(v_1, V_i)$ for $i > 1$.

Suppose that after step i the properties 1-3 are still satisfied. Then, in step $i+1$, a vertex $w \in V_{i+1,i}$ is called *good* if $|N(w, V_{j,i})| \geq \epsilon |V_{j,i}|$ for each $j > i+1$ with $\Psi(i+1, j) = 2$ and $|N(w, V_{j,i})| \geq q |V_{j,i}|$ for each $j > i+1$ with $\Psi(i+1, j) = 1$. If there is no good vertex in $V_{i+1,i}$, then there is a subset $A \subset V_{i+1,i}$ with $|A| \geq \frac{1}{k-i-1} |V_{i+1,i}|$ and $j > i+1$ such that either $\Psi(i+1, j) = 1$ and every vertex $w \in A$ has fewer than $q |V_{j,i}|$ edges of color 1 to $V_{j,i}$ or $\Psi(i+1, j) = 2$ and every vertex $w \in A$ is connected to less than $\epsilon |V_{j,i}|$ vertices in $V_{j,i}$ by edges of color 2. Note that even in the last step when $i+1 = k$ the size of A is still at least $|V_{k,k-1}|/k \geq \epsilon^d q^k |V_k|/k \geq \epsilon^d q^k k^{-2} n$. Therefore, letting $B = V_{j,i}$, we conclude that as before the first case is impossible since the graph of color 1 is $\text{bi-}(q, \epsilon^d q^k k^{-2})$ -dense, while the second case would complete the proof, since A and B would satisfy the assertion of the lemma. Hence, we may assume that there is a good vertex $v_{i+1} \in V_{i+1,i}$, and we define $V_{j,i+1} = N(v_{i+1}, V_{j,i})$ for $j > i+1$. Note that $|V_{j,i+1}| \geq q |V_{j,i}|$ if $\Psi(i+1, j) = 1$ and $|V_{j,i+1}| \geq \epsilon |V_{j,i}|$ if $\Psi(i+1, j) = 2$. This implies that after step $i+1$ we have that $|V_{\ell,i+1}| \geq \epsilon^{D(\ell,i+1)} q^{i+1-D(\ell,i+1)} |V_\ell|$ for all $i+2 \leq \ell \leq k$.

The iterative process must stop at one of the steps $j \leq k-1$, since otherwise the coloring Φ would not be Ψ -free. As we already explained above, when this happens we have two disjoint subsets A and B that satisfy the assertion of the lemma. \square

Notice that if coloring $\Phi : K_n \rightarrow [s]$ is Ψ -free, then so is Φ restricted to any subset of K_n of size αn . Therefore, Lemma 5.1 has the following corollary.

Corollary 5.2 *Let d and k be positive integers and $\Psi : E(K_k) \rightarrow [2]$ be a 2-coloring of the edges of K_k such that the graph of color 2 is d -degenerate. If $q, \alpha, \epsilon \in (0, 1)$ and $\Phi : E(K_n) \rightarrow [s]$ is a Ψ -free*

edge-coloring such that the graph of color 1 is bi- $(q, \alpha\rho)$ -dense with $\rho = \epsilon^d q^k k^{-2}$, then the graph of color 2 is $(\alpha, \rho, \epsilon, 2)$ -sparse.

The next statement follows immediately from Corollary 5.2 (with $\epsilon/4$ instead of ϵ) and Corollary 3.3.

Corollary 5.3 *Let d, k , and h be positive integers and $\Psi : E(K_k) \rightarrow [2]$ be a 2-coloring of the edges of K_k such that the graph of color 2 is d -degenerate. Suppose that $q, \alpha, \epsilon \in (0, 1)$ and $\Phi : E(K_n) \rightarrow [s]$ is a Ψ -free edge-coloring such that the graph of color 1 is bi- $(q, \alpha\rho)$ -dense with $\rho = (\epsilon/4)^d q^k k^{-2}$. Then the graph of color 2 is $((\frac{2}{\rho})^{h-1} \alpha, 2^{1-h} \rho^h, \epsilon, 2^h)$ -sparse.*

Pending one additional lemma, we are now ready to prove the main result of this section, showing that pseudo-random graphs have strong induced Ramsey properties.

Theorem 5.4 *Let $\chi \geq 2$ and G be a (p, λ) -pseudo-random graph with $0 < p \leq 3/4$ and $\lambda \leq ((\frac{p}{10k})^d 2^{-pk})^{20 \log \chi} n$. Then every d -degenerate graph on k vertices with chromatic number at most χ occurs as an induced monochromatic copy in every 2-coloring of the edges of G . Moreover, all of these induced monochromatic copies can be found in the same color.*

Taking $p = 1/k$, $n = k^{cd \log \chi}$ and constant c sufficiently large so that $((\frac{p}{10k})^d 2^{-pk})^{20 \log \chi} > n^{-0.1}$ one can easily see that this result implies Theorem 1.4. To obtain Corollary 1.6, recall that for a prime power n , the Paley graph P_n has vertex set \mathbb{F}_n and distinct vertices $x, y \in \mathbb{F}_n$ are adjacent if $x - y$ is a square. This graph is $(1/2, \lambda)$ -pseudo-random with $\lambda = \sqrt{n}$ (see e.g., [38]). Therefore, for sufficiently large constant c , the above theorem with $n = 2^{ck \log^2 k}$, $p = 1/2$ and $d = \chi = k$ implies that every graph on k vertices occurs as an induced monochromatic copy in all 2-edge-colorings of the Paley graph. Similarly, one can prove that there is a constant c such that, with high probability, the random graph $G(n, 1/2)$ with $n \geq 2^{ck \log^2 k}$ satisfies that every graph on k vertices occurs as an induced monochromatic copy in all 2-edge-colorings of G .

Proof of Theorem 5.4. Suppose for contradiction that there is an edge-coloring Φ_0 of G with colors red and blue, and d -degenerate graphs H_1 and H_2 each having k vertices and chromatic number at most χ such that there is no induced red copy of H_1 and no induced blue copy of H_2 . Since H_1, H_2 are d -degenerate graphs on k vertices we may suppose that their vertex set is $[k]$ and every vertex i has at most d neighbors less than i in both H_1 and H_2 .

Consider the red-blue-green edge-coloring Φ of the complete graph K_n , in which the edges of G have their original coloring Φ_0 , and the edges of the complement \bar{G} are colored green. Let Ψ be the edge-coloring of the complete graph K_k where the red edges form a copy of H_1 and the remaining edges are green. By assumption, the coloring Φ is Ψ -free. Since G is (p, λ) -pseudo-random, we have that the density of edges in \bar{G} between any two disjoint sets A, B of size at least $6p^{-1}\lambda$ is at least

$$d_{\bar{G}}(A, B) = 1 - d_G(A, B) \geq 1 - \left(p + \frac{\lambda}{\sqrt{|A||B|}} \right) \geq 1 - \frac{7}{6}p.$$

Therefore the green graph in coloring Φ is bi- $(q, 6p^{-1}\frac{\lambda}{n})$ -dense for $q = 1 - 7p/6$.

Let $\epsilon = \frac{p}{1000k^6}$, $\rho = (\epsilon/4)^d q^k k^{-2}$, $h = \log \chi$, and $\alpha = (\rho/2)^{h-1}$. Using that $q = 1 - 7p/6$ and $\lambda/n \leq ((\frac{p}{10k})^d 2^{-pk})^{20 \log \chi}$ it is straightforward to check that $6p^{-1}\frac{\lambda}{n} \leq 2^{1-h}\rho^h = \alpha\rho$. By Corollary 5.3

and Definition 3.1, there are $2^h = \chi$ subsets W_1, \dots, W_χ of K_n with $|W_1| = \dots = |W_\chi| \geq 2^{1-h} \rho^h n$, such that the sum of the densities of red edges between all pairs W_i and W_j is at most $\binom{\chi}{2} \epsilon$. Hence, the density between W_i and W_j is also at most $\chi^2 \epsilon$ for all $1 \leq i < j \leq \chi$. Partition every set W_i into k subsets each of size $|W_i|/k \geq \frac{1}{k} 2^{1-h} \rho^h n$. Since the chromatic number of H_2 is at most χ and it has k vertices, we can choose for every vertex i of H_2 one of these subsets, which we call V_i , such that all subsets corresponding to vertices of H_2 in the same color class (of a proper χ -coloring) come from the same set W_ℓ . In particular, for every edge (i, j) of H_2 , the corresponding sets V_i and V_j lie in two different sets $\{W_\ell\}$. Since the size of V_i 's is by a factor k smaller than the size of W_ℓ 's the density of red edges between V_i and V_j corresponding to an edge in H_2 is at most $k^2 \chi^2 \epsilon \leq \frac{p}{1000k^2}$ (note that it can increase by a factor at most k^2 compared to the density between the sets $\{W_\ell\}$). Notice that the subgraph $G' \subset G$ induced by $V_1 \cup \dots \cup V_k$ has $n' \geq 2^{1-h} \rho^h n$ vertices and is also (p, λ) -pseudo-random. By the definitions of ρ and h , and our assumption on λ , we have that

$$\lambda/n' \leq 2^{h-1} \rho^{-h} \lambda/n \leq 2^{h-1} \rho^{-h} \left(\left(\frac{p}{10k} \right)^d 2^{-pk} \right)^{20 \log \chi} \leq \left(\left(\frac{p}{10k} \right)^d 2^{-pk} \right)^{10 \log \chi}.$$

Applying Lemma 5.5 below with $H = H_2$ to the coloring Φ_0 of graph G' with partition $V_1 \cup \dots \cup V_k$, we find an induced blue copy of H_2 , completing the proof. \square

Lemma 5.5 *Let H be a d -degenerate graph with vertex set $[k]$ such that each vertex i has at most i neighbors less than i . Let $G = (V, E)$ be a (p, λ) -pseudo-random graph on n vertices with $0 < p \leq 3/4$, $\lambda \leq \left(\left(\frac{p}{10k} \right)^d 2^{-pk} \right)^{10} n$ and let $V = V_1 \cup \dots \cup V_k$ be a partition of its vertices such that each V_i has size n/k . Suppose that the edges of G are 2-colored, red and blue, such that for every edge (j, ℓ) of H , the density of red edges between the pair (V_j, V_ℓ) is at most $\beta = \frac{p}{1000k^2}$. Then there is an induced blue copy of H in G for which the image of every vertex $i \in [k]$ lies in V_i .*

Proof. For $i < j$, let $D(i, j)$ denote the number of neighbors of j that are at most i . Let $\epsilon_1 = \frac{1}{k}$, $\epsilon_2 = \frac{p}{10k}$, and $\delta = (1 - p)^k p^d$. Since $p \leq 3/4$, notice that $\delta \geq 2^{-3pk} p^d$ and

$$\lambda \leq \left(\left(\frac{p}{10k} \right)^d 2^{-pk} \right)^{10} n \leq \frac{p^8}{(10k)^{10}} \delta^2 n. \quad (6)$$

We construct an induced blue copy of H one vertex at a time. At the end of step i , we will have vertices v_1, \dots, v_i and subsets $V_{j,i} \subset V_j$ for $j > i$ such that the following four conditions hold

1. for $j, \ell \leq i$, if (j, ℓ) is an edge of H , then (v_j, v_ℓ) is a blue edge of G , otherwise v_j and v_ℓ are not adjacent in G ,
2. for $j \leq i < \ell$, if (j, ℓ) is an edge of H , then v_j is adjacent to all vertices in $V_{\ell,i}$ by blue edges, otherwise there are no edges of G from v_j to $V_{\ell,i}$,
3. for $i < j$, we have $|V_{j,i}| \geq (1 - p - \epsilon_2)^{i-D(i,j)} (p - \epsilon_2)^{D(i,j)} |V_j|$,
4. and for $j, \ell > i$ if (j, ℓ) is an edge of H , then the density of red edges between $V_{j,i}$ and $V_{\ell,i}$ is at most $(1 + \epsilon_1)^i \beta$.

Clearly, in the end of the first k steps of this process we obtain a required copy of H . For $i = 0$ and $j \in [k]$, define $V_{j,0} = V_j$. Notice that the above four properties are satisfied for $i = 0$ (the first two properties being vacuously satisfied). We now assume that the above four properties are satisfied at the end of step i , and show how to complete step $i + 1$ by finding a vertex $v_{i+1} \in V_{i+1,i}$ and subsets $V_{j,i+1} \subset V_{j,i}$ for $j > i + 1$ such that the conditions 1-4 still hold.

We need to introduce some notation. For a vertex $w \in V_j$ and a subset $S \subset V_\ell$ with $j \neq \ell$, let

- $N(w, S)$ denote the set of vertices $s \in S$ such that (s, w) is an edge of G ,
- $B(w, S)$ denote the set of vertices $s \in S$ such that (s, w) is a blue edge of G ,
- $R(w, S)$ denote the set of vertices $s \in S$ such that (s, w) is a red edge of G ,
- $\tilde{N}(w, S) = N(w, S)$ if (j, ℓ) is an edge of H and $\tilde{N}(w, S) = S \setminus N(w, S)$ otherwise,
- $\tilde{B}(w, S) = B(w, S)$ if (j, ℓ) is an edge of H and $\tilde{B}(w, S) := S \setminus N(w, S)$ otherwise, and
- $p_{j,\ell} = p$ if (j, ℓ) is an edge of H and $p_{j,\ell} = 1 - p$ if (j, ℓ) is not an edge of H .

Note that since graph G is pseudo-random with edge density p , by the above definitions, for every large subset $S \subset V_\ell$ and for most vertices $w \in V_j$ we expect the size of $\tilde{N}(w, S)$ to be roughly $p_{j,\ell}|S|$. We also have for all $S \subset V_\ell$ and $w \in V_j$ that $\tilde{B}(w, S) = \tilde{N}(w, S) \setminus R(w, S)$.

Call a vertex $w \in V_{i+1,i}$ *good* if for all $j > i + 1$, $\tilde{B}(w, V_{j,i}) \geq (p_{i+1,j} - \epsilon_2)|V_{j,i}|$ and for every edge (j, ℓ) of H with $j, \ell > i + 1$, the density of red edges between $\tilde{B}(w, V_{j,i})$ and $\tilde{B}(w, V_{\ell,i})$ is at most $(1 + \epsilon_1)^{i+1}\beta$. If we find a good vertex $w \in V_{i+1,i}$, then we simply let $v_{i+1} = w$ and $V_{j,i+1} = \tilde{B}(w, V_{j,i})$ for $j > i + 1$, completing step $i + 1$. It therefore suffices to show that there is a good vertex in $V_{i+1,i}$.

We first throw out some vertices of $V_{i+1,i}$ ensuring that the remaining vertices satisfy the first of the two properties of good vertices. For $j > i + 1$ and an edge $(i + 1, j)$ of H , let R_j consist of those $w \in V_{i+1,i}$ for which the number of red edges (w, w_j) with $w_j \in V_{j,i}$ is at least $\frac{\epsilon_2}{2}|V_{j,i}|$. Since the density of red between $V_{i+1,i}$ and $V_{j,i}$ is at most $(1 + \epsilon_1)^i\beta$, then R_j contains at most

$$|R_j| \leq \frac{(1 + \epsilon_1)^i\beta|V_{i+1,i}||V_{j,i}|}{\frac{\epsilon_2}{2}|V_{j,i}|} = 2(1 + \epsilon_1)^i\epsilon_2^{-1}\beta|V_{i+1,i}|$$

vertices. Let V' be the set of vertices in $V_{i+1,i}$ that are not in any of the R_j . Using that $\epsilon_1 = 1/k$, $\epsilon_2 = \frac{p}{10k}$ and $\beta = \frac{p}{1000k^2}$ we obtain

$$\begin{aligned} |V'| &\geq |V_{i+1,i}| - \sum_{j>i+1} |R_j| \geq |V_{i+1,i}| - k \left(2(1 + \epsilon_1)^i\epsilon_2^{-1}\beta|V_{i+1,i}| \right) \\ &\geq \left(1 - 2k(1 + \epsilon_1)^k\epsilon_2^{-1}\beta \right) |V_{i+1,i}| \geq \frac{1}{2}|V_{i+1,i}|. \end{aligned}$$

For $j > i + 1$, let S_j consist of those $w \in V'$ for which $\tilde{N}(w, V_{j,i}) < (p_{i+1,j} - \frac{\epsilon_2}{2})|V_{j,i}|$. Then the density of edges of G between S_j and $V_{j,i}$ deviates from p by at least $\frac{\epsilon_2}{2}$. Since graph G is (p, λ) -pseudo-random, we obtain that $\frac{\epsilon_2}{2} \leq \frac{\lambda}{\sqrt{|V_{j,i}||S_j|}}$ and hence $|S_j| \leq \frac{4\lambda^2}{\epsilon_2^2|V_{j,i}|}$. Also using that $p \leq 3/4$

we have $1 - p - \epsilon_2 = 1 - p - \frac{p}{10k} \geq (1 - \frac{1}{3k})(1 - p)$. Therefore, our third condition, combined with $\delta = (1 - p)^k p^d$ and $(1 - x)^t \geq 1 - xt$ for all $0 \leq x \leq 1$, imply that for $j \geq i + 1$

$$\begin{aligned}
|V_{j,i}| &\geq (1 - p - \epsilon_2)^{i-D(i,j)} (p - \epsilon_2)^{D(i,j)} |V_j| \geq (1 - p - \epsilon_2)^k (p - \epsilon_2)^d |V_j| \\
&\geq \left(\left(1 - \frac{1}{3k}\right)(1 - p) \right)^k \left(p - \frac{p}{10k} \right)^d |V_j| \\
&\geq \left(1 - \frac{1}{3k}\right)^k \left(1 - \frac{1}{10k}\right)^k (1 - p)^k p^d |V_j| \\
&\geq \frac{1}{2} (1 - p)^k p^d |V_i| = \frac{\delta n}{2k}.
\end{aligned} \tag{7}$$

Since $\lambda \leq \frac{p\delta}{100k^3}n$ (see (6)) and $\epsilon_2 = \frac{p}{10k}$, we therefore have $|S_j| \leq \frac{4\lambda^2}{\epsilon_2^2 |V_{j,i}|} \leq \frac{1}{4k} |V_{i+1,i}|$. Let V'' be the set of vertices in V' that are not in any of the sets S_j . The cardinality of V'' is at least

$$|V''| \geq |V'| - \sum_{j>i+1} |S_j| \geq |V'| - k \cdot \left(\frac{1}{4k} |V_{i+1,i}| \right) \geq |V'| - \frac{1}{4} |V_{i+1,i}| \geq \frac{1}{4} |V_{i+1,i}|.$$

Moreover, by definition, for every $j > i + 1$ and every vertex $w \in V''$ there are $|R(w, V_{j,i})| \leq \frac{\epsilon_2}{2} |V_{j,i}|$ red edges from w to $V_{j,i}$ if $(i + 1, j)$ is an edge of H and also $\tilde{N}(w, V_{j,i})$ has size at least $(p_{i+1,j} - \frac{\epsilon_2}{2}) |V_{j,i}|$. This implies that

$$|\tilde{B}(w, V_{j,i})| = |\tilde{N}(w, V_{j,i}) \setminus R(w, V_{j,i})| \geq |\tilde{N}(w, V_{j,i})| - \frac{\epsilon_2}{2} |V_{j,i}| \geq (p_{i+1,j} - \epsilon_2) |V_{j,i}|$$

and therefore the vertices of V'' satisfy the first of the two properties of good vertices.

We have reduced our goal to showing that there is an element of V'' that has the second property of good vertices. For $i + 1 < j < \ell \leq k$ and (j, ℓ) an edge of H , let $T_{j,\ell}$ denote the set of $w \in V''$ such that the density of red edges between $\tilde{B}(w, V_{j,i})$ and $\tilde{B}(w, V_{\ell,i})$ is more than $(1 + \epsilon_1)^{i+1} \beta$. Notice that any vertex of V'' not in any of the sets $T_{j,\ell}$ is good. Therefore, if we show that $|T_{j,\ell}| < \frac{|V''|}{k^2}$ for each $T_{j,\ell}$, then there is a good vertex in V'' and the proof would be complete. To do so we will assume without loss of generality that $p_{i+1,j}$ and $p_{i+1,\ell}$ are both p (the other 3 cases can be treated similarly using the fact that \tilde{G} is $(1 - p, \lambda)$ -pseudo-random). Since by (7) we have that $|V_{\ell,i}|, |V_{j,i}| \geq \frac{\delta n}{2k}$ and $\frac{|V''|}{k^2} \geq \frac{1}{4k^2} |V_{i+1,i}| \geq \frac{\delta n}{8k^3}$, the result follows from the following claim.

Claim 5.6 *Let X, Y and Z be three disjoint subsets of our (p, λ) -pseudo-random graph G such that $|X| \geq \frac{\delta n}{8k^3}$ and $|Y|, |Z| \geq \frac{\delta n}{2k}$. For every $w \in X$ let $B_1(w), B_2(w)$ be the set of vertices in Y and Z respectively connected to w by a blue edge and suppose that $|B_1(w)| \geq (p - \frac{p}{10k})|Y|$ and $|B_2(w)| \geq (p - \frac{p}{10k})|Z|$. Also suppose that the density of red edges between Y and Z is at most η for some $\eta \geq \frac{p}{1000k^2}$. Then there is a vertex $w \in X$ such that the density of red edges between $B_1(w)$ and $B_2(w)$ is at most $\frac{k+1}{k}\eta$.*

Proof. Let m denote the number of triangles (x, y, z) with $x \in X, y \in Y, z \in Z$, such that the edge (y, z) is red. We need an upper bound on m . Let U be the set of vertices in Y that have fewer than $p^3 \delta^3 (10k)^{-10} n$ red edges to Z . So the number m_1 of triangles (x, y, z) which have $y \in U$ and edge (y, z) red is clearly at most $m_1 \leq p^3 \delta^3 (10k)^{-10} n^3$. Let W_1, W_2 denote the subsets of vertices in Y whose number of neighbors in X is at least $(p + \frac{p}{20k})|X|$ or respectively at most $(p - \frac{p}{20k})|X|$. Since the

density of edges between W_i and X deviates from p by at least $\frac{p}{20k}$, using (p, λ) -pseudo-randomness of G , we have $\frac{p}{20k} \leq \frac{\lambda}{\sqrt{|X||W_i|}}$, or equivalently, $|X||W_i| \leq 400k^2p^{-2}\lambda^2$. Therefore, using the upper bound $\lambda \leq \frac{p^8}{(10k)^{10}}\delta^2n$ from (6), the number m_2 of triangles (x, y, z) with $y \in W = W_1 \cup W_2$ and edge (y, z) red is at most

$$m_2 \leq |X||W|n \leq 800k^2p^{-2}\lambda^2n \leq (10k)^{-10}p^4\delta^4n^3.$$

For $y \in Y \setminus (U \cup W)$, we have the number of neighbors of y in X satisfy $\left| \frac{|N(y, X)|}{|X|} - p \right| \leq \frac{p}{20k}$ and the number of red edges from y to Z is at least $p^3\delta^3(10k)^{-10}n$. Recall that $R(y, Z)$ denotes the set of vertices in Z connected to y by red edges, hence we have that $|R(y, Z)| \geq p^3\delta^3(10k)^{-10}n$ for every $y \in Y \setminus (U \cup W)$. We also have that $|N(y, X)| \geq p|X|/2 \geq \frac{p\delta n}{16k^3}$. Since G is (p, λ) -pseudo-random, we can bound the number of edges between $N(y, X)$ and $R(y, Z)$ by $p|N(y, X)||R(y, Z)| + \lambda\sqrt{|N(y, X)||R(y, Z)|}$. Using the above lower bounds on $|N(y, X)|$ and $|R(y, Z)|$, and the upper bound (6) for λ , one can easily check that

$$\frac{\lambda}{\sqrt{|N(y, X)||R(y, Z)|}} \leq \frac{\lambda}{\sqrt{(p\delta n/(16k^3))(p^3\delta^3(10k)^{-10}n)}} \leq \frac{p}{20k}.$$

Hence the number of edges between $N(y, X)$ and $R(y, Z)$ is at most $(p + \frac{p}{20k})|N(y, X)||R(y, Z)|$. Recall that for all $y \in Y \setminus (U \cup W)$ we have that $|N(y, X)| \leq (p + \frac{p}{20k})|X|$. Also, since the density of red edges between Y and Z is at most η , we have that $\sum_y |R(y, Z)| \leq \eta|Y||Z|$. Therefore, the number m_3 of triangles (x, y, z) with $y \in Y \setminus (U \cup W)$, $x \in X$, $z \in Z$ such that the edge (y, z) is red is at most

$$m_3 \leq \left(p + \frac{p}{20k}\right) \sum_{y \in Y \setminus (U \cup W)} |N(y, X)||R(y, Z)| \leq \left(p + \frac{p}{20k}\right)^2 |X| \sum_y |R(y, Z)| \leq \left(p + \frac{p}{20k}\right)^2 \eta |X||Y||Z|.$$

Using the lower bounds on $|X|, |Y|, |Z|, \eta$ from the assertion of the claim we have that

$$p^2\eta|X||Y||Z| \geq \frac{p^3\delta^3}{(10k)^7}n^3 \geq (10k)^3 \max(m_1, m_2).$$

This implies that the total number of triangles (x, y, z) with $x \in X, y \in Y, z \in Z$, such that the edge (y, z) is red is at most

$$\begin{aligned} m &\leq m_1 + m_2 + m_3 \leq 2\frac{p^2\eta|X||Y||Z|}{(10k)^3} + \left(p + \frac{p}{20k}\right)^2 \eta |X||Y||Z| \\ &\leq (1 + 1/(8k))p^2\eta|X||Y||Z|. \end{aligned}$$

Therefore, there is vertex $w \in X$ such that the number of these triangles through w is at most $(1 + 1/(8k))p^2\eta|Y||Z|$. Since $B_1(w) \subset N(w, Y)$ and $B_2(w) \subset N(w, Z)$, then the number of red edges between $B_1(w)$ and $B_2(w)$ is at most $(1 + 1/(8k))p^2\eta|Y||Z|$. Since we have that $|B_1(w)| \geq (p - \frac{p}{10k})|Y|$ and $|B_2(w)| \geq (p - \frac{p}{10k})|Z|$, the density of red edges between $B_1(w)$ and $B_2(w)$ can be at most

$$\frac{(1 + 1/(8k))p^2\eta|Y||Z|}{|B_1(w)||B_2(w)|} \leq \frac{(1 + 1/(8k))p^2\eta}{(p - \frac{p}{10k})^2} \leq \frac{k+1}{k}\eta,$$

completing the proof. \square

6 Trees with superlinear induced Ramsey numbers

In this section we prove Theorem 1.7, that there are trees whose induced Ramsey number is superlinear in the number of vertices. The proof uses Szemerédi's regularity lemma, which we mentioned in the introduction.

A red-blue edge-coloring of the edges of a graph partitions the graph into two monochromatic subgraphs, the *red graph*, which contains all vertices and all red edges, and the *blue graph*, which contains all vertices and all blue edges. The weak induced Ramsey number $r_{\text{weak ind}}(H_1, H_2)$, introduced by Gorgol and Łuczak [33], is the least positive integer n such that there is a graph G on n vertices such that for every red-blue coloring of the edges of G , either the red graph contains H_1 as an induced subgraph or the blue graph contains H_2 as an induced subgraph. Note that this definition is a relaxation of the induced Ramsey numbers since we allow blue edges between the vertices of a red copy of H_1 or red edges between the vertices of a blue copy of H_2 . Therefore a weak induced Ramsey number lies between the usual Ramsey number and the induced Ramsey number. Using this new notion we can strengthen Theorem 1.7 as follows. Recall that $K_{1,k}$ denotes a star with k edges.

Theorem 6.1 *For each $\alpha \in (0, 1)$, there is a constant $k(\alpha)$ such that if H is a graph on $k \geq k(\alpha)$ vertices with maximum independent set of size less than $(1 - \alpha)k$, then $r_{\text{weak ind}}(K_{1,k}, H) \geq \frac{k}{\alpha}$.*

Let T be a tree which is a union of path of length $k/2$ with the star of size $k/2$ such that the end point of the path is the center of the star. Since T contains the path $P_{k/2}$ and the star $K_{1,k/2}$ as induced subgraphs, then $r_{\text{ind}}(T) \geq r_{\text{weak ind}}(P_{k/2}, K_{1,k/2})$. By using the above theorem with $k/2$ instead of k , $H = P_{k/2}$, and sufficiently small α , we obtain that $r_{\text{ind}}(T)/k \rightarrow \infty$. Moreover the same holds for every sufficiently large tree which contains a star and a matching of linear size as subgraphs. We deduce Theorem 6.1 from the following lemma.

Lemma 6.2 *For each $\delta > 0$ there is a constant $c_\delta > 0$ such that if $G = (V, E)$ is a graph on n vertices, then there is a 2-coloring of the edges of G with colors red and blue such that the red graph has maximum degree less than δn and for every subset $W \subset V$, either there are at least $c_\delta n^2$ blue edges in the subgraph induced by W or there is an independent set in W in the blue graph of cardinality at least $|W| - \delta n$.*

Proof. Let $\epsilon = \frac{\delta^2}{100}$. By Szemerédi's regularity lemma, there is a positive integer $M(\epsilon)$ together with an equitable partition $V = \bigcup_{i=1}^k V_i$ of vertices of the graph $G = (V, E)$ into k parts with $\frac{1}{\epsilon} < k < M(\epsilon)$ such that all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ -regular. Recall that a partition is equitable if $||V_i| - |V_j|| \leq 1$ and a pair (V_i, V_j) is called ϵ -regular if for every $X \subset V_i$ and $Y \subset V_j$ with $|X| > \epsilon|V_i|$ and $|Y| > \epsilon|V_j|$, we have $|d(X, Y) - d(V_i, V_j)| < \epsilon$. Let $c_\delta = \epsilon M(\epsilon)^{-2}$. Notice that to prove Lemma 6.2, it suffices to prove it under the assumption that n is sufficiently large. So we may assume that $n \geq \epsilon^{-1} M(\epsilon)$.

If a pair (V_i, V_j) is ϵ -regular with density $d(V_i, V_j)$ at least 2ϵ , then color the edges between V_i and V_j blue. Let G' be the subgraph of G formed by deleting the edges of G that are already colored blue. Let V' be the vertices of G' of degree at least δn . Color blue any edge of G' with a vertex in V' . The remaining edges are colored red. First notice that every vertex has red degree less than δn .

We next show that $|V'|$ is small by showing that G' has few edges. There are at most

$$\sum_{i=1}^k \binom{|V_i|}{2} \leq \frac{n^2}{k} \leq \epsilon n^2$$

edges (v, w) of G with v and w both in the same set V_i . Since at most ϵk^2 of the pairs (V_i, V_j) are not ϵ -regular, then there are at most ϵn^2 edges in such pairs. The ϵ -regular pairs (V_i, V_j) with density less than 2ϵ contain at most a fraction 2ϵ of all possible edges on n vertices. So there are less than ϵn^2 edges of this type. Therefore the number of edges of G' is at most $3\epsilon n^2$, and therefore there are at most $|V'| \leq 2 \frac{\epsilon(G')}{\delta n} \leq 6\epsilon \delta^{-1} n < \frac{\delta n}{10}$ vertices of degree at least δn in it.

Let $W \subset V$. Let $W' = W \setminus V'$, so W' has cardinality at least $|W| - \frac{\delta n}{10}$. Let $W_i = V_i \cap W'$. Let $W'' = \bigcup_{|W_i| \geq \epsilon |V_i|} W_i$. Notice that for any $i \in [k]$ there are at most $\epsilon \frac{n}{k}$ vertices in $(W' \setminus W'') \cap V_i$, so there are at most $k(\epsilon \frac{n}{k}) = \epsilon n = \frac{\delta^2 n}{100}$ vertices in $W' \setminus W''$. Therefore, W'' has at least $|W| - \delta n$ vertices. If there are $i \neq j$ such that $|W_i|, |W_j| \geq \epsilon \frac{n}{k}$ and the pair (V_i, V_j) is ϵ -regular with density at least 2ϵ , then there are at least

$$\epsilon |W_i| |W_j| \geq \frac{\epsilon}{k^2} n^2 \geq \epsilon M(\epsilon)^{-2} n^2 = c_\delta n^2$$

blue edges between W_i and W_j . In this case the blue subgraph induced by W has at least $c_\delta n^2$ edges. Otherwise, all the edges in W'' are red, and W'' is an independent set in the blue graph of cardinality at least $|W| - \delta n$. \square

Proof of Theorem 6.1. Let H be a graph on k vertices with maximum independent set of size less than $(1 - \alpha)k$. Take $\delta = \alpha^2$ and c_δ to be as in Lemma 6.2. Let $G = (V, E)$ be any graph on n vertices, where $n \leq \frac{k}{\alpha}$. If H has at least $c_\delta k^2$ edges, consider a random red-blue coloring of the edges of G such that the probability of an edge being red is $\frac{\alpha}{2}$. The expected degree of a vertex in the red graph is at most $\alpha n/2$. Therefore by the standard Chernoff bound for the Binomial distribution it is easy to see that with probability $1 - o(1)$ the degree of every vertex in the red graph is less than $\alpha n \leq k$, i.e., it contains no $K_{1,k}$. On the other hand, for k sufficiently large, the probability that the blue graph contains a copy of H is at most

$$n^k (1 - \alpha/2)^{e(H)} \leq n^k e^{-\alpha c_\delta k^2/2} \leq e^{-\alpha c_\delta k^2/2 + k \log(k/\alpha)} = o(1).$$

Thus with high probability this coloring has no blue copy of H as well. This implies that we can assume that the number of edges in H is less than $c_\delta k^2$.

By Lemma 6.2, there is a red-blue edge-coloring of the edges of G such that the red graph has maximum degree at most δn and every subset $W \subset V$ contains either an independent set in the blue graph of size at least $|W| - \delta n$ or contains at least $c_\delta n^2$ blue edges. Since $\delta n = \alpha^2 n < k$, then the red graph does not contain $K_{1,k}$ as a subgraph. Suppose for contradiction that there is an induced copy of H in the blue graph, and let W be the vertex set of this copy. The blue graph induced by W has $e(H) < c_\delta k^2 \leq c_\delta n^2$ edges. Therefore it contains an independent set of size at least $|W| - \delta n \geq |W| - \alpha k = (1 - \alpha)k$, contradicting the fact that H has no independent set of size $(1 - \alpha)k$. Therefore, there are no induced copies of H in the blue graph. \square

7 Concluding Remarks

- All of the results in this paper concerning induced subgraphs can be extended to many colors. One such multicolor result was already proved in Section 5 (see Lemma 5.1), and we use here the notation from that section. For example, one can obtain the following generalization of Theorem 1.1. For $k \geq 2$, let $\Psi : E(K_k) \rightarrow [r]$ be an edge-coloring of the complete graph K_k and $\Phi : E(K_n) \rightarrow [s]$ be a Ψ -free edge-coloring of the complete graph K_n . Then there is a constant c so that for every $\epsilon \in (0, 1/2)$, there is a subset $W \subset K_n$ of size at least $2^{-crk(\log \frac{1}{\epsilon})^2} n$ and a color $i \in [r]$ such that the edge density of color i in W is at most ϵ . Since the proofs of this statement and other generalizations can be obtained using our key lemma in essentially the same way as the proofs of the results that we already presented (which correspond to the two color case), we do not include them here.
- It would be very interesting to get a better estimate in Theorem 1.1. This will immediately give an improvement of the best known result for the Erdős-Hajnal conjecture on the size of the maximum homogeneous set in H -free graphs. We believe that our bound can be strengthened as follows.

Conjecture 7.1 *For each graph H , there is a constant $c(H)$ such that if $\epsilon \in (0, 1/2)$ and G is a H -free graph on n vertices, then there is an induced subgraph of G on at least $\epsilon^{c(H)} n$ vertices that has edge density either at most ϵ or at least $1 - \epsilon$.*

This conjecture if true would imply the Erdős-Hajnal conjecture. Indeed, take $\epsilon = n^{-\frac{1}{c(H)+1}}$. Then every H -free graph G on n vertices contains an induced subgraph on at least $\epsilon^{c(H)} n = n^{\frac{1}{c(H)+1}}$ vertices that has edge density at most ϵ or at least $1 - \epsilon$. Note that this induced subgraph or its complement has average degree at most 1, which implies that it contains a clique or independent set of size at least $\frac{1}{2} n^{\frac{1}{c(H)+1}}$.

- One of the main remaining open problems on induced Ramsey numbers is a beautiful conjecture of Erdős which states that there exists a positive constant c such that $r_{\text{ind}}(H) \leq 2^{ck}$ for every graph H on k vertices. This, if true, will show that induced Ramsey numbers in the worst case have the same order of magnitude as ordinary Ramsey numbers. Our results here suggest that one can attack this problem by studying 2-edge-colorings of a random graph with edge probability $1/2$. It looks very plausible that for sufficiently large constant c , with high probability the random graph $G(n, 1/2)$ with $n \geq 2^{ck}$ has the property that any of its 2-edge-colorings contains every graph on k vertices as an induced monochromatic subgraph. Moreover, maybe this is even true for every sufficiently pseudo-random graph with edge density $1/2$.
- The results on induced Ramsey numbers of sparse graphs naturally lead to the following questions. What is the asymptotic behavior of the maximum of induced Ramsey numbers over all trees on k vertices? We have proved $r_{\text{ind}}(T)$ is superlinear in k for some trees T . On the other hand, Beck [8] proved that $r_{\text{ind}}(T) = O(k^2 \log^2 k)$ for all trees T on k vertices.

For induced Ramsey numbers of bounded degree graphs, we proved a polynomial upper bound with exponent which is nearly linear in the maximum degree. Can this be improved further, e.g.,

is it true that the induced Ramsey number of every n -vertex graph with maximum degree d is at most a polynomial in n with exponent independent of d ? It is known that the usual Ramsey numbers of bounded degree graphs are linear in the number of vertices.

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