MAT 307: Combinatorics

Lecture 22: Eigenvalues and expanders

Instructor: Jacob Fox

1 Expander graphs

Expander graphs are graphs with the special property that any set of vertices S (unless very large) has a number of outgoing edges proportional to $|S|$. Expansion can be defined both with respect to the number of the edges or vertices on the boundary of S. We will stick with edge expansion, which is more directly related to eigenvalues.

Definition 1. *The edge expansion (or "Cheeger constant") of a graph is*

$$h(G) = \min_{|S| \leq n/2} \frac{e(S, \bar{S})}{|S|}$$

where $e(S, \bar{S})$ is the number of edges between S and its complement.

Definition 2. *A graph is a (d, ϵ)-expander if it is d-regular and $h(G) \geq \epsilon$.*

Observe that $e(S, \bar{S}) \leq d|S|$ and so ϵ cannot be more than d. Graphs with ϵ comparable to d are very good expanders. Expanders are very useful in computer science. We will mention some applications later.

2 Random graphs

It is known that random graphs are good expanders. It is easier to analyze *bipartite expanders* which are defined as follows.

Definition 3. *A bipartite graph G on $n+n$ vertices $L \cup R$ is called a (d, β)-expander, if the degrees in L are d and any set of vertices $S \subset L$ of size $|S| \leq n/d$ has at least $\beta|S|$ neighbors in R.*

Theorem 1. *Let $d \geq 4$ and G be a random bipartite graph obtained by choosing d random edges for each vertex in L. Then G is a $(d, d/4)$-expander with constant positive probability.*

Proof. For each $S \subseteq L$ and $T \subseteq R$, let $E_{S,T}$ denote the event that all neighbors of S are in T. The probability of this event is

$$\Pr[E_{S,T}] = \left(\frac{|T|}{n}\right)^{d|S|}.$$

Let $\beta = d/4 \geq 1$. By the union bound, and the standard estimate $\binom{n}{k} \leq (\frac{ne}{k})^k$,

$$\Pr[\exists S, T; |S| \leq n/d, |T| < \beta|S|] \leq \sum_{s=1}^{n/d} \binom{n}{s}\binom{n}{\beta s}\left(\frac{\beta s}{n}\right)^{ds} \leq \sum_{s=1}^{n/d} \binom{n}{\beta s}^2 \left(\frac{\beta s}{n}\right)^{ds}$$

$$\leq \sum_{s=1}^{n/d} \left(\frac{ne}{\beta s}\right)^{2\beta s}\left(\frac{\beta s}{n}\right)^{ds} = \sum_{s=1}^{n/d}\left(\frac{4ne}{ds}\right)^{ds/2}\left(\frac{ds}{4n}\right)^{ds} = \sum_{s=1}^{n/d}\left(\frac{eds}{4n}\right)^{ds/2} \leq \sum_{s=1}^{n/d}\left(\frac{e}{4}\right)^{ds/2}.$$

This is bounded by $\sum_{s=1}^{\infty}(e/4)^{ds/2} = (e/4)^{d/2}/(1-(e/4)^{d/2}) < 1$ for $d \geq 4$. □

3 Eigenvalue bounds on expansion

In general, random graphs are very good expanders, so the existence of expanders is not hard to establish. However, the difficult question is how to construct expanders explicitly. For now, we leave this question aside and we explore the connection between expansion and eigenvalues.

Theorem 2. *For any d-regular graph G with second eigenvalue λ_2,*

$$h(G) \geq \frac{1}{2}(d - \lambda_2).$$

Proof. For any subset of vertices S of size s, let $x = (n-s)\mathbf{1}_S - s\mathbf{1}_{\bar{S}}$.[1] We get

$$x^T x = (n-s)^2 s + s^2(n-s) = s(n-s)n$$

and

$$x^T A x = 2 \sum_{(i,j)\in E} x_i x_j = 2(n-s)^2 e(S) - 2s(n-s)e(S,\bar{S}) + 2s^2 e(\bar{S}).$$

To eliminate $e(S)$ and $e(\bar{S})$, observe that every degree is equal to d, and ds can be viewed as counting each edge in $e(S)$ twice and each edge in $e(S,\bar{S})$. Therefore, $ds = 2e(S) + e(S,\bar{S})$. Similarly, $d(n-s) = 2e(\bar{S}) + e(S,\bar{S})$. This yields

$$x^T A x = (n-s)^2(ds - e(S,\bar{S})) - 2s(n-s)e(S,\bar{S}) + s^2(d(n-s) - e(S,\bar{S})) = dns(n-s) - n^2 e(S,\bar{S}).$$

Since $x \cdot 1 = 0$, we can use the variational definition of λ_2 to claim that

$$\lambda_2 \geq \frac{x^T A x}{x^T x} = \frac{dns(n-s) - n^2 e(S,\bar{S})}{s(n-s)n} = d - \frac{n\, e(S,\bar{S})}{s(n-s)}.$$

For any set S of size $s \leq n/2$, we have

$$\frac{e(S,\bar{S})}{|S|} \geq \frac{n-s}{n}(d - \lambda_2) \geq \frac{1}{2}(d - \lambda_2).$$

□

This theorem shows that if $d - \lambda_2$ is large, for example $\lambda_2 \leq d/2$, then the graph is a $(d, d/4)$-expander - very close to best possible. The quantity $d - \lambda_2$ is called the *spectral gap*.

There is also a bound in the opposite direction, although we will not prove it here.

Theorem 3. *For any d-regular graph with second eigenvalue λ_2,*

$$h(G) \leq \sqrt{d(d - \lambda_2)}.$$

[1] Note that we used exactly the same vector to prove our bound on the independence number.

4 How large can the spectral gap be?

We have seen that graphs where the maximum eigenvalue λ_1 dominates all other eigenvalues have very interesting properties. Here we ask the question, how small can the remaining eigenvalues possibly be? We know that the complete graph K_n has eigenvalues $n-1$ and -1 and therefore $\lambda = \max_{i \neq 1} |\lambda_i|$ is dominated by λ_1 by a factor of $n-1$, the degree in K_n. For a constant degree d and large n, this cannot happen.

Theorem 4. *For any constant $d > 1$, any d-regular graph has an eigenvalue $\lambda_i \neq d$ of absolute value*

$$\lambda = \max_{i \neq 1} |\lambda_i| \geq (1 - o(1))\sqrt{d}$$

where $o(1) \to 0$ as $n \to \infty$.

Proof. Consider the square of the adjacency matrix A^2. A^2 has d on the diagonal, and therefore $Tr(A^2) = dn$. On the other hand, the eigenvalues of A^2 are λ_i^2, and so

$$Tr(A^2) = \sum_{i=1}^{n} \lambda_i^2 \leq d^2 + (n-1)\lambda^2.$$

Putting these together, we get

$$\lambda^2 \geq \frac{dn - d^2}{n-1} \geq (1 - d/n)d = (1 - o(1))d.$$

□

So the best possible spectral gap that we can have is roughly between $\sqrt{d}$ and d. More precisely, it is known that the second eigenvalue is always at least $2\sqrt{d-1} - o(1)$. This leads to the definition of *Ramanujan graphs.*

Definition 4. *A d-regular graph is Ramanujan, if all eigenvalues in absolute value are either equal to d or at most $2\sqrt{d-1}$.*

It is known in fact that a random d-regular graph has all non-trivial eigenvalues bounded by $2\sqrt{d-1} + o(1)$ in absolute value. However, it is more difficult to come up with explicit Ramanujan graphs.

5 Explicit expanders

The following graph is a beautiful algebraic construction of Margulis which was the earliest explicit expander known.

Definition 5 (Margulis' graph)**.** *Let $V = Z_n \times Z_n$ and define an 8-regular graph on V as follows. Let*

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Each vertex $v \in Z_n \times Z_n$ is adjacent to $T_1 v, T_2 v, T_1 v + e_1, T_2 v + e_2$ and four other vertices obtained by the inverse tranformations. (This is a multigraph with possible multiedges and loops.)

This graph has maximum eigenvalue $d = 8$, and it can be also computed that the second eigenvalue is $\lambda_2 \leq 5\sqrt{2}$. (We will not show this here.) The spectral gap is $d - \lambda_2 \geq 0.92$ and therefore this graph is a $(8, 0.46)$-expander.

Later, even simpler constructions were found.

Definition 6. *Let p be prime and let $V = Z_p$. We define a 3-regular graph $G = (V, E)$ where the edges are of two types: $(x, x+1)$, and (x, x^{-1}) for each $x \in Z_p$. (We assume that $0^{-1} = 0$ for this purpose.)*

It is known that this is a $(3, \epsilon)$-expander for some fixed $\epsilon > 0$ and any prime p. The proof of this relies on deep results in number theory.

These graphs are not Ramanujan graphs, i.e. their second eigenvalue is not on the order of $\sqrt{d}$. However, even such graphs can be constructed explicitly. The first explicit construction of Ramanujan graphs was found by Lubotzky, Phillips and Sarnak in 1988.