A Bipartite Analogue of Dilworth's Theorem

Jacob Fox

Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract

Let $m(n)$ be the maximum integer such that every partially ordered set P with n elements contains two disjoint subsets A and B, each with cardinality $m(n)$, such that either every element of A is greater than every element of B or every element of A is incomparable with every element of B. We prove that $m(n) = \Theta(\frac{n}{\log n})$. Moreover, for fixed $\epsilon \in (0,1)$ and n sufficiently large, we construct a partially ordered set P with n elements such that no element of P is comparable with n^{ϵ} other elements of P and for every two disjoint subsets A and B of P each with cardinality at least $\frac{14n}{\epsilon \log_2 n}$, there is an element of A that is comparable with an element of B.

1 Introduction

A *chain* in a partially ordered set is a set of pairwise comparable elements and an *antichain* is a set of pairwise incomparable elements. Dilworth's celebrated theorem on partially ordered sets [11,16] implies that every partially ordered set on n elements contains a chain of length a or an antichain of length $\lceil \frac{n}{a} \rceil$. Dilworth's theorem has motivated a great deal of research on partially ordered sets [18,31] and has several applications in combinatorial geometry [14,19,25,30], theoretical computer science [6,21], and set theory [13].

In this paper we develop a bipartite analogue of Dilworth's theorem. Let $m(n)$ be the maximum integer such that every partially ordered set P on n elements contains two disjoint subsets A and B such that $|A| = |B| \geq m(n)$ and every element of A is greater than every element of B or every element of A is incomparable with every element of B. A straightforward application of Dilworth's theorem implies that $m(n) \geq \left\lfloor \frac{n^{1/2}}{2} \right\rfloor$. However, this

Email address: licht@mit.edu (Jacob Fox).

lower bound turns out to be far from the actual order of growth of $m(n)$. Theorem 1 and Theorem 2 below together imply that the order of growth is $m(n) = \Theta(\frac{n}{\log n})$.

Theorem 1 *For every sufficiently large positive integer n and every partially ordered set P on n elements, there are disjoint subsets A and B of P with*

$$|A| = |B| > \frac{n}{4 \log_2 n}$$

such that either every element of A is greater than every element of B or every element of A is incomparable with every element of B.

A benefit of our proof of Theorem 1 is that it easily yields a polynomial time algorithm for finding such sets A and B in a poset P. The following theorem demonstrates that Theorem 1 is tight up to a constant factor.

Theorem 2 *Fix $\epsilon \in (0,1)$. For every sufficiently large positive integer n, there is a partially ordered set P on n elements such that no element of P is comparable with n^ϵ other elements of P and for every two subsets A and B of P with $|A| = |B| \geq \frac{14n}{\epsilon \log_2 n}$, there is an element of A that is comparable with an element of B.*

The proof of Theorem 2 uses expander graphs to give an explicit construction of such partially ordered sets. Theorem 2 implies $m(n) = O(\frac{n}{\log n})$ since $n^\epsilon = O(\frac{n}{\log n})$ for $\epsilon < 1$.

The *dimension* $\dim(P)$ of a poset P is the minimum positive integer such that the partial order of P is the intersection of $\dim(P)$ linear orders. The dimension of a poset is well defined since the partial order of a poset P is the intersection of all linear extensions of P [31]. For a poset P, let $m(P)$ be the maximum integer such that there are disjoint subsets A and B of P with $|A| = |B| \geq m(P)$ and every element of A is greater than every element of B or every element of A is incomparable with every element of B.

It is not difficult to give a nontrivial lower bound on $m(P)$ in terms of the number of elements of P and the dimension of P. Brightwell [8] in a personal communication improved on the bound in Proposition 3 from an earlier draft of this paper. Proposition 3 improves on the lower bound for $m(P)$ for posets P with n elements that satisfy $\dim(P) < 2 \log_2 n$.

Proposition 3 *For every poset P with n elements and dimension d, we have*

$$m(P) \geq \left\lfloor \frac{n}{2d} \right\rfloor.$$

We prove Theorem 1, Theorem 2, and Proposition 3 in Section 2, Section 3, and Section 4, respectively. In Section 5, we consider the k-partite analogue of Dilworth's theorem.

For a family $\mathcal{F}$ of graphs, let $b(\mathcal{F}, n)$ be the largest positive integer m such that for every graph $F \in \mathcal{F}$ with n vertices, we have that $K_{m,m}$ is a subgraph of F or its complement. The rest of this section deals with the function $b(\mathcal{F}, n)$.

The function $b(\mathcal{F}, n)$ has been the topic of considerable research [2,4,5,12,17,24,27]. For example, Alon et al. [4] showed that $b(\mathcal{F}, n)$ is linear in n if $\mathcal{F}$ is a family of intersection graphs of semi-algebraic sets in $\mathbb{R}^d$ of constant description complexity. If $\mathcal{F}$ is the family of all graphs, then it is well known [17] that $b(\mathcal{F}, n) = \Theta(\log n)$. However, until recently, there was no known explicit construction of a family of graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ with G_n having n vertices such that $b(\mathcal{G}, n) = O(n^{\frac{1}{2}-\epsilon})$ for any $\epsilon > 0$. In 2005, Barak et al. [5] gave an explicit construction of a family $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ of graphs with G_n having n vertices such that $b(\mathcal{G}, n) = O(n^{\delta})$ for any $\delta > 0$.

For a poset P, the *comparability graph* $G = (P, E)$ of P is the graph with vertex set P and edges (x, y) whenever x and y are comparable. Let $\mathcal{C}$ denote the family of comparability graphs. Lemma 7 in Section 3 states that $m(n) \leq b(\mathcal{C}, n) \leq 2m(n)$ for all positive integers n. We have the following corollary of Theorem 1, Theorem 2, and Lemma 7.

Corollary 4 *We have*

$$b(\mathcal{C}, n) = \Theta\left(\frac{n}{\log n}\right).$$

The proof of Theorem 2 gives an explicit construction of a family $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ of comparability graphs with G_n having n vertices such that $b(\mathcal{G}, n) = O(\frac{n}{\log n})$.

A graph is *perfect* if the clique number and the chromatic number of each induced subgraph are equal. Comparability graphs and P_4-free graphs are perfect graphs. A perfect graph G is called *basic* if G or $\bar{G}$ is a bipartite graph, a line graph of a bipartite graphs, or a double split graph. The basic perfect graphs have linearly sized complete or empty bipartite subgraph. These results suggest that the following conjecture is plausible.

Conjecture 5 *For $\mathcal{P}$ the family of perfect graphs, we have*

$$b(\mathcal{P}, n) = n^{1-o(1)}.$$

Since every perfect graph on n vertices has a clique or independent set of size at least $n^{\frac{1}{2}}$, we have $b(\mathcal{P}, n) \geq \left\lfloor \frac{n^{\frac{1}{2}}}{2} \right\rfloor$.

Recently, Chudnovsky et al. [10] proved Berge's famous Strong Perfect Graph Conjecture [7], which states that a graph G is perfect if neither G nor $\bar{G}$ contains an odd cycle of length at least 5 as an induced subgraph.

For a graph H, let $\mathrm{Forb}(H)$ denote the family of graphs with no induced subgraph isomorphic to H. Erdős et al. [12] proved that for a fixed graph H on k vertices, we have $b(\mathrm{Forb}(H), n) = \Omega(n^{\frac{1}{k-1}})$. Seinsche [28] proved that a graph G is P_4-free if and only if every induced subgraph H of G satisfies that H or $\bar{H}$ is disconnected. It quickly follows that $b(\mathrm{Forb}(P_4), n) = \Theta(n)$.

For a family $\mathcal{H}$ of graphs, let $\mathrm{Forb}(\mathcal{H})$ denote the family of graphs with no induced subgraph in $\mathcal{H}$. By definition, $b(\mathrm{Forb}(\mathcal{H}), n) = b(\mathrm{Forb}(\bar{\mathcal{H}}), n)$ where $\bar{\mathcal{H}}$ denotes the family of graphs that are the complement of graphs in $\mathcal{H}$. If $\mathcal{H}$ is a finite collection of graphs that each contain a cycle, then by considering sparse random graphs it is easy to show that there is a constant $\epsilon(\mathcal{H}) > 0$ such that $b(\mathrm{Forb}(\mathcal{H}), n) = O(n^{1-\epsilon(\mathcal{H})})$.

2 Proof of Theorem 1

The *dual* P^d of a poset P has the same set of elements as P with the ordering $x \leq_d y$ if and only if $x \geq y$.

Proof of Theorem 1: Suppose P is a poset with n elements such that for every two disjoint subsets A and B of P with $|A| \geq m$ and $|B| \geq m$, there is an element of A that is not greater than all the elements of B, and there is an element of A that is comparable with an element of B. We want to show that $m > \frac{n}{4 \log n}$ if n is sufficiently large.

Take any linear extension $x_1, \ldots, x_n$ of P. Hence, if $x_i < x_j$, then $i < j$. Define $d = \left\lfloor \frac{n}{2m} \right\rfloor$. For $1 \leq l \leq d$, let $C_l = \{x_j : 2m(l-1) < j \leq 2ml\}$. For the rest of the proof, we do not make use of the elements x_j with $2md < j \leq n$. Define the "low set" $L = \bigcup_{1 \leq l < \lceil \frac{d}{2} \rceil} C_l$, the "middle set" $M = C_{\lceil \frac{d}{2} \rceil}$, and the "high set" $H = \bigcup_{\lceil \frac{d}{2} \rceil < l \leq d} C_l$. Notice that L and H each have cardinality greater than $\frac{n}{2} - 3m$, and M has cardinality $2m$.

For an element $x \in M$, define the "up set" $U(x)$ (respectively, "down set" $D(x)$) to be those elements of H (respectively, L) that are greater (respectively, less) than x. For a subset $S \subset M$, let

$$U(S) = \{x : x \in H \text{ and } x > s \text{ for at least one } s \in S\} = \bigcup_{s \in S} U(s)$$

and

$$D(S) = \{x : x \in L \text{ and } x < s \text{ for at least one } s \in S\} = \bigcup_{s \in S} D(s).$$

Let $V = \{x : x \in M \text{ and } |D(x)| < m\}$ and $W = \{x : x \in M \text{ and } |U(x)| < m\}$. If any element $x \in M$ satisfies $U(x) \geq m$ and $D(x) \geq m$, then setting $A = U(x)$ and $B = D(x)$, we have A and B are disjoint, $|A| \geq m$, $|B| \geq m$, and every element of A is greater than every element of B, a contradiction. So for every $x \in P$, we have $|U(x)| < m$ or $|D(x)| < m$. Therefore, we have $M = V \cup W$. Since $|M| = 2m$, then $|V| \geq m$ or $|W| \geq m$. This is illustrated in Figure 1. We can assume without loss of generality that $|V| \geq m$ since otherwise the same argument holds using the dual poset of P and the ordering $x'_i = x_{2md+1-i}$.

If $|D(V)| \leq |L| - m$, then setting $A = V$ and $B = L - D(V)$, we have that $|A| \geq m$, $|B| \geq m$, and every element in A is incomparable with every element in B, a contradiction. Therefore, $|D(V)| > |L| - m$.

Arbitrarily label the elements of V as $v_1, v_2, \ldots, v_{|V|}$. For $0 \leq i \leq \lceil \log_2 |V| \rceil$ and $1 \leq j \leq \left\lceil \frac{|V|}{2^i} \right\rceil$, let $V(i,j) = \{v_k : v_k \in V \text{ and } (j-1)2^i < k \leq j2^i\}$. If $j > \left\lceil \frac{|V|}{2^i} \right\rceil$, then let $V(i,j)$ be the empty set. We note that $V = V(\lceil \log_2 |V| \rceil, 1)$ and $V(i+1, j) = V(i, 2j-1) \cup V(i, 2j)$.

For $0 \leq i < \lceil \frac{d}{2} \rceil$, let $L_i = \bigcup_{i < l < \lceil \frac{d}{2} \rceil} C_l$.

We have two cases to consider.

Case 1: $\lceil \log_2 |V| \rceil < \lceil \frac{d}{2} \rceil - 1$. Since $V = V(\lceil \log_2 |V| \rceil, 1)$, then by Lemma 6 below we have

$$m > |L_{\lceil \log_2 |V| \rceil} \cap D(V)| \geq |C_{\lceil \log_2 |V| \rceil + 1} \cap D(V)|.$$

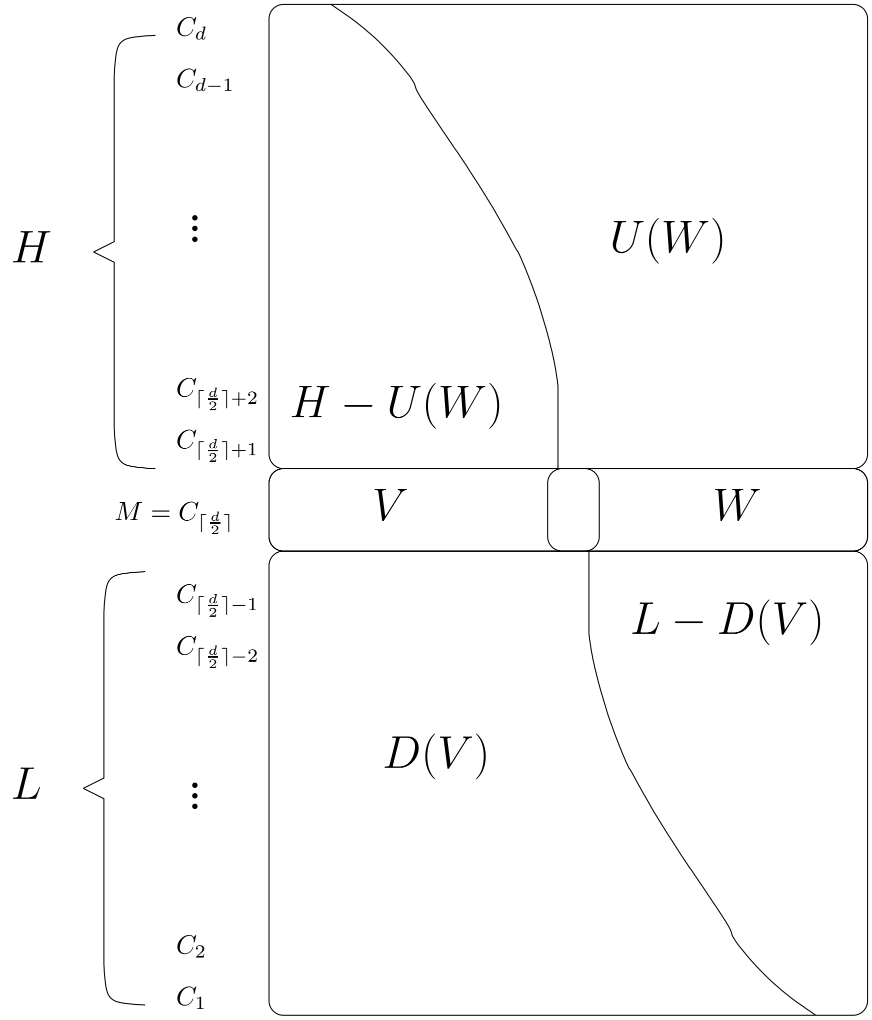


Fig. 1. An illustration of a general poset on $2md$ elements partitioned into a lower set L, middle set M, and upper set U (as discussed in the proof of Theorem 1). The set $U(W)$ consists of the elements in H that are above at least one element of W, and the set $D(V)$ consists of the elements in L that are below at least one element of V. The sets C_l each have cardinality $2m$.

Since $|C_{\lceil \log_2 |V| \rceil+1}| = 2m$, then $|C_{\lceil \log_2 |V| \rceil+1} - D(V)| > m$. But this contradicts the inequality

$$m > |L| - |D(V)| \geq |C_{\lceil \log_2 |V| \rceil+1} - D(V)|.$$

Case 2: $\lceil \log_2 |V| \rceil \geq \left\lceil \frac{d}{2} \right\rceil - 1$. Recall that $\left\lceil \frac{d}{2} \right\rceil - 1 = \frac{|L|}{2m} \geq \frac{n-6m}{4m}$. Also notice that $\lceil \log_2 |V| \rceil < 1 + \log_2 2m = \log_2 4m$ since $V \subset M$ and $|M| = 2m$. Combining these inequalities and simplifying, we have

$$6m + 4m \log_2 4m \geq n.$$

To satisfy this inequality, we must have

$$m \geq \left(1 + (1 - o(1)) \frac{\log_2 \log_2 n}{\log_2 n}\right) \frac{n}{4 \log_2 n}. \qquad \square$$

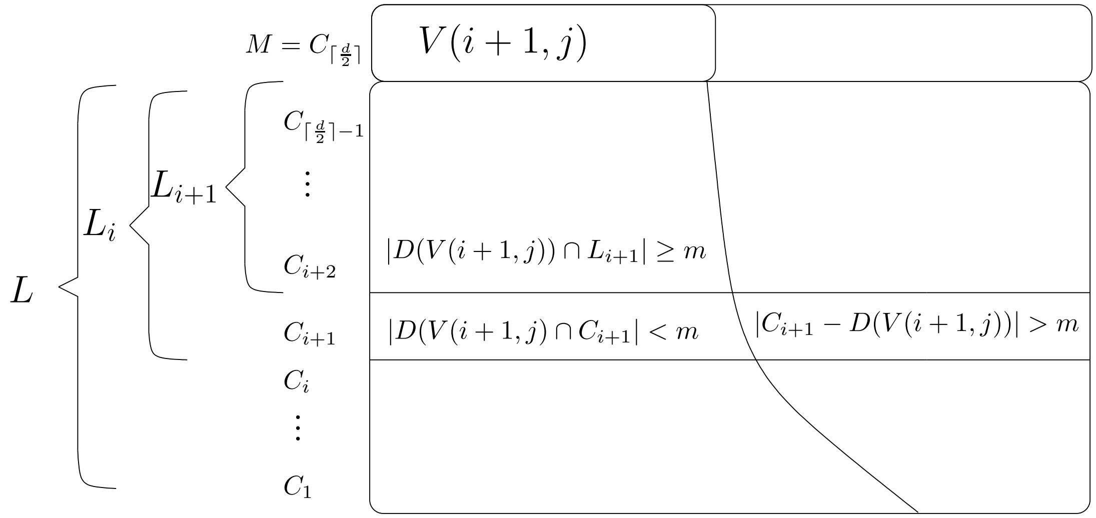


Fig. 2. An illustration of the contradiction to the assumption $|D(V(i+1,j)) \cap L_{i+1}| \geq m$ in the proof of Lemma 6. Setting $A = D(V(i+1,j)) \cap L_{i+1}$ and $B = C_{i+1} - D(V(i+1,j))$, we have $|A| \geq m$, $|B| \geq m$, and every element of A is incomparable with every element of B.

Thus we have reduced our goal to prove the following lemma.

Lemma 6 *Let V, $V(i,j)$, and L_i be as in the proof of Theorem 1. For* $0 \leq i \leq \min(\lceil \log_2 |V| \rceil, \lceil \frac{d}{2} \rceil - 1)$ *and* $1 \leq j \leq \left\lceil \frac{|V|}{2^i} \right\rceil$*, we have*

$$|D(V(i,j)) \cap L_i| < m.$$

Proof: The proof will be by induction on i. The lemma is true in the base case $i = 0$ since $|V(0,j)| = 1$ and $|D(x)| < m$ for $x \in V$, so $|D(V(0,j))| < m$.

Assume the result is true for i. Since $V(i+1,j) = V(i,2j-1) \cup V(i,2j)$, then

$$|D(V(i+1,j)) \cap L_i| \leq |D(V(i,2j-1)) \cap L_i| + |D(V(i,2j)) \cap L_i| < 2m.$$

For the sake of contradiction, assume that

$$|D(V(i+1,j)) \cap L_{i+1}| \geq m.$$

By the last two inequalities, we have

$$|D(V(i+1,j)) \cap C_{i+1}| < m.$$

Hence, $|C_{i+1} - D(V(i+1,j))| > m$. Set $A = D(V(i+1,j)) \cap L_{i+1}$ and $B = C_{i+1} - D(V(i+1,j))$; then A and B are disjoint, $|A| \geq m$, $|B| \geq m$, and every element of A is incomparable with every element of B, a contradiction. This contradiction is illustrated in Figure 2.

Hence, by induction on i, we have proved that for $0 \leq i \leq \min(\lceil \log_2 |V| \rceil, \lceil \frac{d}{2} \rceil - 1)$ and $1 \leq j \leq \lceil \frac{|V|}{2^i} \rceil$, we have $|D(V(i,j)) \cap L_i| < m$. □

3 Proof of Theorem 2

The height $h(x)$ of an element x of a poset P is the length of the longest chain with largest element x. The following lemma shows that $m(n)$ and $b(\mathcal{C}, n)$ are within a constant factor of each other.

Lemma 7 *(i) For all positive integers n, we have*

$$m(n) \leq b(\mathcal{C}, n) \leq 2m(n).$$

(ii) If $K_{m,m}$ is a subgraph of a comparability graph G, then there are subsets A and B of the associated partially ordered set P with $|A| \geq m/2$, $|B| \geq m/2$, and every element of A is greater than every element of B.
(iii) If $K_{m,m}$ is a subgraph of the complement of a comparability graph G, then there are subsets A and B of the associated partially ordered set P with $|A| \geq m/2$, $|B| \geq m/2$, and for every $x \in A$ and $y \in B$, x and y are incomparable and $h(x) \geq h(y)$.

Proof: The lower bound $m(n) \leq b(\mathcal{C}, n)$ is trivially true. The upper bound $b(\mathcal{C}, n) \leq 2m(n)$ follows from (ii) and (iii). Since the proof of (ii) and (iii) are essentially the same, we only prove (ii) here. Let A' and B' be disjoint subsets of P of size m such that every element in A' is comparable with every element of B'. Label the elements of $A' \cup B'$ in order $a_1, \ldots, a_{2m}$ by increasing height, breaking ties arbitrarily when necessary.

If $|A' \cap \{a_{m+1}, \ldots, a_{2m}\}| \geq m/2$, then setting $A = A' \cap \{a_{m+1}, \ldots, a_{2m}\}$ and $B = B' \cap \{a_1, \ldots, a_m\}$, we have $|A| \geq m/2$, $|B| \geq m/2$, and every element of A is greater than every element of B.

If $|A' \cap \{a_{m+1}, \ldots, a_{2m}\}| < m/2$, then setting $A = B' \cap \{a_{m+1}, \ldots, a_{2m}\}$ and $B = A' \cap \{a_1, \ldots, a_m\}$, we have $|A| > m/2$, $|B| > m/2$, and every element of A is greater than every element of B. □

An (n, r, d)-*expander* is a finite r-regular undirected graph $G = (V, E)$ such that $|V| = n$ and for every subset S of V, the set $N(S) = \{v \in V : (v, u) \in E \text{ for some } u \in S\}$ of neighbors of S satisfies

$$|N(S)| \geq (1 + d(1 - |S|/n))\,|S|. \tag{1}$$

In particular, we have

$$|N(S)| \geq (1 + d/2)|S| \tag{2}$$

for all subsets $S \subset V$ with $|S| \leq |V|/2$.

For a graph G, let $\lambda = \lambda(G)$ denote the second largest eigenvalue of the adjacency matrix for G. One of the most fundamental results concerning expander graphs relates the second largest eigenvalue of a graph to its expansion d. Alon and Milman [1], [3] showed that if G is an r-regular graph on n vertices, then G is an $(n, r, 1 - \frac{\lambda^2}{r^2})$-expander.

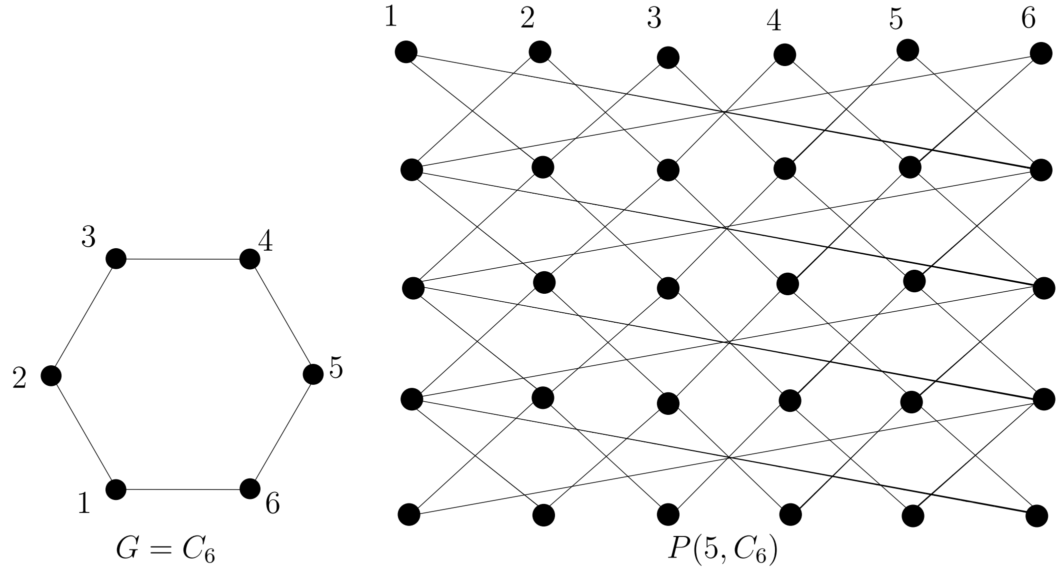


Fig. 3. Illustration of the Hasse Diagram of the partially ordered set $P(5, C_6)$.

A *Ramanujan graph* is a r-regular graph G whose second largest eigenvalue $\lambda(G)$ is at most $2\sqrt{r-1}$. Ramanujan graphs are excellent expanders in that they have large expansion. Infinite families of r-regular Ramanujan graphs have been constructed for $r-1$ being any prime power [9,20,22,23].

For a positive integer a and graph G on b vertices with vertex set $V = \{1, \ldots, b\}$, define the poset $P(a, G)$ with elements $\{(j, k)\colon 1 \le j \le a \text{ and } 1 \le k \le b\}$ generated by the comparisons $(j_1, k_1) < (j_2, k_2)$ whenever $j_2 = j_1 + 1$ and (k_1, k_2) is an edge of G. Figure 3 illustrates the Hasse diagram of the poset $P(5, C_6)$.

We will use the following two lemmas to prove Theorem 2.

Lemma 8 *If G is an r-regular graph, then every element of the poset $P(a, G)$ is comparable with at most $\frac{r^a-1}{r-1}$ elements of $P(a, G)$.*

Proof: Since the degree of every vertex in G is r, then the number of elements of $P(a, G)$ that are at least (j, k) is at most $1 + r + \cdots + r^{a-j} = \frac{r^{a-j+1}-1}{r-1}$. The number of elements of $P(a, G)$ that are at most (j, k) is at most $1 + r + \cdots + r^{j-1} = \frac{r^j-1}{r-1}$. Hence, the number of elements of $P(a, G)$ that are comparable with (j, k) is at most

$$\frac{r^{a-j+1} + r^j - r - 1}{r-1}.$$

This upper bound is maximized at $j = 1$ and $j = a$. Hence, every element of the poset $P(a, G)$ is comparable with at most $\frac{r^a-1}{r-1}$ elements of $P(a, G)$. □

Lemma 9 *If G is a (b, r, d)-expander, then for every pair A and B of disjoint subsets of $P(a, G)$ with $|A| \ge \frac{b(2+d)}{d}$ and $|B| \ge \frac{b(2+d)}{d}$, there is an element of A that is comparable with an element of B.*

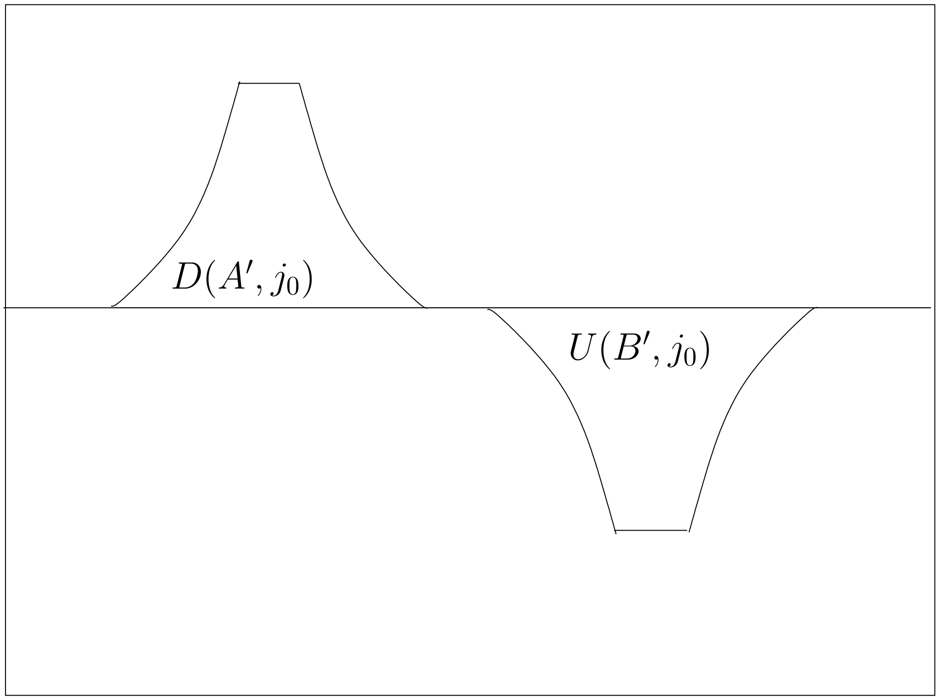
A_{j_0} 

Fig. 4. Illustration of the incomparable sets $D(A', j_0)$ and $U(B', j_0)$.

Proof: Assume there are disjoint subsets A and B of size m such that every element of A is incomparable with every element of B. By Lemma 7, there are disjoint subsets A' and B' satisfying $|A'| \geq \frac{m}{2}$, $|B'| \geq \frac{m}{2}$ and such that every element of A' is incomparable with every element of B' and $h(x) \geq h(y)$ for all $x \in A'$ and $y \in B'$.

Let $A_j = \{(j, k) : 1 \leq k \leq b\}$. So $P(a, G) = \bigcup_{j=1}^{a} A_j$ is a partition of $P(a, G)$ into antichains of size b. Let j_0 be an integer such that $h(x) \geq j_0 \geq h(y)$ for all $x \in A'$ and $y \in B'$.

For a subset S of $P(a, G)$, let

$$U(S, j) = \{(j', k') : j' \leq j \text{ and } (j', k') \geq s \text{ for at least one } s \in S\}.$$

Likewise, let

$$D(S, j) = \{(j', k') : j' \geq j \text{ and } (j', k') \leq s \text{ for at least one } s \in S\}.$$

Notice that $D(A', j_0)$ and $U(B', j_0)$ have the property that for every $x \in D(A', j_0)$ and $y \in U(B', j_0)$, x is incomparable with y and $h(x) \geq j_0 \geq h(y)$. Moreover, $A' \subset D(A', j_0)$ and $B' \subset U(B', j_0)$. This is illustrated in Figure 4.

Notice that $|D(A', j_0) \cap A_{j_0}| \leq \frac{b}{2}$ or $|U(B', j_0) \cap A_{j_0}| \leq \frac{b}{2}$ since the sets $D(A', j_0)$ and $U(B', j_0)$ are disjoint. By considering the dual poset if necessary, we may assume without loss of generality that $|D(A', j_0) \cap A_{j_0}| \leq \frac{b}{2}$.

For each nonnegative integer i with $i \leq n - j_0$, let $S_i = \{k : (j_0 + i, k) \in D(A', j_0)\}$. Notice that $|S_0| = |D(A', j_0) \cap A_{j_0}| \leq \frac{b}{2}$. We view S_i as a subset of vertices of G. Therefore, for every nonnegative integer i with $i < n - j_0$, we have $N(S_{i+1}) \subset S_i$. By the expansion

Inequality (2) for G, we have

$$|S_i| \leq |S_0| \left(1+\frac{d}{2}\right)^{-i} \leq \frac{b}{2}\left(1+\frac{d}{2}\right)^{-i}.$$

Hence, we have

$$\frac{m}{2} \leq |A'| \leq |D(A', j_0)| = \sum_{i=0}^{n-j_0} |S_i| < \sum_{i=0}^{\infty} \frac{b}{2}\left(1+\frac{d}{2}\right)^{-i} = \frac{b(2+d)}{2d}.$$

Hence, $m < \frac{b(2+d)}{d}$. We conclude that for every pair A and B of disjoint subsets of $P(a, G)$ with $|A| \geq \frac{b(2+d)}{d}$ and $|B| \geq \frac{b(2+d)}{d}$, there is an element of A that is comparable with an element of B. □

We now prove Theorem 2.

Proof of Theorem 2:

Let $r = 9$, $n > r^{\frac{1}{\epsilon}}$, $a = \left\lfloor \frac{\epsilon \log n}{\log r} \right\rfloor$, and let b be the smallest positive integer such that $b > \frac{n}{a}$ and there is a 9-regular Ramanujan graph G on b vertices. By the constructions of Morgenstern [22], such a b exists with $b = (1 + o(1))\frac{n \log 9}{\epsilon \log n}$. Let P be any subposet of $P(a, G)$ with $|P| = n$. By Lemma 8, every element of P is comparable with at most $\frac{r^a-1}{r-1}$ elements of P. Notice that the inequality $\frac{r^a-1}{r-1} < r^a$ holds. Substituting in $a = \left\lfloor \frac{\epsilon \log n}{\log r} \right\rfloor$, every element of P is comparable with at most n^{ϵ} elements of P. By Lemma 9, for every pair A and B of disjoint subsets of P with $|A| \geq \frac{b(2+d)}{d}$ and $|B| \geq \frac{b(2+d)}{d}$, there is an element of A that is comparable with an element of B. Substituting in

$$d = 1 - \frac{\lambda^2}{r^2} \geq 1 - \frac{4r-4}{r^2} = \frac{49}{81},$$

we have that for every pair A and B of disjoint subsets of P satisfying

$$|A| = |B| \geq (13.66 + o(1))\frac{n}{\epsilon \log_2 n},$$

there is an element of A that is comparable with an element of B. □

We note that the constant factor 13.66 in the proof above can be improved significantly by using the full strength of Inequality (1) instead of using Inequality (2).

4 Dimension and the bipartite analogue of Dilworth's theorem

In this section we prove Proposition 3. The proof is similar to the proof of Lemma 7.

Proof of Proposition 3:

Let P be a poset with n elements and dimension d. Let L_1,...,L_d be linear extensions of P whose intersection is P. Let X_i be the largest $\lfloor \frac{n}{2} \rfloor$ elements in the linear extension L_i and Y_i

be the smallest $\lceil \frac{n}{2} \rceil$ elements in the linear extension L_i. Notice that $|X_1 \cap Y_i| = |Y_1 \cap X_i|$ and every element of $X_1 \cap Y_i$ is incomparable with every element of $Y_1 \cap X_i$. If there is an i such that $|X_1 \cap Y_i| \geq \lfloor \frac{n}{2d} \rfloor$, then setting $A = X_1 \cap Y_i$ and $B = Y_1 \cap X_i$, then $|A| = |B| \geq \lfloor \frac{n}{2d} \rfloor$ and every element of A is incomparable with every element of B. If there is no such i satisfying $|X_1 \cap Y_i| \geq \lfloor \frac{n}{2d} \rfloor$, then the two sets $X_1 \cap \ldots \cap X_d$ and $Y_1 \cap \ldots \cap Y_d$ each have cardinality at least $\lfloor \frac{n}{2d} \rfloor$ and the elements of the first set are larger than the elements of the second set. □

5 Concluding Remarks

Pach and Toth [26] showed that a graph G is a comparability graph if and only if there is a family of n continuous real functions defined on $[0, 1]$ whose intersection graph is G. They show, as a corollary of this result and Theorem 2, that for every $\epsilon > 0$ there is a positive constant $c(\epsilon)$ such that for every integer $n > 1$, there is a family of n continuous real functions defined on $[0, 1]$ such that their intersection graph G contains no complete bipartite graph with at least $c(\epsilon)\frac{n}{\log_2 n}$ vertices in each of its vertex classes, and every vertex in G is adjacent to all but at most n^ϵ other vertices.

Let $p_r(n)$ be the largest integer such that for every set P on n elements and partial orders $<_1, \ldots, <_r$ on P, there are two disjoint subsets $A, B \subset P$, each with at least $p_r(n)$ elements such that either there is an i with $a <_i b$ for all $a \in A$ and $b \in B$ or every element of A is incomparable with every element of B for all r partial orders $<_1, \ldots, <_r$. Pach and the author [15] prove that for each fixed r, there is a constant c_r such that

$$\frac{n}{2^{(1+o(1))(\log_2 \log_2 n)^r}} \leq p_r(n) \leq c_r n \left(\log_2 n\right)^{-r} \left(\log \log n\right)^{r-1}.$$

Larman et al. [19] showed that there are four partial orders $<_1, \ldots, <_4$ on convex compact sets in the plane such that two convex compact sets intersect if and only if they are incomparable in all four partial orders. It follows from repeated application of Dilworth's theorem that every intersection graph G of n convex compact sets in the plane contains a clique or independent set on $n^{1/5}$ elements. From the lower bound on $p_4(n)$ and the result of Larman et al., it follows that for every intersection graph G of n convex compact sets in the plane, either G or $\bar{G}$ contains a complete bipartite graph with $\frac{n}{2^{(1+o(1))(\log_2 \log_2 n)^4}}$ vertices in each of its vertex classes. Similar results do not hold in 3 dimensions, as Tietze [29] showed a century ago that every graph is the intersection graph of convex compact sets in 3 dimensions.

Let $m_k(n)$ denote the largest integer such that for every partially ordered set P on n elements, there are pairwise disjoint subsets $A_1, \ldots, A_k$ of P with $|A_i| \geq m_k(n)$ for $1 \leq i \leq k$, and every element of A_i is greater than every element of A_j if $i > j$ or every element of A_i is incomparable with every element of A_j if $i \neq j$.

By iterating Theorem 1, we get the following corollary of Theorem 1 and Theorem 2.

Corollary 10 *Fix a positive integer k. We have*

$$\Omega\left(\frac{n}{(\log n)^{2k-1}}\right) \leq m_{2^k}(n) \leq O\left(\frac{n}{\log n}\right).$$

It would be nice to tighten Corollary 10.

Question 11 *Does* $m_3(n) = \Omega\left(\frac{n}{\log n}\right)$*?*

6 Acknowledgements

This research was performed under the supervision of Joseph A. Gallian at the University of Minnesota Duluth in his REU program. The program was supported by the NSF (DMS 0447070) and the NSA (H98230-04-1-0050). The author would like to thank Reid Barton, Graham Brightwell, Joseph A. Gallian, Joshua Greene, Daniel J. Kleitman, Phil Matchett, Janos Pach, Rados Radoičić, Csaba Toth, and William T. Trotter for helpful advice and comments.

References

[1] N. Alon, Eigenvalues and expanders. Theory of computing (Singer Island, Fla., 1984). Combinatorica 6 (1986), no. 2, 83–96.

[2] N. Alon, Ramsey graphs cannot be defined by real polynomials. J. Graph Theory 14 (1990), no. 6, 651–661.

[3] N. Alon and V. D. Milman, λ_1, isoperimetric inequalities for graphs, and superconcentrators. J. Combin. Theory Ser. B 38 (1985), no. 1, 73–88.

[4] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir, Crossing patterns of semi-algebraic sets, Journal of Combinatorial Theory, to appear.

[5] B. Barak, G. Kindler, R. Shaltiel, B. Sudakov and A. Wigderson, Simulating Independence: New Constructions of Condensers, Ramsey Graphs, Dispersers and Extractors, Proc. of the 37-th ACM STOC (2005), 1-10.

[6] A. Benczúr, A. András, J. Förster and Z. Király, Dilworth's theorem and its application for path systems of a cycle—implementation and analysis. Algorithms—ESA '99 (Prague), 498–509, Lecture Notes in Comput. Sci., 1643, Springer, Berlin, 1999

[7] C. Berge, Les problmes de coloration en théorie des graphes, Publ. Inst. Stat. Univ. Paris 9 (1960) 123–160.

[8] G. Brightwell, personal communication.

[9] P. Chiu, Cubic Ramanujan graphs. Combinatorica 12 (1992), no. 3, 275–285.

[10] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Annals of Math., to appear.

[11] R. P. Dilworth, A decomposition theorem for partially ordered sets. Ann. of Math. (2) 51, (1950) 161–166.

[12] P. Erdős, A. Hajnal, and J. Pach, Ramsey-type theorem for bipartite graphs, Geombinatorics 10 (2000) 64–68.

[13] P. Erdős and J. Komlós, On a problem of Moser. Combinatorial theory and its applications, I (Proc. Colloq., Balatonfred, 1969), pp. 365–367. North-Holland, Amsterdam, 1970.

[14] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Mathematica 2 (1935) 463–470.

[15] J. Fox and J. Pach, A bipartite analogue of Dilworth's theorem for multiple partial orders.

[16] Ira Gessel and Gian-Carlo Rota (ed.), *Classic papers in combinatorics.* Birkhuser Boston, Inc., Boston, MA, 1987.

[17] R.L. Graham, B.L. Rothschild, J. Spencer, *Ramsey Theory*. John Wiley & Sons Inc., New York, second edition 1990.

[18] C. Greene and D. J. Kleitman, The structure of Sperner k-families. J. Combinatorial Theory Ser. A 20 (1976), no. 1, 41–68.

[19] D. Larman, J. Matoušek, J. Pach and J. Töröcsik, A Ramsey-type result for convex sets. Bull. London Math. Soc. 26 (1994), no. 2, 132–136.

[20] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs. Combinatorica 8 (1988), no. 3, 261–277.

[21] J. Matoušek and E. Welzl, Good splitters for counting points in triangles. J. Algorithms 13 (1992), no. 2, 307–319.

[22] M. Morgenstern, Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power q. J. Combin. Theory Ser. B 62 (1994), no. 1, 44–62.

[23] M. R. Murty, Ramanujan graphs, J. Ramanujan Math. Soc. 18, No. 1, (2003) 1–20. .

[24] J. Pach and J. Solymosi, Crossing patterns of segments, J. Combin. Theory Ser. A 96 (2001) 316–325.

[25] J. Pach and J. Törőcsik, Some geometric applications of Dilworth's theorem. Discrete Comput. Geom. 12 (1994), no. 1, 1–7.

[26] J. Pach and G. Tóth, Comments on Fox News, Geombinatorics 15 (2006), 150–154.

[27] P. Pudlák and V. Rödl, Pseudorandom sets and explicit constructions of Ramsey graphs. Complexity of computations and proofs, 327–346, Quad. Mat., 13, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.

[28] D. Seinsche, On a property of the class of n-colorable graphs, J. Combinatorial Theory Ser. B 16 (1974) 191–193.

[29] H. Tietze, Über das Problem der Nachbargeibiete im Raum, Monatshefte Math. 16 (1905) 211–216.

[30] G. Tóth and P. Valtr, Geometric graphs with few disjoint edges. 14th Annual ACM Symposium on Computational Geometry (Minneapolis, MN, 1998). Discrete Comput. Geom. 22 (1999), no. 4, 633–642.

[31] W. T. Trotter, *Combinatorics and partially ordered sets. Dimension theory.* Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.