ON THE ASCENT OF THE ALMOST AND QUASI-ACCP

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Abstract. The quasi-ACCP is a notion weaker than the ACCP that is defined in terms of certain common divisors. On the other hand, the property of being almost ACCP refers to the property of being atomic and satisfying the quasi-ACCP simultaneously. Motivated by extending the class of atomic domains not satisfying the ACCP, the first author and Li recently studied the almost ACCP in connection to atomicity. It is well known that if M is a submonoid of a rank-one torsion-free abelian group, then the fact that M satisfies the ACCP implies that the monoid algebra $F[M]$ over any field F also satisfies the ACCP (as an integral domain). In this paper we prove that, unlike the ACCP, the quasi-ACCP and the almost ACCP properties do not ascend from rank-one torsion-free commutative monoids to their corresponding monoid algebras over fields.

1. Introduction

The monoid M is atomic if every nonunit of M factors into finitely many atoms, also called irreducible elements. One can readily verify that if M satisfies the ascending chain condition on principal ideals (ACCP), then M must be atomic. The converse does not hold, and none of the known atomic domains not satisfying the ACCP has been obtained in a trivial manner. The first of such integral domains was constructed by Grams [24] back in the seventies. Since then, several alternative constructions have appeared in the literature (see [27, 30] and the more recent constructions in [4, 5, 19, 20]. Atomic non-commutative rings not satisfying the ACCP have been recently constructed in [2]. One of the properties we study here sits between atomicity and the ACCP, and will allow us to provide interesting examples of atomic monoids not satisfying the ACCP. We say that a common divisor $d \in M$ of a nonempty $S \subseteq M$ is a maximal common divisor (MCD) of S if the only common divisors of the set $\left\{\frac{s}{d} : s \in S\right\}$ are the invertible elements of M. Monoids where every nonempty finite subset has an MCD are called MCD monoids, while monoids where every nonempty subset has an MCD were recently called in [21] strongly MCD: it turns out that every monoid that satisfies the ACCP is strongly MCD monoid [21, Proposition 3.1]. We devote this paper to study two conditions weaker than the ACCP, the quasi-ACCP and the almost ACCP. We investigate these conditions in connection to both atomicity and the existence of MCDs.

In Section 2 we introduce most of the notation and terminology we shall be using later. We also provide a quick revision of the background needed to fully understand the results we establish here.

In Section 3, we start our investigation of two properties weaker than the ACCP: the quasi-ACCP and the almost ACCP. These properties were recently introduced and investigated in [20]. We say that the monoid M satisfies the *quasi-ACCP* if for every nonempty finite subset S of M there exists a common divisor $d \in M$ of S and an element $s \in S$ such that $s - d$ satisfies the ACCP in M. If a monoid is atomic and satisfies the quasi-ACCP, then we say it satisfies the almost ACCP. As every monoid satisfying the ACCP is atomic, every monoid satisfying the ACCP also satisfies the almost ACCP. Also, it follows directly from the definitions that if a monoid satisfies the almost ACCP it also satisfies the quasi-ACCP. In the study of these properties provided in [20], the emphasis is put on the setting of integral domains. Here we focus on the larger class of commutative monoids, and the only class of

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integral domains we consider is that of monoid algebras over fields. In Section 3, we provide several examples of monoids satisfying the quasi-ACCP (some of them not satisfying the almost ACCP), and we connect these properties with the existence of MCDs, proving that every monoid satisfying the quasi-ACCP must be an MCD monoid.

In Section 4 is dedicated to determine whether the quasi-ACCP and the almost ACCP ascend from the class of rank-one torsion-free monoids to their corresponding monoid algebras (over fields). The ascend of atomicity was first posed by Gilmer [15, page 189] in the eighties as the following question: given a pair (M, R) of monoid and domain such that $R[M]$ is an integral domain, does the fact that M and R are atomic suffices to ensure that the monoid algebra $R[M]$ is atomic. It first answer was given by Roitman [27] ten years later with the construction of an atomic domain R whose polynomial extension is not atomic. The dual question was not solved by Coykendall and the first author in the more recent paper [11], where they constructed an atomic rank-one torsion-free monoid whose monoid algebra over $\mathbb{F}_2$ (the field of two elements) is not atomic (this result was recently generalized in [22]). We provide negative answers to the ascent of both quasi-atomicity (in Theorem 4.1) and almost atomicity (Theorem 4.7) from rank-one torsion-free monoids to their corresponding monoid algebras. The non-ascent of almost atomicity is more subtle, and our approach is highly motivated by that one given in [11] to establish the non-ascent of atomicity. It is worth emphasizing that our approach improve upon that given in [11] as we construct a rank-one torsion-free monoid satisfying the almost ACCP (and so being atomic) whose monoid algebra over $\mathbb{F}_2$ is not even atomic.

2. Background

2.1. **General Notation.** Let us first introduce some general notation we shall be using throughout this paper. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. In addition, we let $\mathbb{P}$ denote the set of primes. As it is customary, we let $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of rational numbers and the set of real numbers, respectively. For a subset S of $\mathbb{R}$ and $r \in \mathbb{R}$, we set $S_{\ge r} := \{s \in S : s \ge r\}$ and $S_{>r} := \{s \in S : s > r\}$. For $b, c \in \mathbb{Z}$, we set

$$[\![b, c]\!] := \{n \in \mathbb{Z} : b \le n \le c\},$$

allowing the discrete interval $[\![b, c]\!]$ to be empty when $b > c$. For a nonzero $q \in \mathbb{Q}$, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ denote, respectively, the unique $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ such that $q = n/d$ and $\gcd(n, d) = 1$. For $p \in \mathbb{P}$ and a nonzero $n \in \mathbb{Z}$, we let $v_p(n)$ denote the maximum $m \in \mathbb{N}_0$ such that $p^m \mid n$. Then for each $p \in \mathbb{P}$, we let $v_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ denote the p-adic valuation map, which means that for every nonzero $q \in \mathbb{Q}$

$$v_p(q) = v_p(\mathsf{n}(q)) - v_p(\mathsf{d}(q)).$$

2.2. **Commutative Monoids.** Let us introduce now some terminology about commutative monoids. Throughout this paper we tacitly assume that every monoid is commutative and cancellative. Let M be a monoid that is additively written (and with identity element denoted by 0). We say that M is *nontrivial* if $M \neq \{0\}$. The group consisting of all the invertible elements of M (also called units) will be denoted by $\mathscr{U}(M)$, and we say that M is *reduced* if $\mathscr{U}(M)$ is the trivial group. A subset N of M is called a *submonoid* if N contains 0 and is closed under the operation of M. The arbitrary intersection of submonoids of M is also a submonoid of M. If S is a subset of M, then we let $\langle S \rangle$ denote the submonoid of M generated by S; that is, $\langle S \rangle$ is the intersection of all the submonoids of M containing S. The *Grothendieck group* of M, denoted by $\text{gp}(M)$, is the abelian group consisting of all the formal differences of elements of M. As M is assumed to be cancellative, we can identify M with a submonoid of $\text{gp}(M)$ (indeed, it turns out that $\text{gp}(M)$ is the unique abelian group up to isomorphism that minimally contains an isomorphic copy of M). The monoid M is called *torsion-free* provided that $\text{gp}(M)$ is a torsion-free abelian group or, equivalently, for each $n \in \mathbb{N}$ and $b, c \in M$, the equality $nb = nc$ implies that $b = c$.

A non-invertible element $a \in M$ is called an *atom* (or *irreducible*) if for all $b, c \in M$ the equality $a = b + c$ implies that either $b \in \mathscr{U}(M)$ or $c \in \mathscr{U}(M)$. The set consisting of all the atoms of M is denoted by $\mathscr{A}(M)$. Following Cohn [9], we say that the monoid M is *atomic* if every element in $M \setminus \mathscr{U}(M)$ can be written as a sum of finitely many atoms (allowing repetitions). The monoid M is called a *unique factorization monoid* (UFM) if it is atomic and every non-invertible element of M can be written as a sum of finitely many atoms in a unique way (up to order and associates). For progress on atomicity in the setting of integral domains, see the recent survey [10] by Coykendall and Gotti. Ascending chains of principal ideals are often studied in connection to atomicity. A subset I of M is called an *ideal* if

$$I + M := \{b + c : b \in I \text{ and } c \in M\} \subseteq I.$$

An ideal of M is called *principal* if it has the form $b + M$ for some $b \in M$. The monoid M is called a *valuation monoid* if for all $b, c \in M$ either $b + M \subseteq c + M$ or $c + M \subseteq b + M$. A sequence $(I_n)_{n \ge 1}$ of subsets of M is called *ascending* if $I_n \subseteq I_{n+1}$ for all $n \in \mathbb{N}$, while a sequence $(J_n)_{n \ge 1}$ of subsets of M is said to *stabilize* if we can take $N \in \mathbb{N}$ such that $J_n = J_N$ for all $n \in \mathbb{N}$ with $n \ge N$. Then we say that M satisfies the *ascending chain condition on principal ideals* (ACCP) if every ascending chain of principal ideals eventually stabilizes. Every monoid satisfying the ACCP is atomic [14, Proposition 1.1.4], but the converse is not true in general. Some examples of atomic monoids that do not satisfy the ACCP, along with recent progress on the connection between the ACCP and atomicity, are provided in [19]. Also, we will discuss further examples throughout this paper. Finally, nontrivial elementary examples of monoids satisfying the ACCP are the additive submonoids of $\mathbb{R}_{\ge 0}$ whose corresponding sets of nonzero elements are bounded below by a positive real number [17, Proposition 4.5].

Additive submonoids of $\mathbb{Q}_{\ge 0}$ are the most relevant algebraic objects in this paper; they are called *Puiseux monoids*. The atomicity and arithmetic of factorizations of Puiseux monoids have been systematically investigated during the past decade ([7, 22] provides a friendly survey in this direction). The *rank* of the monoid M is, by definition, the rank of its Grothendieck group or, equivalently, the dimension of the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$ over the field $\mathbb{Q}$ obtained by tensoring the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\text{gp}(M)$. Observe that every nontrivial Puiseux monoid has rank 1. It is known that nontrivial Puiseux monoids account, up to isomorphism, for all rank-1 torsion-free monoids that are not abelian groups [13, Theorem 3.12] (see also [12, Section 24] and [15, Theorem 2.9]). Finally, observe that every finitely generated Puiseux monoid is an additive submonoid of $\mathbb{N}_0$ (or a numerical monoid) up to isomorphism, and so it must satisfy the ACCP. More generally, every additive submonoid of $\mathbb{R}_{\ge 0}$ (and so every Puiseux monoid)

2.3. **Divisibility.** Even though it is more natural to write monoids multiplicatively when dealing with notions related to divisibility (as we did in the introduction), we will keep writing the monoid M additively. The reason for this is that the monoids we are primarily interested in the scope of this paper are Puiseux monoids, which are additive by nature.

For $b, c \in M$, we say that c *divides* b and write $c \mid_M b$ provided that $b = c + d$ for some $d \in M$. For a nonempty subset S of M, an element d is called a *common divisor* of S if $d \mid_M s$ for all $s \in S$, in which case, we let $S - d$ denote the set $\{s - d : s \in S\}$. A *greatest common divisor* (GCD) of a nonempty subset S of M is a common divisor $d \in M$ of S such that any other common divisor of S divides d in M. A *maximal common divisor* (MCD) of a nonempty subset S of M is a common divisor $d \in M$ of S such that the only common divisors of $S - d$ are the invertible elements of M. In addition, the monoid M is called a *GCD monoid* (resp., an *MCD monoid*) provided that every nonempty finite subset of M has a GCD (resp., an MCD). It is clear that every greatest common divisor is a maximal common divisor, and so every GCD monoid is an MCD monoid. For $k \in \mathbb{N}$, we say that M is a *k-MCD monoid* if every subset of M of cardinality k has a maximal common divisor. Clearly, every monoid is a 1-MCD monoid. Also, observe that M is an MCD monoid if and only if M is a k-MCD monoid for every $k \in \mathbb{N}$. The notion of k-MCD was introduced by Roitman in [27].

3. Two Weaker Notions of the ACCP

The almost ACCP was recently introduced and studied in [20]. In this section we consider the quasi-ACCP property, which is a notion weaker than the almost ACCP.

Definition 3.1. Let M be a commutative monoid. We say that M satisfies the *quasi-ACCP* if for every nonempty finite subset S of M there exists a common divisor $d \in M$ of S and an element $s \in S$ such that $s - d$ satisfies the ACCP in M.

If a monoid satisfies the ACCP, then it follows immediately from the previous definition that it also satisfies the quasi-ACCP. Therefore we can consider the quasi-ACCP property a weaker notion of the standard ACCP. As the following example illustrates, there are Puiseux monoids that satisfy the quasi-ACCP but not the ACCP.

Example 3.2. Every totally orderable valuation monoid M satisfies the quasi-ACCP: indeed, if S is a nonempty finite subset of M, then we can take both d and s to be $\min S$ to obtain that $s - d = 0$ satisfies the ACCP. Thus, the nonnegative cone of every linearly ordered abelian group is a quasi-ACCP monoid. In particular, the additive monoids $\mathbb{Z}[\frac{1}{n}]_{\ge 0}$ and $\mathbb{R}_{\ge 0}$ are weak-ACCP monoids that are not atomic.

3.1. A Class of Puiseux Monoids Satisfying the Quasi-ACCP.

For each positive rational q, the atomic structure and arithmetic of length of the Puiseux monoid $\mathbb{N}_0[q]$ have been considered in various recent papers, including [7]. In this section we prove that $\mathbb{N}_0[q]$ satisfies the quasi-ACCP for all $q \in \mathbb{Q}_{>0}$. Let us start by arguing the following lemma.

Lemma 3.3. *Take $q \in \mathbb{Q} \cap (0,1)$, and consider the additive monoid $\mathbb{N}_0[q]$. For each $b \in M$, the following statements hold.*

(1) *There exist unique nonnegative integer coefficients $c_0, c_1, c_2, \dots$ with $c_n < \mathsf{d}(q)$ for every $n \in \mathbb{N}$ and $c_n = 0$ for all but finitely many $n \in \mathbb{N}$ such that*

$$b = c_0 + \sum_{n \in \mathbb{N}} c_n q^n. \tag{3.1}$$

(2) *If the monoid $\mathbb{N}_0[q]$ is atomic, then the following conditions are equivalent.*
   (a) $|\mathsf{Z}(b)| = 1$.
   (b) $b = c_0 + \sum_{n=1}^m c_n q^n$ *for some* $c_0 \in [\![0, \mathsf{n}(q) - 1]\!]$ *and* $c_1, \dots, c_n \in [\![0, \mathsf{d}(q) - 1]\!]$.

Proof. (1) To prove the existence, take $b \in M$. As $\{q^n : n \in \mathbb{N}_0\}$ is a generating set of M, we can choose nonnegative integer coefficients $c_0, c_1, \dots$ such that $b = c_0 + \sum_{n \in \mathbb{N}} c_n q^n$, where $c_n = 0$ for all but finitely many $n \in \mathbb{N}_0$. If for at least one choice of coefficients, $c_n < d$ for every $n \in \mathbb{N}_0$, we are done. Thus, we can assume that the set $S := \{n \in \mathbb{N}_0 : c_n \ge d\}$ is nonempty for any possible choice of the coefficients $c_0, c_1, \dots$. Then we can further assume that we have chosen the coefficients $c_0, c_1, \dots$ minimizing $m := \max S$. We claim that $m = 0$. Suppose, by way of contradiction, that $m \ge 1$. Accordingly, we can write $c_m = c\mathsf{d}(q) + r$ for some $c \in \mathbb{N}$ and $r \in [\![0, \mathsf{d}(q) - 1]\!]$. Because

$$c_m q^m = (c\mathsf{d}(q) + r)q^m = rq^m + (c\mathsf{n}(q))q^{m-1} + rq^m,$$

we can replace $c_{m-1}q^{m-1} + c_m q^m$ by $(c_{m-1} + c\mathsf{n}(q))q^{m-1} + rq^m$ in $b = \sum_{n \in \mathbb{N}_0} c_n q^n$ to contradict the minimality of m. Thus, $m = 1$, and so the decomposition $r = c_0 + \sum_{n \in \mathbb{N}} c_n q^n$ satisfies the conditions of that in (3.1) with $c_0 = N_r$. We proceed to argue the uniqueness of the same decomposition. Assume that

$$b = N_b + \sum_{n \in \mathbb{N}} c_n q^n \quad \text{and} \quad b = N_b' + \sum_{n \in \mathbb{N}} d_n q^n$$

are two sum decompositions of b satisfying the desired conditions on the coefficients. Consider the set $T := \{n \in \mathbb{N} : d_n \neq c_n\}$, and observe that the given sum decomposition of b are the same if and only if T is empty. Suppose, by way of contradiction, that T is not empty, and set $m := \max T$. Then we can write

$$(c_m - d_m)\mathsf{n}(q)^m = (N_b' - N_b)\mathsf{d}(q)^m + \sum_{n=1}^{m-1} (d_n - c_n)\mathsf{n}(q)^n \mathsf{d}(q)^{m-n}.$$

Since the left-hand side of the previous equality is divisible by $\mathsf{d}(q)$, so is $(c_m - d_m)\mathsf{n}(q)^m$, from which we infer that $\mathsf{d}(q) \mid c_m - d_m$. Thus, we obtain that $c_m = d_m$, which is a contradiction. Hence the sum decomposition in (3.1) is unique.

(2) Now assume that $\mathbb{N}_0[q]$ is an atomic monoid.

(a) $\Rightarrow$ (b): Suppose that $|\mathsf{Z}(b)| = 1$. It follows from part (1) that $b = c_0 + \sum_{n=1}^m c_n q^n$ for some $c_0, \dots, c_m \in \mathbb{N}_0$ such that $c_1, \dots, c_m \in [\![1, m]\!]$. Thus, $z := \sum_{n=0}^m c_n q^n$ is the only factorization of b in $\mathbb{N}_0[q]$. Now observe that $c_0 < \mathsf{n}(q)$ as, otherwise, we could replace $c_0 + c_1 q$ by $(c_0 - \mathsf{n}(q)) + (\mathsf{d}(q)c_0 + c_1)q$ in $c_0 + \sum_{n=1}^m c_n q^n$ to obtain a factorization of b different from z.

(b) $\Rightarrow$ (a): Write $b = c_0 + \sum_{n=1}^m c_n q^n$ for some $c_0 \in [\![0, \mathsf{n}(q) - 1]\!]$ and $c_1, \dots, c_n \in [\![0, \mathsf{d}(q) - 1]\!]$. Then $z := \sum_{n=0}^m c_n q^n$ is a factorization of b in $\mathbb{N}_0[q]$. We proceed to show that z is the only factorization of b in $\mathbb{N}_0[q]$. Assume, towards a contradiction, that $|\mathsf{Z}(b)| \geq 2$, and let $z' := \sum_{n=0}^m d_n q^n$ be a factorization of b in $\mathbb{N}_0[q]$ such that $z' \neq z$. Then it follows from the uniqueness in of part (1) that $d_n \geq \mathsf{d}(q)$ for some $n \in [\![1, m]\!]$. Set $j := \max\{n \in [\![1, m]\!] : d_n \geq \mathsf{d}(q)\}$, and then write $d_j = k_j \mathsf{d}(q) + r_j$ for some $k_j \in \mathbb{N}$ and $r_j \in [\![0, \mathsf{d}(q) - 1]\!]$. Now observe that after replacing $d_{j-1}q^{j-1} + d_j q^j$ by $(d_{j-1} + k_j \mathsf{n}(q))q^{j-1} + r_j q^j$ in $\sum_{n=0}^m d_m q^m$, we obtain a factorization z_1 of b such that $j - 1 := \max\{n \in [\![1, m]\!] : d_n \geq \mathsf{d}(q)\}$ and $d_{j-1} \geq \mathsf{n}(q)$. After repeating this replacement process j times, we obtain a chain $z' = z_0, z_1, \dots, z_j$ of factorizations of b in $\mathbb{N}_0[q]$ such that for each $i \in [\![1, j]\!]$,

$$z_i = d_{i,0} + \sum_{n=1}^m d_{i,n} q^n$$

for some coefficients $d_{i,0}, \dots, d_{i,m} \in \mathbb{N}_0$ with $d_{i,j-i} \geq \mathsf{n}(q)$ and $d_{i,n} < \mathsf{d}(q)$ for every $n \in [\![j - i + 1, m]\!]$. In particular, $z_j = d_{j,0} + \sum_{n=1}^m d_{j,n} q^n$ is a factorization of b such that $d_{j,0} \geq \mathsf{n}(q)$ and $d_{j,n} < \mathsf{d}(q)$ for every $n \in [\![1, m]\!]$. Thus, it follows from the uniqueness in part (1) that $z = z_j$ and so $c_0 = d_0 \geq \mathsf{n}(d)$. However, this contradicts the fact that $c_0 \in [\![0, \mathsf{n}(q) - 1]\!]$. Thus, $|\mathsf{Z}(b)| = 1$. $\square$

We are in position to show that the monoid $\mathbb{N}_0[q]$ satisfies the quasi-ACCP for all $q \in \mathbb{Q}_{>0}$.

Proposition 3.4. *For each $q \in \mathbb{Q}_{>0}$, the Puiseux monoid $\mathbb{N}_0[q]$ satisfies the quasi-ACCP.*

Proof. Take $q \in \mathbb{Q}_{>0}$, and set $M := \mathbb{N}_0[q]$. If $q \geq 1$, then 0 is not a limit point of $M \setminus \{0\}$, and so it follows from [17, Proposition 4.5] that M satisfies the ACCP and so the quasi-ACCP. Therefore we assume that $q \in (0, 1) \cap \mathbb{Q}$. If $q = \frac{1}{n}$ for some $n \in \mathbb{N}$ with $n \geq 2$, then M is a valuation monoid and, as we have seen in Example 3.2, M satisfies the quasi-ACCP. Thus, we further assume that $q \neq \frac{1}{n}$ for any $n \in \mathbb{N}$. In light of this assumption, it follows from [18, Theorem 6.2] that M is atomic with $\mathscr{A}(M) = \{q^n : n \in \mathbb{N}_0\}$.

To argue that M satisfies the quasi-ACCP, let S be a finite nonempty subset of M. By virtue of part (1) of Lemma 3.3, for each $s \in S$, we can write

$$s = c_{s,0} + \sum_{n=1}^{m} c_{s,n} q^n$$

for some $c_{s,0}, c_{s,1}, \dots, c_{s,m} \in \mathbb{N}_0$ such that $c_{s,n} \in [\![0, \mathsf{d}(q) - 1]\!]$ for every $n \in [\![1, m]\!]$. We can further assume that $c_{s,m} \ge 1$ for some $s \in S$. Now for each $n \in [\![0, m]\!]$, set $c_n := \min\{c_{s,n} : s \in S\}$ and then consider the element

$$d := \sum_{n=0}^{m} d_n q^n \in M.$$

It is clear that d is a common divisor of S in M. On the other hand, we can take $s \in S$ such that $d_0 = c_{s,0}$. In this case, $s - d = \sum_{n=1}^{m} (c_{s,n} - d_n) q^n$. Now because $c_{s,n} - d_n \in [\![0, \mathsf{d}(q) - 1]\!]$ for every $n \in [\![1, m]\!]$, it follows from part (2) of Lemma 3.3 that $|\mathsf{Z}(s - d)| = 1$. This, together with the fact that M is atomic, guarantees that every ascending chain of principal ideals of M starting at $s - d$ must stabilize. Hence M satisfies the quasi-ACCP. $\square$

3.2. Quasi-ACCP and MCDs.

We proceed to prove that every monoid that satisfies the quasi-ACCP is an MCD monoid.

Proposition 3.5. *Let M be a commutative monoid. If M satisfies the quasi-ACCP, then M is an MCD monoid.*

Proof. Suppose that M satisfies the quasi-ACCP. Then assume, by way of contradiction, that M is not an MCD monoid. Let S be a nonempty finite subset of M having no MCD, which means that for every common divisor d of S in M there exists a non-invertible common divisor of $S - d$ in M. Since M satisfies the quasi-ACCP, we can take a common divisor d_0 of S in M such that $s_0 - d_0$ satisfies the ACCP for some $s_0 \in S$. As d_0 is not an MCD of S, we can take a common divisor $d_1 \in M \setminus \mathscr{U}(M)$ of $S - d_0$, and so $d_0 + d_1$ is a common divisor of S in M. Similarly, as $d_0 + d_1$ is a common divisor of S in M, we can take a common divisor $d_2 \in M \setminus \mathscr{U}(M)$ of $S - (d_0 + d_1)$ in M, and so $d_0 + d_1 + d_2$ is a common divisor of S in M. We can repeat this process indefinitely to obtain a sequence $(d_n)_{n \ge 1}$ with terms in $M \setminus \mathscr{U}(M)$ such that for each $n \in \mathbb{N}_0$ the element $c_n := d_0 + d_1 + \dots + d_n$ is a common divisor of S in M and, in particular, $c_n \mid_M s_0$. Finally, from the fact that $(s_0 - c_n) - (s_0 - c_{n+1}) = d_{n+1} \in M \setminus \mathscr{U}(M)$ for every $n \in \mathbb{N}_0$, we obtain that $(s_0 - c_n + M)_{n \ge 0}$ is an ascending chain of principal ideals of M starting at $s_0 - d_0$ that does not stabilize. However, this contradicts that M satisfies the quasi-ACCP. $\square$

The converse of Proposition 3.5 does not hold. Indeed, the monoid in the following example is a GCD monoid that does not satisfy the quasi-ACCP.

Example 3.6. Consider the additive valuation monoids $V_2 := \mathbb{N}_0[\frac{1}{2}]$ and $V_3 := \mathbb{N}_0[\frac{1}{3}]$, and then set $S := V_2 + V_3$. We first prove that each element $q \in M$ can be written uniquely as follows

$$q = b_0 + \sum_{n \in \mathbb{N}} \frac{b_n}{2^n} + \sum_{n \in \mathbb{N}} \frac{c_n}{3^n}, \tag{3.2}$$

for nonnegative integer coefficients $b_0, b_1, b_2, \dots$ and $c_1, c_2, \dots$ almost all being zero such that $b_n \le 1$ and $c_n \le 2$ for every $n \in \mathbb{N}$. As $\left\{ \frac{1}{2^n}, \frac{1}{3^n} : n \in \mathbb{N} \right\}$ is a generating set of S, we can express q as in (3.2) only assuming that the coefficients $b_0, b_1, b_2, \dots$ and $c_1, c_2, \dots$ are nonnegative integers almost all being zero. We further assume that, among all choices of coefficients, we have picked one minimizing the sum $\sum_{n \in \mathbb{N}} b_n + \sum_{n \in \mathbb{N}} c_n$. Then observe that $b_n \le 1$ for every $n \in \mathbb{N}$: indeed, if $b_m \ge 2$ for some $m \in \mathbb{N}$, then replacing $\frac{b_{m-1}}{2^{m-1}} + \frac{b_m}{2^m}$ by $\frac{b_{m-1}+1}{2^{m-1}} + \frac{b_m-2}{2^m}$ in (3.2) would contradict the minimality of $\sum_{n\in\mathbb{N}} b_n + \sum_{n\in\mathbb{N}} c_n$. In a similar way, we can argue that $c_n \le 2$ for every $n \in \mathbb{N}$. For the uniqueness, suppose that

$$q = b'_0 + \sum_{n\in\mathbb{N}} \frac{b'_n}{2^n} + \sum_{n\in\mathbb{N}} \frac{c'_n}{3^n},$$

is also a sum decomposition of q satisfying the same conditions as the one in (3.2). If there is an index $m \in \mathbb{N}$ such that $b'_m \neq b_m$, after assuming that m is as large as it could possible be and applying the 2-adic valuation map to both sides of the equality

$$\frac{b'_m - b_m}{2^m} = \sum_{n=0}^{m-1} \frac{b_n - b'_n}{2^n} + \sum_{n\in\mathbb{N}} \frac{c_n - c'_n}{3^n},$$

we obtain that $2 \mid b'_m - b_m$, which is not possible. Thus, $b'_n = b_n$ for every $n \in \mathbb{N}$. Similarly, we can check that $c'_n = c_n$ for every $n \in \mathbb{N}$. Thus, the sum decomposition in (3.2) is unique.

To argue that S is a GCD monoid, we fix nonzero $q, r \in S$ and proceed to show that the set $\{q, r\}$ has a GCD in S. Write q as in (3.2) and set $d_q := \sum_{n\in\mathbb{N}_0} \frac{b_n}{2^n} \in V_2$. As V_2 is a valuation monoid, the uniqueness of (3.2) implies that $d_q = \max\{d \in V_2 : d \mid_S q\}$. Similarly, we can take d_r to be the maximum of $\{d \in V_2 : d \mid_S r\}$. Write $q = d_q + e_q$ and $r = d_r + e_r$, and observe that $e_q, e_r \in V_3 \cap [0, 1)$. As $d_q, d_r \in V_2$, which is a valuation monoid, $d := \max\{d_q, d_r\}$ is a common divisor of $\{d_q, d_r\}$, and so after subtracting d from both q and r we can assume that $0 \in \{d_q, d_r\}$. Similarly, as $e_q, e_r \in V_3$, which is also a valuation monoid, we can assume that $0 \in \{e_q, e_r\}$. Since $qr \neq 0$, one can assume, without loss of generality, that $q = d_q$ and $r = e_r$. If $1 \mid_{V_2} d_q$, then the fact that $e_r \in V_3 \cap (0, 1)$ implies that $e_r \mid_{V_3} 1$ and so $r \mid_S q$, whence r is the GCD of $\{r, q\}$ in S. If $1 \nmid_{V_2} d_q$, then $d_q < 1$ (also, $e_r < 1$), so the uniqueness of (3.2) ensures that 0 is the only common divisor of $\{d_q, e_r\} = \{q, r\}$ in S, which implies that 0 is the GCD of $\{q, r\}$ in S. Hence S is a GCD monoid and, therefore, an MCD monoid.

We proceed to argue that M does not satisfy the quasi-ACCP. By the uniqueness of (3.2), it follows that the only common divisor of $\{\frac{1}{2}, \frac{1}{3}\}$ in S is 0. This, along with the fact that both $\left(\frac{1}{2^n} + S\right)_{n\ge1}$ and $\left(\frac{1}{3^n} + S\right)_{n\ge1}$ are ascending chains of principal ideals of S respectively starting at $\frac{1}{2}$ and $\frac{1}{3}$ that do not stabilize, implies that S does not satisfy the quasi-ACCP.

4. Ascent of Almost and Quasi-ACCP to Monoid Algebras

The primary purpose of this section is to decide whether the quasi-ACCP and the almost ACCP properties ascend to monoid algebras over fields. First, we consider the quasi-ACCP, which is significantly easier.

4.1. Ascent of Quasi-ACCP to Monoid Algebras.

For a perfect field $\mathbb{F}$, we proceed to prove that the property of satisfying the quasi-ACCP does not ascend from rank-one torsion-free monoids to their corresponding monoid algebras over $\mathbb{F}$.

Theorem 4.1. *For $p \in \mathbb{P}$, let $\mathbb{F}$ be a perfect field of characteristic $p \in \mathbb{P}$. Then there exists a rank-one torsion-free monoid M that satisfies the quasi-ACCP such that $\mathbb{F}[M]$ does not satisfy the quasi-ACCP.*

Proof. Consider the Puiseux monoid $M := \mathbb{N}_0[\frac{1}{p}]$. Observe that M is also a valuation monoid; indeed, for any $q, r \in M$ the divisibility relation $q \mid_M r$ holds if and only if $q \le r$. Therefore M satisfies the quasi-ACCP. We proceed to argue that the nonzero elements of the monoid algebra $\mathbb{F}[M]$ that satisfy the ACCP are precisely the nonzero constant polynomials, that is, the elements of $\mathbb{F}$.

The set consisting of all nonzero constant polynomial expressions of $\mathbb{F}[M]$ is $\mathbb{F}[M]^\times$, and units clearly satisfy the ACCP. For the converse, it suffices to show that for any nonconstant polynomial expression $g(x) \in \mathbb{F}[M]$, there exists a nonconstant polynomial expression $f(x) \in \mathbb{F}[M]$ with $\deg f(x) < \deg g(x)$ such that $g(x)\mathbb{F}[M] \subsetneq f(x)\mathbb{F}[M]$. For this, take a nonzero $g(x) \in \mathbb{F}[M] \setminus \mathbb{F}$, and write

$$g(x) = \sum_{i=1}^{n} c_i x^{r_i} \in \mathbb{F}_p[M]$$

with coefficients $c_1, \dots, c_n \in \mathbb{F}$ (not all zeros) and (pairwise distinct) exponents $r_1, \dots, r_n \in M$. The fact that $\mathbb{F}$ is a perfect field guarantees the existence of $b_1, \dots, b_n$ such that $b_i^p = c_i$ for every $i \in [\![1, n]\!]$. Now observe that M is a p-divisible monoid: therefore we can pick $q_1, \dots, q_n \in M$ such that $pq_i = r_i$ for every $i \in [\![1, n]\!]$. Now set $f(x) := \sum_{i=1}^n b_i x^{q_i} \in \mathbb{F}[M]$ and note that

$$f(x)^p = \bigg(\sum_{i=1}^{n} b_i x^{q_i} \bigg)^p = \sum_{i=1}^{n} b_i^p x^{pq_i} = \sum_{i=1}^{n} c_i x^{r_i} = g(x).$$

It is clear that $f(x)$ is a nonconstant polynomial expression with $\deg f(x) = \frac{1}{p} \deg g(x) < \deg g(x)$. Therefore the only elements of $\mathbb{F}[M] \setminus \{0\}$ that satisfy the ACCP are the constant polynomials.

It follows from Proposition 3.5 that the multiplicative monoid $\mathbb{F}[M]^*$ cannot be satisfy the quasi-ACCP if it is not an MCD monoid. Thus, we are done once we prove the following claim.

CLAIM. For any common divisor d of $\{x, x+1\}$ in $\mathbb{F}[M]^*$, neither $\frac{x}{d}$ nor $\frac{x+1}{d}$ satisfy the ACCP in $\mathbb{F}[M]^*$.

PROOF OF CLAIM. Suppose, towards a contradiction, that there exists a common divisor d of $\{x, x+1\}$ in the multiplicative monoid $\mathbb{F}[M]$ such that either $\frac{x}{d}$ or $\frac{x+1}{d}$ satisfies the ACCP. From the relations $d \mid_{\mathbb{F}[M]} x$ and $d \mid_{\mathbb{F}[M]} x + 1$, we deduce that $d \mid_{\mathbb{F}[M]} 1$, which implies that $d \in \mathbb{F}[M]^\times \mathbb{F}^\times$. Therefore either x or $x+1$ must satisfy the ACCP in $\mathbb{F}[M]$. However, this is not possible as we have already seen that only constant polynomial expressions satisfy the ACCP in $\mathbb{F}[M]$. Hence the claim is established, which concludes our proof. □

4.2. **Ascent of Almost ACCP to Monoid Algebras.** This section is devoted to show that the almost ACCP property does not ascend to monoid algebras (over fields). We will produce a rank-one torsion-free monoid M that satisfies the almost ACCP such that the monoid algebra $\mathbb{F}_2[M]$ does not.

4.2.1. *An Atomic Monoid not Satisfying the ACCP.* Let us introduce the primary ingredient of this section, which is the following rank-one torsion-free commutative monoid

$$M := \left\langle \frac{3^n - (-2)^n}{5 \cdot 6^n} : n \in \mathbb{N}_{\geq 2} \right\rangle. \tag{4.1}$$

Now we set $a_n := \frac{3^n - (-2)^n}{5 \cdot 6^n}$ for each $n \in \mathbb{N}$ with $n \geq 2$, and we observe that

$$\begin{aligned} a_{n-2} + a_{n-1} &= \frac{3^{n-2} - (-2)^{n-2}}{5 \cdot 6^{n-2}} + \frac{3^{n-1} - (-2)^{n-1}}{5 \cdot 6^{n-1}} \\ &= \frac{2 \cdot 3^{n-1} + 3 \cdot (-2)^{n-1} + 3^{n-1} - (-2)^{n-1}}{5 \cdot 6^{n-1}} \\ &= \frac{3^n - (-2)^n}{5 \cdot 6^{n-1}} = 6a_n \end{aligned}$$

provided that $n \geq 4$. We will record this recurrence for future reference:

$$6a_n = a_{n-1} + a_{n-2} \tag{4.2}$$

for every $n \in \mathbb{N}$ with $n \geq 4$. Here is another preliminary fact we need.

Lemma 4.2. $\mathbb{Z}[\frac{1}{2}] \subseteq M \subseteq \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$.

Proof. Now, for each $n \in \mathbb{N}_{\geq 2}$, set $a_n := \frac{3^n-(-2)^n}{5\cdot 6^n}$. Let us start by verifying that $\mathbb{Z}[\frac{1}{2}] \subseteq M \subseteq \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$. For the first inclusion, it suffices to verify that $\{\frac{1}{2^n} : n \in \mathbb{N}_{\geq 3}\} \subseteq M$, which amounts to observing that for every $n \in \mathbb{N}_{\geq 2}$,

$$a_n + 3a_{n+1} = \frac{1}{5}\left(\frac{6\cdot 3^n - 6(-2)^n}{6^{n+1}} + \frac{9\cdot 3^n + 6(-2)^n}{6^{n+1}}\right) = \frac{1}{5}\left(\frac{15\cdot 3^n}{6^{n+1}}\right) = \frac{1}{2^{n+1}}.$$

To argue the second inclusion, we take $n \in \mathbb{N}_{\geq 2}$ and notice that the divisibility relation $5 \mid 3^n - (-2)^n$ holds, and so $a_n \in \langle \frac{1}{6^n} : n \in \mathbb{N}_{\geq 2}\rangle$. This implies that $M \subseteq \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$. □

Our next goal is to argue that the monoid M is atomic with set of atoms $\mathscr{A}(M) = \{a_n : n \in \mathbb{N}_{\geq 2}\}$. In order to do so, we need to describe how we can decompose canonically every element of M as a sum of the defining generators $\{a_n : n \in \mathbb{N}_{\geq 2}\}$.

Proposition 4.3. *Each $q \in M$ can be uniquely written as follows:*

$$q = \sum_{k\geq 2} c_k a_k = \frac{c_2}{6^2} + \sum_{k\geq 3} c_k \frac{3^k - (-2)^k}{5\cdot 6^k}, \tag{4.3}$$

where $c_2 \in \mathbb{N}_0$, $c_k \in [\![0,5]\!]$ for every $k \in \mathbb{N}_{\geq 3}$, and $c_k = 0$ for almost all $k \in \mathbb{N}_{\geq 2}$. In addition, for each $n \in \mathbb{N}_{\geq 3}$ the following identity holds:

$$6a_n = a_{n-1} + a_{n-2}. \tag{4.4}$$

Proof. First, we argue the existence of the sum decomposition in (4.3). Fix $q \in M$. Since the set $\{a_n : n \in \mathbb{N}_{\geq 2}\}$ generates M, we can write

$$q = \sum_{k\geq 2} c_k a_k \tag{4.5}$$

for some nonnegative integer coefficients $c_2, c_3, \dots$ such that $c_k = 0$ for almost all $k \in \mathbb{N}_{\geq 2}$. If there is a choice of the coefficients $c_2, c_3, \dots$ such that $c_k \leq 5$ for every $k \in \mathbb{N}_{\geq 3}$, then we are done. Suppose otherwise, and assume that we have chosen the coefficients in (4.5) that minimize the index

$$m := \max\{k \in \mathbb{N}_{\geq 2} : c_k \geq 6\}.$$

In addition, we can further assume that among all the choices of coefficients in (4.5) minimizing m, the one we picked yields the minimum possible c_m. We claim that $m = 2$. Suppose, by way of contradiction, that $m \geq 3$. We split the rest of the existence into the following two cases.

CASE 1: $m = 3$. In this case, we see that

$$(c_2+7)a_2 + (c_3-6)a_3 = c_2a_2 + c_3a_3 + 7\frac{3^2-(-2)^2}{5\cdot 6^2} - 6\frac{3^3-(-2)^3}{5\cdot 6^3} = c_2a_2 + c_3a_3.$$

Therefore, after replacing $c_2a_2 + c_3a_3$ by $(c_2+7)a_2 + (c_3-6)a_3$ in (4.5), we would obtain another decomposition of q that generates a contradiction with either the minimality of m or the minimality of c_m.

CASE 2: $m \geq 4$. In light of (4.2), the equality $6a_m = a_{m-1} + a_{m-2}$ holds. As a consequence, since $c_m \geq 6$ we can replace $c_{m-2}a_{m-2} + c_{m-1}a_{m-1} + c_m a_m$ by $(c_{m-2}+1)a_{m-2} + (c_{m-1}+1)a_{m-1} + (c_m-6)a_m$ in (4.5) to obtain a decomposition of q that generates a contradiction with either the minimality of m or the minimality of c_m.

Finally, we prove that the decomposition in (4.3) is unique. Suppose, towards a contradiction, that there exists $q \in M$ with two distinct decompositions $q = \sum_{k \ge 2} c_k a_k = \sum_{k \ge 2} d_k a_k$, where $c_2, d_2 \in \mathbb{N}_0$, $c_k, d_k \in [\![0,5]\!]$ for all $k \in \mathbb{N}_{\ge 3}$, and $c_k = d_k = 0$ for almost all $k \in \mathbb{N}_{\ge 2}$. Now set

$$N := \max\{k \in \mathbb{N} : c_k \neq d_k\},$$

which must exist because $c_k = d_k = 0$ for almost all $k \in \mathbb{N}_{\ge 2}$. Observe that $N \ge 3$. We can now subtract the two given decomposition of q to obtain that

$$\sum_{k=2}^{N} (c_k - d_k) \frac{3^k - (-2)^k}{5 \cdot 6^k} = 0. \tag{4.6}$$

After multiplying by 6^N both sides of (4.6), we obtain that $6 \mid (c_N - d_N)(3^N - (-2)^N)$. This, along with the fact that 6 is relatively prime with $3^N - (-2)^N$ guarantees that $6 \mid c_N - d_N$. Now the fact that $c_N, d_N \in [\![0,5]\!]$ guarantees that $c_N = d_N$, which is a contradiction. □

Now, we can readily deduce that M is an atomic monoid.

Corollary 4.4. *The monoid M in (4.1) is atomic with $\mathscr{A}(M) = \{a_n : n \in \mathbb{N}_{\ge 2}\}$.*

Proof. This immediately follows from the uniqueness of the decomposition (4.3). □

The unique decomposition described in (4.3) plays a crucial role in what follows.

Definition 4.5. For each $q \in M$, we call the unique decomposition $q = \sum_{k \ge 2} c_k a_k$ in (4.3) the *canonical sum decomposition* of q in M. Then we say that the *height* of q is h provided that

$$h = \max\{k \in \mathbb{N}_{\ge 2} : c_k \neq 0\};$$

that is, the height of q is the maximum index n such that the atom a_n appears in the canonical sum decomposition of q.

We conclude this section with the following lemma, which gives another interpretation of the height of an element of M.

Lemma 4.6. *Let M be the monoid in (4.1), and take a nonzero $q \in M$ with height at least 3. Then the height of q is $\min\{k \in \mathbb{N} : 6^k q \in \mathbb{Z}\}$.*

Proof. Assume that the height of q is h, and write the canonical sum decomposition of q as follows: $q = c_2 a_2 + \cdots + c_h a_h$, where $c_2 \in \mathbb{N}_0$ and $c_3, \dots, c_h \in [\![0,5]\!]$ with $c_h \neq 0$. Since $h \ge 3$, it follows that $c_h < 6$ and so $6 \nmid c_h$. Now take $k \in [\![1,h]\!]$ such that $m := 6^k q \in \mathbb{Z}$. Then

$$c_h \frac{3^h - (-2)^h}{5} = 6^{h-k} m - \sum_{i=2}^{h-1} c_i 6^{h-i} \frac{3^i - (-2)^i}{5}. \tag{4.7}$$

Observe that the inequality $k < h$ cannot hold as, otherwise, each summand in the right-hand side of (4.7) would be divisible by 6, and so 6 would also divide the coefficient c_h. Hence $k = h$, which means that h is the minimum $k \in \mathbb{N}$ such that $6^k q \in \mathbb{Z}$. □

4.2.2. *A Monoid Algebra that Does not Satisfy the Almost ACCP.* We are in a position to prove that the almost ACCP property does not ascend to monoid algebras. Indeed, we will prove that M satisfies the almost ACCP but its monoid algebra $\mathbb{F}_2[M]$ does not. Before establishing this, we need the following known result.

Let us establish the main result of this section.

Theorem 4.7. *For the monoid M introduced in (4.1), the following statements hold.*

(1) *M satisfies the almost ACCP but not the ACCP.*

(2) *$\mathbb{F}_2[M]$ does not satisfy the almost ACCP.*

Proof. (1) First, observe that M does not satisfy the ACCP: indeed, it follow from Lemma 4.2 that $\mathbb{Z}[\frac{1}{2}] \subseteq M$, and so $\big(\frac{1}{2^n} + M)_{n \ge 2}$ is an ascending chain of principal ideals of M that does not stabilize. Before proving that M satisfies the quasi-ACCP, we need to argue the following claim.

Claim 1. If an element $q \in M$ does not satisfy the ACCP, then there exists $N \in \mathbb{N}$ such that $a_n \mid_M q$ for every $n \ge N$.

Proof of Claim 1. Take $q \in M$ such that q does not satisfy the ACCP. Let $(b_n + M)_{n \ge 0}$ be an ascending chain of principal ideals of M starting at q that does not stabilize. We can further assume that $b_n + M \subsetneq b_{n+1} + M$ for every $n \in \mathbb{N}_0$. As M is atomic with $\mathscr{A}(M) = \{a_n : n \in \mathbb{N}_{\ge 2}\}$, for each $n \in \mathbb{N}_0$ we can take $i_n \in \mathbb{N}$ such that $a_{i_n} \mid_M b_n - b_{n+1}$. As $b_0 = q$, it follows that $a_{i_1} + \cdots + a_{i_j} \mid_M q$ for every $j \in \mathbb{N}$. As for each $a \in \mathscr{A}(M)$, the set $\{j \in \mathbb{N} : a_{i_j} = a\}$ is finite, we can pick a strictly increasing sequence $(i_n)_{n \ge 1}$ such that $a_{i_n} \mid_M q$ for every $n \in \mathbb{N}$. Now set $n := \min\{k \in \mathbb{N}_{\ge 2} : 6^k q \in \mathbb{Z}\}$, and then take $N \in \mathbb{N}$ such that $N > n + 5$. We claim that $a_n \mid_M q$ for every $n \ge N$.

Suppose, by way of contraction that $a_k \nmid_M q$ for some $k \ge N$, and assume that k has been picked as smallest as it could possibly be. As $(i_n)_{n \ge 1}$ is a strictly increasing sequence such that $a_{i_n} \mid_M q$, we can pick $\ell \in \mathbb{N}$ with $\ell > k$ such that $a_i \nmid_M q$ for every $i \in [\![k, \ell - 1]\!]$, but $a_\ell \mid_M q$. Take the maximum $c \in \mathbb{N}$ such that $c a_\ell \mid_M q$. Now observe that if $c < 6$, then $6^\ell(q - c a_\ell)$ is an integer that is not divisible by 6 and, therefore, $\ell := \min\{6^k(q - c a_\ell) \in \mathbb{Z}\}$ and so it follows form Lemma 4.6 that $a_\ell \mid_M q - c a_\ell$, which is not possible because of the maximality of c. Hence $6a_\ell \mid_M q$.Then it follows that $6a_\ell = a_{\ell-1} + a_{\ell-2}$ and, as a consequence, we obtain that $a_{\ell-1} \mid_M q$, which is a contradiction. Hence the claim is established.

In order to argue that M satisfies the almost ACCP, fix a nonempty finite subset S of M. For each $s \in S$, use the canonical sum decomposition of s to write

$$s = \sum_{i=2}^{n} c_{i,s} a_i,$$

where $c_{2,s} \in \mathbb{N}_0$ and $c_{i,s} \in [\![0, 5]\!]$ for every $i \in [\![2, n]\!]$. Now take the maximum common divisor of S of the form

$$d := d_2 a_2 + \cdots + d_n a_n. \tag{4.8}$$

Claim 2. d is an MCD of S in M.

Proof of Claim 2. Suppose, towards a contradiction that this is not the case. Then there exists $k \in \mathbb{N}$ such that $d + a_k$ is a common divisor of S. It follows from the maximality of d that $k \ge n + 1$. Now we take coefficients $d_{n+1}, \dots, d_k \in \mathbb{N}_0$ such that $e := d + d_{n+1} a_{n+1} + \cdots + d_k a_k$ is a common divisor of S in such a way that the value of c is as large as possible. In addition, among all the possible choices of the coefficients $d_{n+1}, \dots, d_k$ maximizing e, we can further assume that we have picked one maximazing the sum $d_{n+1} + \cdots + d_k$. Since $d + a_k$ is a common divisor of S, the maximality of e guarantees that $d_{n+1} + \cdots + d_k \ge 1$. In addition, after replacing k by the smallest index $j \in [\![n+1, k]\!]$ with $d_j \neq 0$, we can further assume that $d_k \neq 0$. On the other hand, observe that $d_k \le 5$ as otherwise after replacing $6a_k$ by $a_{k-1} + a_{k-2}$ in (4.8) we would contradict the maximality of $d_{n+1} + \cdots + d_k$.

For each $s \in S$, the fact that $d_k \le 5$ implies that $6^k(s-e)$ is an integer not divisible by 6, and so it follows from Lemma 4.6 that $a_k \mid_M s - e$. As a consequence, $e + a_k$ is a common divisor of S, which contradicts the maximality of e. Hence d is an MCD of S.

Because d is a common divisor of S, in order to conclude that M satisfies the weak-ACCP it suffices to argue that $s-d$ satisfies the ACCP for some $s \in S$. However, note that if $s-d$ does not satisfy the ACCP for any $s \in S$, then in light of Claim 1 we would be able to take an index $\ell \in \mathbb{N}$ large enough so that the atom a_ℓ divides $s-d$ for every $s \in S$, which is not possible because, according to Claim 2, the element d is a maximal common divisor of S.

(2) Finally, we prove that $\mathbb{F}_2[M]$ does not satisfy the almost ACCP. Indeed, we will argue that $\mathbb{F}_2[M]$ is not even atomic. It suffices to prove that the polynomial expression x^2+x+1 of $\mathbb{F}_2[M]$ is not divisible by any irreducible. Suppose, by way of contradiction, that this is not the case, and let $f \in \mathbb{F}[M]$ be an irreducible dividing x^2+x+1. Now take $g \in \mathbb{F}_2[M]$ such that $x^2+x+1 = f(x)g(x)$. Since $M \subseteq \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$, we can take $m \in \mathbb{N}$ sufficiently large so that $f\big(x^{6^m}\big)$ and $g\big(x^{6^m}\big)$ both belong to the polynomial ring $\mathbb{F}_2[x]$. Now observe that

$$\big(x^{2\cdot 3^m} + x^{3^m} + 1\big)^{2^m} = x^{2\cdot 6^m} + x^{6^m} + 1 = f\big(x^{6^m}\big)g\big(x^{6^m}\big). \tag{4.9}$$

It follows from [11, Lemma 5.3] that the polynomial $x^{2\cdot 3^m} + x^{3^m} + 1$ is irreducible in $\mathbb{F}_2[x]$. Thus, by virtue of (4.9) and the fact that $\mathbb{F}_2[x]$ is a UFD, there exists $k \in \llbracket 1, 2^m \rrbracket$ such that

$$f\big(x^{6^m}\big) = \big(x^{2\cdot 3^m} + x^{3^m} + 1\big)^k.$$

As a result, we can write

$$f(x) = \big(x^{2\cdot\frac{1}{2^m}} + x^{\frac{1}{2^m}} + 1\big)^k = \big(x^{2\cdot\frac{1}{2^{m+1}}} + x^{\frac{1}{2^{m+1}}} + 1\big)^{2k}$$

in $\mathbb{F}_2[\mathbb{Q}_{\ge 0}]$. Now the inclusion $\mathbb{Z}[\frac{1}{2}] \subseteq M$ ensures that $x^{2\cdot\frac{1}{2^{m+1}}} + x^{\frac{1}{2^{m+1}}} + 1 \in \mathbb{F}_2[M]$, and so the inequality $2k > 1$ contradicts the irreducibility of f in $\mathbb{F}_2[M]$. Hence $\mathbb{F}_2[M]$ is not an atomic domain. □

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