MAT 307: Combinatorics

Lecture 13: Graphs of high girth and high chromatic number

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1 Markov's inequality

Another simple tool that's often useful is *Markov's inequality*, which bounds the probability that a random variable X is too large, based on the expectation $\mathbf{E}[X]$.

Lemma 1. *Let X be a nonnegative random variable and $t > 0$. Then*

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

Proof.

$$\mathbf{E}[X] = \sum_a a \Pr[X = a] \geq \sum_{a \geq t} t \Pr[X = a] = t \Pr[X \geq t].$$

□

Working with expectations is usually easier than working directly with probabilities or more complicated quantities such as variance. Recall that $\mathbf{E}[X_1 + X_2 + \ldots + X_n] = \sum_{i=1}^n \mathbf{E}[X_i]$ for any collection of random variables.

2 Graphs of high girth and high chromatic number

We return to the notion of a chromatic number $\chi(G)$. Observe that for a graph that does not contain any cycles, $\chi(G) \leq 2$ because every component is a tree that can be colored easily by 2 colors. More generally, consider graphs of *girth* ℓ, which means that the length of the shortest cycle is ℓ. If ℓ is large, this means that starting from any vertex, the graph looks like a tree within distance $\ell/2 - 1$. One might expect that such graphs can be also colored using a small number of colors, since locally they can be colored using 2 colors. However, this is far from being true, as shown by a classical application of the probabilistic method.

Theorem 1. *For any k and ℓ, there is a graph of chromatic number $> k$ and girth $> \ell$.*

Proof. We start by generating a random graph $G_{n,p}$, where each edge appears independently with probability p. We fix a value $\lambda \in (0, 1/\ell)$ and we set $p = n^{\lambda - 1}$. Let X be the number of cycles of length at most ℓ in $G_{n,p}$. The number of potential cycles of length j is certainly at most n^j, and each of them appears with probability p^j, therefore

$$\mathbf{E}[X] \leq \sum_{j=3}^{\ell} n^j p^j = \sum_{j=3}^{\ell} n^{\lambda j} \leq \frac{n^{\lambda \ell}}{1 - n^{-\lambda}}.$$

Because $\lambda\ell < 1$, this is less than $n/4$ for n sufficiently large. By Markov's inequality, $\Pr[X \geq n/2] \leq 1/2$. Note that we are not able to prove that there are *no short cycles* in $G_{n,p}$, but we will deal with this later.

Now let us consider the chromatic number of $G_{n,p}$. Rather than the chromatic number $\chi(G)$ itself, we analyze the *independence number* $\alpha(G)$, i.e. the size of the largest independent set in G. Since every color class forms an independent set, it's easy to see that $\chi(G) \geq |V(G)|/\alpha(G)$. We set $a = \lceil \frac{3}{p} \ln n \rceil$ and consider the event that there is an independent set of size a. By the union bound, we get

$$\Pr[\alpha(G) \geq a] \leq \binom{n}{a}(1-p)^{\binom{a}{2}} \leq n^a e^{-pa(a-1)/2} \leq n^a n^{-3(a-1)/2} \to 0.$$

For n sufficiently large, this probability is less than $1/2$. Hence, again by the union bound, we get

$$\Pr[X \geq n/2 \text{ or } \alpha(G) \geq a] < 1.$$

Therefore there is a graph where the number of short cycles is $X < n/2$ and the independence number $\alpha(G) < a$. We can just delete one vertex from each short cycle arbitrarily, and we obtain a graph G' on at least $n/2$ vertices which has no cycles of length at most ℓ, and $\alpha(G') < a$. The chromatic number of this graph is

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{3n^{1-\lambda} \ln n} = \frac{n^\lambda}{6 \ln n}.$$

By taking n sufficiently large, we get $\chi(G') > k$. □

We should mention that constructing such graphs explicitly is not easy. We present a construction for triangle-free graphs, which is quite simple.

Proposition 1. *Let G_2 be a graph consisting of a single edge. Given $G_n = (V, E)$, construct G_{n+1} as follows. The new set of vertices is $V \cup V' \cup \{z\}$, where V' is a copy of V and z is a single new vertex. $G_{n+1}[V]$ is isomorphic to G_n. For each vertex $v' \in V$ which is a copy of $v \in V$, we connect it by edges to all vertices $w \in V$ such that $(v, w) \in E$. We also connect each $v' \in V'$ to the new vertex z.*

Then G_n is triangle-free and $\chi(G_n) = n$.

Proof. The base case $n = 2$ is trivial. Assuming that G_n is triangle-free, it is easy to see that G_{n+1} is triangle-free as well. Any triangle would have to use one vertex from V' and two vertices from V, because there are no edges inside V'. However, by the construction of G_{n+1}, this would also imply a triangle in G_n, which is a contradiction.

Finally, we deal with the chromatic number. We assume $\chi(G_n) = n$. Note that it's possible to color V and V' in the same way, and then assign a new color to z, hence $\chi(G_{n+1}) \leq n + 1$. We claim that this is essentially the best way to color G_{n+1}. Consider any n-coloring of V. For each color c, there is a vertex v_c of color c, which is connected to vertices of all other colors - otherwise, we could re-color all vertices of color c and decrease the number of colors in G_n. Therefore, there is also a vertex $v'_c \in V'$ which is connected to all other colors different from c. If we want to color G_{n+1} using n colors, we must use color c for v'_c. But then, V' uses all n colors and z cannot use any of them. Therefore, $\chi(G_{n+1}) = n + 1$. □