ACCIO SQUARES!
SUMS OF SQUARES IN RATIONAL MONOGENIC SEMIDOMAINS

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Abstract. Lagrange's four-square theorem states that every nonnegative integer is the sum of four squares of integers. In this paper, we investigate the same question in rational monogenic semidomains, which are the semirings consisting of the values that polynomials with nonnegative integer coefficients take at a fixed positive rational number. We prove that every element of such a semiring is a sum of squares of elements of the same semiring if and only if the fixed rational number or its reciprocal is an integer; in this case, four squares always suffice, and four squares are sometimes needed. In all the remaining cases, we show that there are elements that can be written as sums of squares, but only using as many squares as we wish, which means that the Pythagoras number of the semiring is infinite. Finally, for the rational number two thirds, we describe the elements that are sums of squares, provide an algorithm to decide whether a given element is a sum of squares, and bound the number of squares needed in terms of the denominator of the element.

1. Introduction: Lagrange's Ancient Spell

Which positive integers can be written as sums of squares of integers, and how many squares are needed? Questions of this kind are among the oldest in number theory. In his 1621 edition of Diophantus' *Arithmetica*, Bachet conjectured that every positive integer is a sum of four squares of integers, and Lagrange proved this conjecture in 1770. Before that, Fermat had claimed and Euler had proved that an odd prime is a sum of two squares exactly when it leaves remainder 1 upon division by 4, and around 1800 Legendre and Gauss characterized the positive integers that are sums of three squares. Since 7 is not a sum of three squares, the number four in Lagrange's theorem cannot be improved. We refer the reader to [12, 14] for proofs of these classical results and for some of their history, and to [3] for their history in the wizarding world.

One can ask the same questions for every commutative ring R: which elements of R are sums of squares, and is there a positive integer n such that every sum of squares in R is a sum of at most n squares? The smallest such n, when it exists, is called the *Pythagoras number* of R. For instance, the Pythagoras number of the field of real numbers is 1, while the Pythagoras number of the field of rational numbers is 4.

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Pythagoras numbers of fields and rings have been systematically investigated in the literature (see [6] and the references therein, and see [20] for an arithmantic point of view).

In this paper, we study sums of squares in a class of semirings. A *semidomain* is a subset of an integral domain that contains 0 and 1 and is closed under both addition and multiplication. For instance, the set of nonnegative integers is a semidomain, and so is the set of nonnegative rational numbers. In a semidomain consisting of nonnegative real numbers, every element is a candidate to be a sum of squares, and so it is natural to ask whether every element is a sum of squares, exactly as Lagrange's theorem does for the nonnegative integers. Our main objects of study are the *rational monogenic semidomains*: for a positive rational number q, the rational monogenic semidomain $\mathbb{N}_0[q]$ is the set of all the values $f(q)$, where f ranges over all polynomials with nonnegative integer coefficients. Equivalently, $\mathbb{N}_0[q]$ is the smallest semidomain containing q. For instance, $\mathbb{N}_0[1/2]$ consists of the nonnegative rational numbers whose denominators are powers of 2, while $\mathbb{N}_0[2/3]$ is a more mysterious semidomain: it contains $2/3$ and $4/9$, but it does not contain $1/3$. The additive and multiplicative structures of rational monogenic semidomains (and of some of their generalizations) have been investigated in recent years from the point of view of factorization theory [1, 5, 7, 11].

Here is the question that motivated this paper: for which positive rational numbers q is every element of $\mathbb{N}_0[q]$ a sum of squares of elements of $\mathbb{N}_0[q]$, and how many squares are needed in this case? An important subtlety is that the square roots must belong to $\mathbb{N}_0[q]$: just as underage wizards are not allowed to use magic outside Hogwarts, our square roots are not allowed to leave $\mathbb{N}_0[q]$. For instance, the equality

$$\frac{2}{3} = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$$

does not express $2/3$ as a sum of squares in $\mathbb{N}_0[2/3]$ because $1/3 \notin \mathbb{N}_0[2/3]$. In fact, we will see that $2/3$ is not a sum of squares in $\mathbb{N}_0[2/3]$ at all. The main result of this paper is the following theorem, which completely answers our motivating question.

Main Theorem. *Let $q = a/b$, where a and b are relatively prime positive integers.*

(1) *If either $a = 1$ or $b = 1$ then every element of $\mathbb{N}_0[q]$ is a sum of at most four squares of elements of $\mathbb{N}_0[q]$, and the element 7 is not a sum of fewer than four such squares.*

(2) *If $a > 1$ and $b > 1$ then q is not a sum of squares of elements of $\mathbb{N}_0[q]$. Moreover, for each $N \in \mathbb{N}$, there exists an element of $\mathbb{N}_0[q]$ that is a sum of N squares of elements of $\mathbb{N}_0[q]$ but is not a sum of fewer than N such squares.*

In terms of Pythagoras numbers, the Main Theorem says that the Pythagoras number of $\mathbb{N}_0[q]$ is 4 when either $a = 1$ or $b = 1$, and it is infinite otherwise. Observe that part (2) is much stronger than saying that not every element of $\mathbb{N}_0[q]$ is a sum of squares: it

says that even the elements that are sums of squares may need as many squares as we wish.

When $a > 1$ and $b > 1$, it is natural to ask which elements of $\mathbb{N}_0[q]$ are sums of squares, and how many squares each of them needs. For $q = 2/3$, we completely answer the first question: an element of $\mathbb{N}_0[2/3]$ is a sum of squares if and only if it belongs to the smaller semidomain $\mathbb{N}_0[4/9]$, and this can be decided by a simple algorithm. We also bound the number of squares needed in terms of the denominator of the element, and we show that whether an element is a sum of one or two squares can be decided by a finite computation. The second question, namely, computing the least number of squares needed by an arbitrary element of $\mathbb{N}_0[2/3]$, remains open.

This paper is organized as follows. In Section 2, we introduce the notation and terminology we will use throughout this paper, and we recall the classical theorems on sums of squares of integers that we will need. In Section 3, we prove part (1) of the Main Theorem together with the first statement of part (2), and we illustrate with some examples the role played by the ambient semidomain. In Section 4, we establish a rigidity property of polynomials whose coefficients belong to $\{0, 1\}$, and we use it to complete the proof of the Main Theorem. Finally, in Section 5, we focus on the semidomain $\mathbb{N}_0[2/3]$, and we conclude with some open questions.

2. Background: The Standard Book of Spells

In this section, we introduce the notation and terminology we will use throughout this paper, and we recall the classical results on sums of squares of integers that we need later. For background on semirings, see [9]. For the basics of witchcraft and wizardry, see [16], and for the standard spells and their underlying theory, see [10, 21].

Following standard notation, we let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of integers, rational numbers, and real numbers, respectively. Additionally, we let $\mathbb{N}$ denote the set of positive integers, and we set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Given $r \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, we set $S_{\geq r} := \{s \in S : s \geq r\}$, and we define $S_{>r}$ in a similar way. For $m, n \in \mathbb{Z}$ with $m \leq n$, we denote by $[\![m, n]\!]$ the discrete interval from m to n; that is, $[\![m, n]\!] := \{k \in \mathbb{Z} : m \leq k \leq n\}$. For $q \in \mathbb{Q}_{>0}$, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ denote the unique positive integers such that $q = \mathsf{n}(q)/\mathsf{d}(q)$ and $\gcd(\mathsf{n}(q), \mathsf{d}(q)) = 1$. For $b \in \mathbb{N}$, we let $\mathbb{Z}[1/b]$ denote the ring of rational numbers that can be written as m/b^k for some $m \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. As usual, for $m, n \in \mathbb{N}$, we write $m \mid n$ when m divides n, while for elements x and y of a semidomain S, we write $x \mid_S y$ when $y = xz$ for some $z \in S$. Finally, for $v_0, \ldots, v_h \in \mathbb{N}_0$, we set

$$\langle v_0, \ldots, v_h \rangle := \{c_0 v_0 + \cdots + c_h v_h : c_0, \ldots, c_h \in \mathbb{N}_0\},$$

which is the additive submonoid of $\mathbb{N}_0$ *generated* by $v_0, \ldots, v_h$.

2.1. **The Wizarding World: Semidomains and Rational Monogenic Semidomains.** Recall that a commutative ring with identity is an *integral domain* provided that it contains no nonzero zero-divisors.

Definition 2.1. Let R be an integral domain. A subset S of R is called a *semidomain* if $0, 1 \in S$ and S is closed under both addition and multiplication. A semidomain S is called *positive* if $S \subseteq \mathbb{R}_{\geq 0}$.

Here are some basic examples of semidomains.

Example 2.2. Every integral domain is a semidomain. The sets $\mathbb{N}_0$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ are positive semidomains. On the other hand, the set $\mathbb{N}_0[X]$ of polynomials with nonnegative integer coefficients is a semidomain of the integral domain $\mathbb{Z}[X]$, but it is not a positive semidomain. ■

Semidomains have been studied from the point of view of factorization theory in [11]. In this paper we focus on the following class of positive semidomains.

Definition 2.3. For $q \in \mathbb{Q}_{>0}$, the *rational monogenic semidomain* generated by q is

$$S_q := \mathbb{N}_0[q] := \{f(q) : f \in \mathbb{N}_0[X]\}.$$

Since $f(q) + g(q) = (f + g)(q)$ and $f(q)g(q) = (fg)(q)$ for all $f, g \in \mathbb{N}_0[X]$, the set S_q is a positive semidomain containing q. Moreover, every semidomain of $\mathbb{R}$ containing q contains $f(q)$ for each $f \in \mathbb{N}_0[X]$, and so S_q is the smallest semidomain of $\mathbb{R}$ containing q. The additive monoid of S_q is the Puiseux monoid generated by the geometric sequence $(q^n)_{n\geq 0}$, whose factorization invariants were studied in [5], while the semidomains S_q were investigated from the point of view of atomicity in [1], where they are called exponential Puiseux semirings.

Let us look at three examples of rational monogenic semidomains.

Example 2.4.

(1) For $n \in \mathbb{N}$, we see that $S_n = \mathbb{N}_0$.
(2) For $b \in \mathbb{N}$, we see that $S_{1/b} = \{m/b^k : m, k \in \mathbb{N}_0\} = \mathbb{Z}[1/b]_{\geq 0}$.
(3) The semidomain $S_{2/3}$ contains 1, $2/3$, $4/9$, $5/3 = 1+2/3$, and $13/9 = 1+(2/3)^2$. As we will see in Example 2.6, the semidomain $S_{2/3}$ does not contain $1/3$. ■

The following lemma collects three basic properties of rational monogenic semidomains that we will use repeatedly throughout this paper.

Lemma 2.5. *Let $q = a/b$, where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Then the following statements hold.*

(1) *The inclusion $S_q \subseteq \mathbb{Z}[1/b]$ holds.*
(2) *If $q > 1$ then every element of S_q that is smaller than q belongs to $\mathbb{N}_0$.*

(3) *If $q < 1$ then every element of S_q that is smaller than 1 belongs to $qS_q := \{qx : x \in S_q\}$.*

Proof. (1) Take $f = \sum_{k=0}^{d} c_k X^k \in \mathbb{N}_0[X]$. After clearing denominators, we see that $f(q) = \big(\sum_{k=0}^{d} c_k a^k b^{d-k}\big)/b^d \in \mathbb{Z}[1/b]$.

(2) Assume that $q > 1$, and take $f = \sum_{k=0}^{d} c_k X^k \in \mathbb{N}_0[X]$ such that $f(q) < q$. If $c_k \neq 0$ for some $k \in [\![1, d]\!]$ then $f(q) \geq c_k q^k \geq q$ because $q > 1$, which is not possible. Hence $f = c_0$, and so $f(q) = c_0 \in \mathbb{N}_0$.

(3) Assume that $q < 1$, and take $f \in \mathbb{N}_0[X]$ such that $f(q) < 1$. If the constant coefficient of f were positive then $f(q) \geq 1$. Hence $f = Xg$ for some $g \in \mathbb{N}_0[X]$, which implies that $f(q) = q\, g(q) \in qS_q$. □

The inclusion in part (1) of Lemma 2.5 is not always an equality, as the following example illustrates.

Example 2.6. Set $q = 2/3$. By part (1) of Lemma 2.5, the inclusion $S_q \subseteq \mathbb{Z}[1/3]$ holds. However, $1/3 \notin S_q$. To argue this, suppose, by way of contradiction, that $1/3 \in S_q$. Since $1/3 < 1$, it follows from part (3) of Lemma 2.5 that $1/3 = \sum_{k=1}^{d} c_k (2/3)^k$ for some $d \in \mathbb{N}$ and $c_1, \ldots, c_d \in \mathbb{N}_0$. After multiplying both sides of this equality by 3^d, we obtain that

$$3^{d-1} = \sum_{k=1}^{d} c_k 2^k 3^{d-k},$$

which is not possible because the left-hand side is odd while the right-hand side is even. On the other hand, observe that $3/2 \notin S_q$ because $3/2 \notin \mathbb{Z}[1/3]$. ■

2.2. **Ancient Runes: Classical Results over the Nonnegative Integers.** We proceed to recall the classical results on sums of squares of integers that motivated this paper (see [2] for their translation into ancient runes). Only parts (2) and (3) of the following theorem will be used in the proofs of our results.

Theorem 2.7. *For each $n \in \mathbb{N}$, the following statements hold.*

(1) Fermat–Euler [12, Chapter XX]*: n is a sum of two squares of integers if and only if every prime congruent to 3 modulo 4 appears with an even exponent in the prime factorization of n.*

(2) Legendre–Gauss [14, Chapter 1]*: n is a sum of three squares of integers if and only if $n \neq 4^h(8j + 7)$ for all $h, j \in \mathbb{N}_0$.*

(3) Lagrange [12, Chapter XX]*: n is a sum of four squares of integers.*

Let us take a look at an example.

Example 2.8. Observe that $13 = 2^2 + 3^2$ is a sum of two squares, which is consistent with the fact that $13 \equiv 1 \pmod 4$ is a prime. Since 3 appears with exponent 1 in the prime factorization of $21 = 3 \cdot 7$, the integer 21 is not a sum of two squares, but it is a sum of three squares: $21 = 4^2+2^2+1^2$. On the other hand, $7 = 2^2+1^2+1^2+1^2$ is not a sum of three squares because it has the form $4^0(8\cdot 0+7)$. Similarly, $28 = 5^2+1^2+1^2+1^2$ is not a sum of three squares because it has the form $4^1(8 \cdot 0 + 7)$. ■

The number 7, which Tom Riddle considered the most powerfully magical number [17], will play a crucial role in Section 3. We conclude this subsection with two further celebrated results, which we will not use but which illustrate how much is known about sums of squares of integers.

Remark 2.9. For each $n \in \mathbb{N}$, let $r_4(n)$ be the number of quadruples $(x_1, \dots, x_4) \in \mathbb{Z}^4$ such that $x_1^2 + \cdots + x_4^2 = n$, where the order of the entries matters. Then Jacobi's four-square theorem (see [12, Chapter XX]) states that

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d.$$

On the other hand, if $B(x)$ denotes the number of positive integers at most x that are sums of two squares then Landau's theorem states that $B(x) \sim Kx/\sqrt{\log x}$ as $x \to \infty$, where $K \approx 0.7642$ is the Landau–Ramanujan constant (see the survey [4]).

2.3. **Ollivander's Measurements: Sums of Squares in Positive Semidomains.** We are now in a position to introduce the main definitions of this paper. Just as Mr. Ollivander carefully measures every wand before it chooses its wizard [15], we measure each element of a positive semidomain by the number of squares it needs.

Definition 2.10. Let S be a positive semidomain. We set

$$\Sigma S^2 := \left\{ \sum_{i=1}^{r} x_i^2 : r \in \mathbb{N}_0 \text{ and } x_1, \dots, x_r \in S \right\},$$

where the empty sum is 0. For each $s \in S$, the *square length* of s is

$$\ell_S(s) := \min\{r \in \mathbb{N}_0 : s = x_1^2 + \cdots + x_r^2 \text{ for some } x_1, \dots, x_r \in S\},$$

where $\ell_S(s) := \infty$ if $s \notin \Sigma S^2$. The *universal square bound* and the *Pythagoras number* of S are, respectively,

$$u(S) := \sup\{\ell_S(s) : s \in S\} \quad \text{and} \quad p(S) := \sup\{\ell_S(s) : s \in \Sigma S^2\}.$$

For $q \in \mathbb{Q}_{>0}$, we write $u(q)$ and $p(q)$ instead of $u(S_q)$ and $p(S_q)$, respectively.

Since $0 = 0^2$, repeated and zero summands are allowed in our representations, and so an element is a sum of at most r squares if and only if it is a sum of exactly r squares. Since $\Sigma S^2 \subseteq S$, the inequality $p(S) \leq u(S)$ holds. Also, if $u(S)$ is finite then $\ell_S(s) < \infty$ for every $s \in S$, which means that $S = \Sigma S^2$.

Let us compute these invariants for three familiar positive semidomains.

Example 2.11.

(1) For $S = \mathbb{N}_0$, it follows from Lagrange's theorem that $\Sigma S^2 = S$ and $u(S) \leq 4$. By Example 2.8, we see that $\ell_S(13) = 2$, $\ell_S(21) = 3$, and $\ell_S(7) = \ell_S(28) = 4$. Hence $u(\mathbb{N}_0) = p(\mathbb{N}_0) = 4$.

(2) For $S = \mathbb{Q}_{\geq 0}$, take $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. By Lagrange's theorem, there exist $v_1, \ldots, v_4 \in \mathbb{N}_0$ such that $mn = v_1^2 + \cdots + v_4^2$, and so

$$\frac{m}{n} = \frac{mn}{n^2} = \left(\frac{v_1}{n}\right)^2 + \cdots + \left(\frac{v_4}{n}\right)^2.$$

Hence $u(\mathbb{Q}_{\geq 0}) \leq 4$. We will see in Proposition 3.1 that $u(\mathbb{Q}_{\geq 0}) = p(\mathbb{Q}_{\geq 0}) = 4$.

(3) For $S = \mathbb{R}_{\geq 0}$, every element is the square of its square root, and so $u(\mathbb{R}_{\geq 0}) = p(\mathbb{R}_{\geq 0}) = 1$. ■

The set ΣS^2 is not only closed under addition, but also under multiplication.

Lemma 2.12. *If S is a positive semidomain then ΣS^2 is a semidomain contained in S.*

Proof. Since $0 = 0^2$ and $1 = 1^2$, both 0 and 1 belong to ΣS^2. In addition, the sum of two sums of squares is again a sum of squares, and so ΣS^2 is closed under addition. To see that ΣS^2 is also closed under multiplication, take $x_1, \ldots, x_r, y_1, \ldots, y_n \in S$, and observe that

$$\left(\sum_{i=1}^{r} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) = \sum_{i=1}^{r}\sum_{j=1}^{n}(x_i y_j)^2 \in \Sigma S^2. \qquad \square$$

3. The Order of the Four Squares: Universal Square Bounds

In this section, we determine the rational monogenic semidomains in which every element is a sum of squares, which yields part (1) of the Main Theorem and the first statement of part (2). We start by showing that the number 7 needs four squares in every positive semidomain consisting of rational numbers.

Proposition 3.1. *Let S be a semidomain contained in $\mathbb{Q}_{\geq 0}$. Then $\ell_S(7) = 4$. In particular, $p(S) \geq 4$.*

Proof. Since $\mathbb{N}_0 \subseteq S$, the equality $7 = 2^2 + 1^2 + 1^2 + 1^2$ ensures that $\ell_S(7) \leq 4$. To prove the reverse inequality, it suffices to show that 7 is not a sum of three squares of rational numbers. Suppose, by way of contradiction, that $7 = x_1^2 + x_2^2 + x_3^2$ for some $x_1, x_2, x_3 \in \mathbb{Q}$. After clearing a common positive denominator w, we can write $7w^2$ as a sum of three squares of integers. Now write $w = 2^h v$, where $h \in \mathbb{N}_0$ and v is an odd positive integer. Since $v^2 \equiv 1 \pmod 8$, we see that $7v^2 \equiv 7 \pmod 8$, and so $7w^2 = 4^h(7v^2)$ has the form excluded by part (2) of Theorem 2.7. This contradiction shows that $\ell_S(7) = 4$. Since $7 \in \Sigma S^2$, we conclude that $p(S) \geq \ell_S(7) = 4$. □

We are now ready to prove the main result of this section.

Theorem 3.2. *Let $q = a/b$, where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Then the following conditions are equivalent.*

(a) *The universal square bound $u(q)$ is finite.*
(b) *Every element of S_q is a sum of squares of elements of S_q.*
(c) *The generator q is a sum of squares of elements of S_q.*
(d) *Either $a = 1$ or $b = 1$.*

Moreover, if the conditions above hold then $u(q) = p(q) = 4$.

Proof. (a) ⇒ (b) ⇒ (c): If $u(q)$ is finite then $\ell_{S_q}(s) < \infty$ for every $s \in S_q$, which is condition (b), and (b) implies (c) because $q \in S_q$.

(c) ⇒ (d): Suppose, by way of contradiction, that $a > 1$ and $b > 1$, and that

$$q = x_1^2 + \cdots + x_r^2$$

for some $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in S_q$. Since all the summands are nonnegative, $x_i \leq \sqrt{q}$ for every $i \in [\![1, r]\!]$. Observe that $q \neq 1$ because $b > 1$. We split the rest of the proof into two cases.

Case 1: $q > 1$. In this case, $x_i \leq \sqrt{q} < q$ for every $i \in [\![1, r]\!]$, and so it follows from part (2) of Lemma 2.5 that $x_1, \ldots, x_r \in \mathbb{N}_0$. As a consequence, $q = x_1^2 + \cdots + x_r^2 \in \mathbb{N}_0$, which contradicts that $b > 1$.

Case 2: $q < 1$. In this case, $x_i \leq \sqrt{q} < 1$ for every $i \in [\![1, r]\!]$, and so it follows from part (3) of Lemma 2.5 that, for each $i \in [\![1, r]\!]$, we can write $x_i = qy_i$ for some $y_i \in S_q$. Dividing both sides of $q = q^2 \sum_{i=1}^r y_i^2$ by q^2, we obtain that

$$\frac{b}{a} = \frac{1}{q} = y_1^2 + \cdots + y_r^2 \in S_q.$$

By part (1) of Lemma 2.5, we see that $b/a \in \mathbb{Z}[1/b]$, and so $b/a = c/b^k$ for some $c \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. This implies that $a \mid b^{k+1}$, which contradicts that $a > 1$ and $\gcd(a, b) = 1$.

From Case 1 and Case 2, we conclude that either $a = 1$ or $b = 1$.

(d) $\Rightarrow$ (a): If $b = 1$ then $S_q = \mathbb{N}_0$, and so Lagrange's theorem guarantees that $u(q) \leq 4$. Suppose now that $a = 1$, and so $S_q = \{m/b^k : m, k \in \mathbb{N}_0\}$. Fix $s = m/b^k \in S_q$, and take $h \in \mathbb{N}_0$ such that $2h \geq k$. By Lagrange's theorem, there exist $v_1, \ldots, v_4 \in \mathbb{N}_0$ such that $mb^{2h-k} = v_1^2 + \cdots + v_4^2$, which implies that

$$s = \frac{mb^{2h-k}}{b^{2h}} = \left(\frac{v_1}{b^h}\right)^2 + \cdots + \left(\frac{v_4}{b^h}\right)^2.$$

Since $v_i/b^h \in S_q$ for every $i \in [\![1, 4]\!]$, we obtain that $u(q) \leq 4$.

Finally, assume that the equivalent conditions (a)–(d) hold. Then $S_q = \Sigma S_q^2$, and so $u(q) = p(q)$. In addition, we have seen that $u(q) \leq 4$, and it follows from Proposition 3.1 that $p(q) \geq 4$. Hence $u(q) = p(q) = 4$. □

As an immediate consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *For each $q \in \mathbb{Q}_{>0}$,*

$$u(q) = \begin{cases} 4 & \text{if } \mathsf{n}(q) = 1 \text{ or } \mathsf{d}(q) = 1, \\ \infty & \text{if } \mathsf{n}(q) > 1 \text{ and } \mathsf{d}(q) > 1. \end{cases}$$

The following remark gives a second argument for one of the equivalences in Theorem 3.2.

Remark 3.4. The equivalence (b) $\Leftrightarrow$ (c) in Theorem 3.2 also follows directly from Lemma 2.12: if the semidomain ΣS_q^2 contains q then it must contain every polynomial expression in q with nonnegative integer coefficients, and so $\Sigma S_q^2 = S_q$.

3.1. **No Magic Outside Hogwarts: The Role of the Ambient Semidomain.** We conclude this section with some examples illustrating Theorem 3.2 and the role played by the semidomain S_q in which the square roots are required to live.

Example 3.5. Set $q = 1/3$. By Theorem 3.2, every element of S_q is a sum of at most four squares of elements of S_q. For instance,

$$\frac{7}{9} = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2.$$

In addition, $\ell_{S_q}(7/9) = 4$: indeed, if $7/9$ were a sum of three squares of elements of S_q then after multiplying by 9, we would obtain that 7 is a sum of three squares of elements of $\mathbb{Q}_{\geq 0}$, which is not possible by Proposition 3.1 applied to $S = \mathbb{Q}_{\geq 0}$. ■

Example 3.6. Set $q = 3/2$. It follows from part (2) of Lemma 2.5 that every element of S_q that is smaller than q is an integer. Therefore every square root appearing in a sum-of-squares representation of q in S_q must be an integer because it is at most $\sqrt{q} < q$. As q is not an integer, q is not a sum of squares of elements of S_q. This argument is a special case of Case 1 in the proof of Theorem 3.2. ■

Example 3.7. Set $q = 2/3$, and suppose that $q = x_1^2 + \cdots + x_r^2$ for some $x_1, \ldots, x_r \in S_q$. Since $x_i \leq \sqrt{q} < 1$ for every $i \in [\![1, r]\!]$, it follows from part (3) of Lemma 2.5 that $x_i = qy_i$ for some $y_i \in S_q$, and dividing by q^2 yields $3/2 = y_1^2 + \cdots + y_r^2 \in S_q$, which is not the case by Example 2.6. Hence q is not a sum of squares of elements of S_q. Thus, the equality

$$\frac{2}{3} = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$$

does not contradict our conclusion, as $1/3 \notin S_q$. This equality is the source of the claim in [13] that $2/3$ is a sum of three squares in S_q, a claim that overlooks the fact that $1/3 \notin S_q$. ■

These examples show that a representation of an element of S_q as a sum of squares in $\mathbb{Q}_{\geq 0}$ does not automatically yield a representation as a sum of squares in S_q. Indeed, it follows from part (2) of Example 2.11 that every element of S_q is a sum of four squares of nonnegative rational numbers, but the corresponding square roots do not necessarily belong to the semidomain S_q.

4. The Room of Requirement: Pythagoras Numbers

In this section, we determine the Pythagoras number of every rational monogenic semidomain, which completes the proof of the Main Theorem. Unlike the universal square bound, the Pythagoras number only concerns the elements that are sums of squares. Like the Room of Requirement, which provides whatever its visitors need, the semidomain S_q will provide elements needing as many squares as we wish whenever $\mathsf{n}(q) > 1$ and $\mathsf{d}(q) > 1$, and, unlike many of the creatures described in [18], such elements are easy to find. The key observation is that, when $\mathsf{n}(q) > 1$ and $\mathsf{d}(q) > 1$, a polynomial with coefficients in $\{0, 1\}$ is uniquely determined, among all the polynomials in $\mathbb{N}_0[X]$, by its value at q. To prove this, we need the following definition.

Definition 4.1. Let $q = a/b$, where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. We say that a polynomial $f \in \mathbb{N}_0[X]$ is *q-reduced* if, for every $k \in \mathbb{N}$, the coefficient of X^k in f belongs to $[\![0, b-1]\!]$ (the constant coefficient of f is unrestricted).

Let us illustrate this definition with an example.

Example 4.2. Set $q = 2/3$. The polynomial $1 + 2X + X^2$ is q-reduced, while the polynomial $g = 3X^3 + 1$ is not. Since $3q^3 = 2q^2$, the polynomial g and the q-reduced polynomial $2X^2 + 1$ have the same value at q, namely, $17/9$. ■

As Example 4.2 suggests, the identity $bq^k = aq^{k-1}$ allows us to replace every polynomial in $\mathbb{N}_0[X]$ by a q-reduced polynomial with the same value at q; in the terminology of [8, Chapter 4], each such replacement is an elementary transfiguration.

Lemma 4.3. *Let $q = a/b$, where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. For each $g \in \mathbb{N}_0[X]$, there exists a unique q-reduced polynomial $f \in \mathbb{N}_0[X]$ such that $f(q) = g(q)$. Moreover, f can be obtained from g by applying finitely many replacements of the form $bX^k \mapsto aX^{k-1}$ for $k \in \mathbb{N}$, starting at the highest degree and proceeding downward.*

Proof. Since $bq^k = aq^{k-1}$ for every $k \in \mathbb{N}$, the replacement $bX^k \mapsto aX^{k-1}$ preserves the value at q and keeps all coefficients nonnegative. Let d be the degree of g. For each k from d down to 1, we apply the replacement $bX^k \mapsto aX^{k-1}$ as many times as needed to make the coefficient of X^k smaller than b; since these replacements only increase the coefficient of X^{k-1}, they do not modify the coefficients of degree larger than k, which have already been reduced. As a result, this process terminates in a q-reduced polynomial f with $f(q) = g(q)$.

To argue uniqueness, suppose that f_1 and f_2 are two distinct q-reduced polynomials such that $f_1(q) = f_2(q)$, and write $f_1 - f_2 = \sum_{k=0}^{d} e_k X^k$ with $e_d \neq 0$. If $d = 0$ then $f_1(q) - f_2(q) = e_0 \neq 0$, which is not possible. Therefore $d \geq 1$, and so $0 < |e_d| < b$ because both f_1 and f_2 are q-reduced. Multiplying the equality $\sum_{k=0}^{d} e_k q^k = 0$ by b^d, we obtain that

$$\sum_{k=0}^{d} e_k a^k b^{d-k} = 0.$$

After reducing this equality modulo b, we see that $b \mid_{\mathbb{Z}} e_d a^d$. Since $\gcd(a, b) = 1$, this implies that $b \mid_{\mathbb{Z}} e_d$, which contradicts that $0 < |e_d| < b$. □

We are now in a position to establish the rigidity property mentioned at the beginning of this section.

Lemma 4.4. *Let $q = a/b$, where $a, b \in \mathbb{N}_{\geq 2}$ and $\gcd(a, b) = 1$. If $f \in \mathbb{N}_0[X]$ has all its coefficients in $\{0, 1\}$ and $g \in \mathbb{N}_0[X]$ satisfies $g(q) = f(q)$ then $g = f$.*

Proof. Since $b \geq 2$, the polynomial f is q-reduced. Therefore it follows from Lemma 4.3 that f is obtained from g after applying finitely many replacements of the form $bX^k \mapsto aX^{k-1}$. Suppose, by way of contradiction, that at least one replacement is applied, and let $H \in \mathbb{N}_0[X]$ be the polynomial right before the last replacement. Then

$$f = H - bX^k + aX^{k-1}$$

for some $k \in \mathbb{N}$. If $c \in \mathbb{N}_0$ is the coefficient of X^{k-1} in H then the coefficient of X^{k-1} in f is $c + a \geq a \geq 2$, which contradicts that all the coefficients of f belong to $\{0, 1\}$. Hence no replacement is applied, and so $g = f$. □

The hypothesis that both a and b are at least 2 cannot be dropped in Lemma 4.4.

Example 4.5. For $q = 1/2$, the polynomials X and $2X^2$ are distinct and have the same value at q, namely, $1/2$. Similarly, for $q = 2$, the polynomials X and 2 are distinct and have the same value at q. ■

We are now ready to prove the main result of this section, which says that, when $\mathsf{n}(q) > 1$ and $\mathsf{d}(q) > 1$, for each $N \in \mathbb{N}$, there are elements of S_q whose square length is N.

Theorem 4.6. *Let $q = a/b$, where $a, b \in \mathbb{N}_{\geq 2}$ and $\gcd(a, b) = 1$. For each $N \in \mathbb{N}$, the element*

$$s_N := \sum_{j=0}^{N-1} q^{2j}$$

of S_q satisfies $\ell_{S_q}(s_N) = N$. In fact, every representation of s_N as a sum of nonzero squares of elements of S_q has exactly N summands. As a consequence, $p(q) = \infty$.

Proof. Since s_N is, by definition, a sum of N squares of elements of S_q, the inequality $\ell_{S_q}(s_N) \leq N$ holds. Now suppose that

$$s_N = \sum_{i=1}^{r} x_i^2$$

for some $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in S_q \setminus \{0\}$. For each $i \in [\![1, r]\!]$, take a nonzero polynomial $g_i \in \mathbb{N}_0[X]$ such that $g_i(q) = x_i$. Since all the coefficients of $\sum_{j=0}^{N-1} X^{2j}$ belong to $\{0, 1\}$, it follows from Lemma 4.4 that

$$\sum_{i=1}^{r} g_i(X)^2 = \sum_{j=0}^{N-1} X^{2j}. \tag{4.1}$$

All the coefficients of the left-hand side of (4.1) are nonnegative, while all the coefficients of its right-hand side are at most 1. If, for some $i \in [\![1, r]\!]$, the polynomial g_i had a coefficient at least 2 then g_i^2 would have a coefficient at least 4. Similarly, if g_i had two distinct nonzero terms then the corresponding cross term would yield a coefficient at least 2 in g_i^2. Since both conclusions contradict (4.1), each g_i must be a monomial with coefficient 1. Thus, (4.1) expresses the N distinct monomials on its right-hand side as a sum of r monomials with coefficient 1, which implies that $r = N$. Finally, since N was arbitrarily taken, $p(q) = \infty$. □

We now specialize Theorem 4.6 to the generator 2/3.

Example 4.7. Set $q = 2/3$. In this case, Theorem 4.6 guarantees that

$$\ell_{S_q}\left(\sum_{j=0}^{N-1} \left(\frac{2}{3}\right)^{2j}\right) = N$$

for every $N \in \mathbb{N}$. The first four elements of this sequence are 1, 13/9, 133/81, and 1261/729, whose square lengths are 1, 2, 3, and 4, respectively. Moreover,

$$s_N = \frac{9}{5}\left(1 - \left(\frac{4}{9}\right)^N\right) < \frac{9}{5}$$

for every $N \in \mathbb{N}$. Thus, square lengths are unbounded even among the sums of squares of elements of S_q in a bounded interval. Observe that this conclusion is stronger than the fact that $2/3$ is not a sum of squares of elements of S_q (see Example 3.7) because it concerns elements that are, indeed, sums of squares. ■

Putting together Theorem 3.2 and Theorem 4.6, we obtain the following classification.

Corollary 4.8. *For each $q \in \mathbb{Q}_{>0}$,*

$$p(q) = \begin{cases} 4 & \text{if } \mathsf{n}(q) = 1 \text{ or } \mathsf{d}(q) = 1, \\ \infty & \text{if } \mathsf{n}(q) > 1 \text{ and } \mathsf{d}(q) > 1. \end{cases}$$

Proof of the Main Theorem. Part (1) follows from Theorem 3.2 and Proposition 3.1. The first statement of part (2) follows from Theorem 3.2, while the second statement of part (2) follows from Theorem 4.6. □

Comparing Corollary 3.3 and Corollary 4.8, we see that, although the universal square bound and the Pythagoras number have different definitions, the equality $u(q) = p(q)$ holds for every $q \in \mathbb{Q}_{>0}$.

5. The Chamber of Two Thirds: The Semidomain $\mathbb{N}_0[2/3]$

Throughout this section, we set

$$q := \frac{2}{3}, \qquad S := S_q = \mathbb{N}_0\left[\frac{2}{3}\right], \qquad \text{and} \qquad t := q^2 = \frac{4}{9}.$$

By Theorem 3.2, not every element of S is a sum of squares of elements of S. Our purpose here is to identify the elements of S that are sums of squares, to give an algorithm deciding whether a given element of S is a sum of squares, and to bound its square length.

5.1. Revelio: The Semiring of Sums of Squares.

Our first result identifies the semidomain ΣS^2. The crucial identity is $3q^2 = 2q$, which allows us to rewrite every cross term of the form $2q^{2k+1}$ as an integer multiple of a power of t.

Proposition 5.1. *The equality $\Sigma S^2 = \mathbb{N}_0[t]$ holds.*

Proof. Every element of $\mathbb{N}_0[t]$ is a finite sum of nonnegative integer multiples of the elements $t^j = (q^j)^2$, which implies that $\mathbb{N}_0[t] \subseteq \Sigma S^2$. For the reverse inclusion, it suffices to show that the square of every element of S belongs to $\mathbb{N}_0[t]$ because $\mathbb{N}_0[t]$ is closed under addition. Write an element of S as $x = \sum_{i=0}^{d} c_i q^i$, where $d \in \mathbb{N}_0$ and $c_0, \ldots, c_d \in \mathbb{N}_0$. Then

$$x^2 = \sum_{i=0}^{d} c_i^2 q^{2i} + \sum_{0 \le i < j \le d} 2c_i c_j q^{i+j}.$$

The terms in the first sum and the terms in the second sum with $i+j$ even belong to $\mathbb{N}_0[t]$. For the remaining terms, observe that the identity

$$2q^{2k+1} = q^{2k}(2q) = q^{2k}(3q^2) = 3t^{k+1}$$

holds for every $k \in \mathbb{N}_0$. Therefore $x^2 \in \mathbb{N}_0[t]$, and so $\Sigma S^2 \subseteq \mathbb{N}_0[t]$. □

Let us illustrate the proof of Proposition 5.1 with an example.

Example 5.2. For $x = 1 + q = 5/3$, the proof of Proposition 5.1 yields

$$x^2 = 1 + 2q + q^2 = 1 + 3t + t = 1 + 4t,$$

which is consistent with the fact that $25/9 = 1 + 4 \cdot (4/9)$. On the other hand, it follows from Proposition 5.1 that $2/3 \notin \mathbb{N}_0[t]$, since $2/3$ is not a sum of squares of elements of S (Example 3.7). ■

5.2. **The Sorting Hat: Deciding Whether an Element Is a Sum of Squares.** Every element of S can be written as $m/9^h$ for some $m, h \in \mathbb{N}_0$ because, by part (1) of Lemma 2.5, its denominator is a power of 3. The next proposition reduces the problem of deciding whether $m/9^h$ is a sum of squares to deciding whether m belongs to a finitely generated additive submonoid of $\mathbb{N}_0$.

Proposition 5.3. *For all $m, h \in \mathbb{N}_0$,*

$$\frac{m}{9^h} \in \Sigma S^2 \quad \Longleftrightarrow \quad m \in \langle 4^i 9^{h-i} : i \in [\![0, h]\!] \rangle.$$

Proof. For the reverse implication, if $m = \sum_{i=0}^{h} e_i 4^i 9^{h-i}$ for some $e_0, \dots, e_h \in \mathbb{N}_0$ then $m/9^h = \sum_{i=0}^{h} e_i t^i \in \mathbb{N}_0[t]$, and so $m/9^h \in \Sigma S^2$ by Proposition 5.1.

For the direct implication, suppose that $m/9^h \in \Sigma S^2$. By Proposition 5.1, there exists $F \in \mathbb{N}_0[X]$ such that $F(t) = m/9^h$. Since $t = 4/9$ and $\gcd(4, 9) = 1$, we can assume, by virtue of Lemma 4.3, that F is t-reduced, that is, all the coefficients of positive degree of F belong to $[\![0, 8]\!]$. We claim that $\deg F \leq h$. Suppose, by way of contradiction, that $d := \deg F > h$, and write $F = \sum_{i=0}^{d} e_i X^i$, where $e_d \in [\![1, 8]\!]$. After multiplying the equality $m/9^h = F(t)$ by 9^d, we obtain that

$$m9^{d-h} = \sum_{i=0}^{d} e_i 4^i 9^{d-i}.$$

Reducing this equality modulo 9, we see that $9 \mid e_d 4^d$. Since $\gcd(4, 9) = 1$, this implies that $9 \mid e_d$, which contradicts that $e_d \in [\![1, 8]\!]$. Hence $\deg F \leq h$, and so we can write $m/9^h = \sum_{i=0}^{h} e_i t^i$ for some $e_0, \dots, e_h \in \mathbb{N}_0$. Multiplying this equality by 9^h, we conclude that $m = \sum_{i=0}^{h} e_i 4^i 9^{h-i}$. □

The finite test in Proposition 5.3 can be carried out by the following algorithm, which uses exactly h iterations. Throughout the algorithm, the residue of an integer modulo 4 is always taken in $[\![0,3]\!]$, including the case when the integer is negative.

ALGORITHM. Input: $m, h \in \mathbb{N}_0$.

(1) Set $M_0 := m$.

(2) For each $i \in [\![0, h-1]\!]$, let $c_i \in [\![0,3]\!]$ be the residue of M_i modulo 4, and set

$$M_{i+1} := \frac{M_i - c_i 9^{h-i}}{4}.$$

(3) Set $c_h := M_h$. If $c_h \geq 0$ then output "sum of squares" together with the coefficients $c_0, \ldots, c_h$. Otherwise, output "not a sum of squares".

The next proposition shows that the algorithm is correct.

Proposition 5.4. *For all $m, h \in \mathbb{N}_0$, the coefficients $c_0, \ldots, c_h$ computed by the algorithm above are integers satisfying*

$$\frac{m}{9^h} = \sum_{i=0}^{h} c_i t^i \quad \text{and} \quad c_i \in [\![0,3]\!] \ \text{ for every } i \in [\![0, h-1]\!], \tag{5.1}$$

and they are the only integers satisfying (5.1). Moreover, $m/9^h \in \Sigma S^2$ if and only if $c_h \geq 0$.

Proof. Since $9^{h-i} \equiv 1 \pmod 4$, we see that $M_i - c_i 9^{h-i} \equiv M_i - c_i \equiv 0 \pmod 4$, and so $M_{i+1} \in \mathbb{Z}$ for every $i \in [\![0, h-1]\!]$. Iterating the defining recurrence, we obtain that

$$m = \sum_{i=0}^{h-1} c_i 4^i 9^{h-i} + 4^h c_h, \tag{5.2}$$

and dividing by 9^h yields the equality in (5.1). As for uniqueness, suppose that integers $c_0', \ldots, c_h'$ also satisfy (5.1). Then they also satisfy (5.2), and reducing (5.2) modulo 4, we see that $c_0' \equiv m \equiv c_0 \pmod 4$, which implies that $c_0' = c_0$ because $c_0, c_0' \in [\![0,3]\!]$. Subtracting $c_0 9^h$ from both sides of (5.2) and dividing by 4, we can repeat the same argument to show that $c_1' = c_1$, and so on, until we obtain that $c_h' = c_h$.

If $c_h \geq 0$ then all the coefficients $c_0, \ldots, c_h$ are nonnegative, and so $m/9^h \in \mathbb{N}_0[t] = \Sigma S^2$ by Proposition 5.1. Conversely, suppose that $m/9^h \in \Sigma S^2$. By Proposition 5.3, we can write $m = \sum_{i=0}^{h} e_i 4^i 9^{h-i}$, that is, $m/9^h = \sum_{i=0}^{h} e_i t^i$, for some $e_0, \ldots, e_h \in \mathbb{N}_0$. Since $4t^i = 9t^{i+1}$ for every $i \in \mathbb{N}_0$, we can apply the replacements $4X^i \mapsto 9X^{i+1}$, for i from 0 up to $h-1$, to make every coefficient of degree less than h smaller than 4. These replacements preserve the value at t and keep all the coefficients nonnegative. By uniqueness, the resulting coefficients are precisely $c_0, \ldots, c_h$, and so $c_h \geq 0$. □

We now run the algorithm on some elements of S.

s	(m, h)	$(c_0, \ldots, c_h)$	output
$1/9$	$(1, 1)$	$(1, -2)$	not a sum of squares
$2/3$	$(6, 1)$	$(2, -3)$	not a sum of squares
$2/3$	$(54, 2)$	$(2, 1, -9)$	not a sum of squares
$13/9$	$(13, 1)$	$(1, 1)$	sum of squares: $1 + t$
$25/9$	$(25, 1)$	$(1, 4)$	sum of squares: $1 + 4t$
$37/9$	$(37, 1)$	$(1, 7)$	sum of squares: $1 + 7t$
$17/3$	$(51, 1)$	$(3, 6)$	sum of squares: $3 + 6t$
$133/81$	$(133, 2)$	$(1, 1, 1)$	sum of squares: $1 + t + t^2$
$1261/729$	$(1261, 3)$	$(1, 1, 1, 1)$	sum of squares: $1 + t + t^2 + t^3$

TABLE 1. The output of the algorithm on some elements of $S = \mathbb{N}_0[2/3]$.

Example 5.5. In Table 1, we show the output of the algorithm for some elements of S. For instance, for $s = 2/3 = 6/9$, the algorithm yields $c_0 = 2$ and $c_1 = (6 - 2 \cdot 9)/4 = -3$, and so $2/3$ is not a sum of squares of elements of S, as we already know. For $s = 37/9$, the algorithm yields $c_0 = 1$ and $c_1 = (37 - 9)/4 = 7$, and so $37/9 = 1 + 7t$ is a sum of squares of elements of S. Observe that the element $2/3$ appears twice in the table, as the algorithm does not require the fraction $m/9^h$ to be written in lowest terms. ■

5.3. **The Unbreakable Vow: A Bound for the Square Length.** The coefficients computed by the algorithm also yield an explicit bound for the square length of each sum of squares, which only depends on the power of 9 in its denominator.

Corollary 5.6. *Let $s = m/9^h \in \Sigma S^2$ for some $m, h \in \mathbb{N}_0$, and let $c_0, \ldots, c_h$ be the coefficients computed by the algorithm. Then*

$$\ell_S(s) \leq \sum_{i=0}^{h-1} c_i + \ell_{\mathbb{N}_0}(c_h) \leq 3h + 4.$$

Proof. By Proposition 5.4, the coefficients $c_0, \ldots, c_h$ are nonnegative. For each $i \in [\![0, h-1]\!]$, the term $c_i t^i$ is the sum of c_i copies of $(q^i)^2$. On the other hand, by Lagrange's theorem, we can write $c_h = v_1^2 + \cdots + v_r^2$, where $v_1, \ldots, v_r \in \mathbb{N}_0$ and $r = \ell_{\mathbb{N}_0}(c_h) \leq 4$. Therefore

$$s = \sum_{i=0}^{h-1} \sum_{j=1}^{c_i} (q^i)^2 + \sum_{j=1}^{r} (v_j q^h)^2,$$

and all the square roots belong to S, which proves the first inequality. The second inequality follows from the fact that $c_i \leq 3$ for every $i \in [\![0, h-1]\!]$. □

Let us compare the bound in Corollary 5.6 with the actual square lengths of two elements of S.

Example 5.7. For $s = 133/81$, Corollary 5.6 yields $\ell_S(s) \leq 3$, and we know from Example 4.7 that $\ell_S(s) = 3$. Thus, the bound in Corollary 5.6 can be attained. On the other hand, for $s = 37/9$, Corollary 5.6 yields $\ell_S(s) \leq 1 + \ell_{\mathbb{N}_0}(7) = 5$, but we will see in Example 5.10 that $\ell_S(37/9) = 3$. ■

Observe that Corollary 5.6 is compatible with the fact that $p(2/3) = \infty$, as the exponent h is not bounded as s ranges over ΣS^2.

5.4. **The Weasley Twins: Sums of One or Two Squares.** The bound in Corollary 5.6 is not always attained, and so it is natural to ask whether the square length of each element of ΣS^2 can be computed. For sums of one or two squares, the answer is yes (and, as Fred and George would say, two are always better than one [22]). We first need a finite test to decide whether a fraction whose denominator is a power of 3 belongs to S.

Lemma 5.8. *For all $n, h \in \mathbb{N}_0$,*

$$\frac{n}{3^h} \in S \quad \iff \quad n \in \langle 2^i 3^{h-i} : i \in [\![0, h]\!] \rangle.$$

Proof. For the reverse implication, if $n = \sum_{i=0}^{h} e_i 2^i 3^{h-i}$ for some $e_0, \dots, e_h \in \mathbb{N}_0$ then $n/3^h = \sum_{i=0}^{h} e_i q^i \in S$. For the direct implication, suppose that $n/3^h = f(q)$ for some $f \in \mathbb{N}_0[X]$. By Lemma 4.3, we can assume that f is q-reduced, that is, all the coefficients of positive degree of f belong to $\{0, 1, 2\}$. Now we can mimic the proof of Proposition 5.3: if $d := \deg f > h$ then after clearing denominators and reducing modulo 3, we obtain that $3 \mid e_d 2^d$, where $e_d \in \{1, 2\}$ is the leading coefficient of f, which is not possible. Hence $\deg f \leq h$, and multiplying the equality $n/3^h = f(q)$ by 3^h yields the desired membership. □

With Lemma 5.8 in hand, we can decide whether an element of S is a sum of at most two squares.

Proposition 5.9. *Let $m, h \in \mathbb{N}_0$, and suppose that $m/9^h = x^2 + y^2$ for some $x, y \in S$. Then $3^h x$ and $3^h y$ are nonnegative integers. As a consequence, one can decide whether $\ell_S(m/9^h) \leq 2$ by enumerating all the pairs $(n_1, n_2) \in \mathbb{N}_0^2$ such that $n_1^2 + n_2^2 = m$ and checking whether $n_1/3^h, n_2/3^h \in S$ using Lemma 5.8. The same finite search, with $n_2 = 0$, decides whether $\ell_S(m/9^h) \leq 1$.*

Proof. By part (1) of Lemma 2.5, we can write $x = n_1/3^D$ and $y = n_2/3^D$ for some $n_1, n_2 \in \mathbb{N}_0$ and $D \in \mathbb{N}_0$ with $D \geq h$. Then $n_1^2 + n_2^2 = m9^{D-h}$. Suppose that $D > h$. Then $n_1^2 + n_2^2 \equiv 0 \pmod 3$. Since every square is congruent to either 0 or 1 modulo 3, both n_1 and n_2 must be divisible by 3, and so we can write $x = (n_1/3)/3^{D-1}$ and $y = (n_2/3)/3^{D-1}$. Repeating this argument as many times as necessary, we can assume that $D = h$, which means that $3^h x$ and $3^h y$ are nonnegative integers. Hence the pair $(n_1, n_2) = (3^h x, 3^h y)$ satisfies $n_1^2 + n_2^2 = m$, and there are only finitely many such pairs. The rest of the statement follows from Lemma 5.8. □

We now use Proposition 5.9 to compute the square length of 37/9.

Example 5.10. We claim that $\ell_S(37/9) = 3$. First, observe that

$$\frac{37}{9} = \left(\frac{8}{9}\right)^2 + \left(\frac{10}{9}\right)^2 + \left(\frac{13}{9}\right)^2,$$

where $8/9 = 2q^2$, $10/9 = q + q^2$, and $13/9 = 1 + q^2$ all belong to S. Therefore $\ell_S(37/9) \leq 3$. On the other hand, the only pairs $(n_1, n_2) \in \mathbb{N}_0^2$ with $n_1^2 + n_2^2 = 37$ are $(1, 6)$ and $(6, 1)$. Since $1/3 \notin S$ (Example 2.6), it follows from Proposition 5.9 that 37/9 is not a sum of at most two squares of elements of S. Hence $\ell_S(37/9) = 3$.

Observe that if we require all the square roots to be fractions with denominator 3 then 37/9 needs four squares. Indeed, by Lemma 5.8, the numerators of such square roots are at most 6 and belong to $\langle 2, 3\rangle$, so they belong to $\{0, 2, 3, 4, 5, 6\}$. In a representation of 37 as a sum of three squares of such numerators, reducing modulo 3 forces exactly one of the numerators to be nondivisible by 3, and then its square must be congruent to $37 \equiv 1$ modulo 9. However, the available numerators 2, 4, and 5 have squares congruent to 4, 7, and 7 modulo 9, respectively. On the other hand, $37/9 = (5/3)^2 + 3(2/3)^2$, where $5/3 = 1 + q \in S$. Thus, a shortest representation may require larger denominators than the ones appearing in the element we aim to represent. ■

The phenomenon illustrated in the second part of Example 5.10 can be much more pronounced, as the following example shows.

Example 5.11. We claim that $\ell_S(17/3) = 3$. Since $17/3 = 51/9$ and $51 \equiv 0 \pmod 3$, every pair $(n_1, n_2) \in \mathbb{N}_0^2$ satisfying $n_1^2 + n_2^2 = 51$ would consist of multiples of 3, which would imply that $9 \mid_{\mathbb{N}_0} 51$. Thus, it follows from Proposition 5.9 that $\ell_S(17/3) \geq 3$. On the other hand, the equality $66985^2 + 69865^2 + 101933^2 = 17 \cdot 3^{19}$ shows that

$$\frac{17}{3} = \left(\frac{66985}{3^{10}}\right)^2 + \left(\frac{69865}{3^{10}}\right)^2 + \left(\frac{101933}{3^{10}}\right)^2,$$

where the three square roots belong to S because

$$\begin{aligned}
\frac{66985}{3^{10}} &= 1 + 2q^7 + q^{10},\\
\frac{69865}{3^{10}} &= 1 + q^6 + 2q^8 + q^{10},\\
\frac{101933}{3^{10}} &= 1 + q^2 + 2q^7 + 2q^8 + 2q^9 + 2q^{10}.
\end{aligned}$$

Hence $\ell_S(17/3) = 3$. Furthermore, an exhaustive computer search (using Lemma 5.8 to enumerate all the candidate square roots) shows that 17/3 cannot be written as a sum of three squares of elements of S whose denominators divide 3^9. Thus, every shortest representation of 17/3 requires a square root whose denominator is at least 3^{10}, even though 17/3 only has denominator 3. ■

5.5. **Divination: Open Questions.** Even without the help of [19], it is not hard to foresee which questions arise next. For the semidomain $S = \mathbb{N}_0[2/3]$, we have seen how to decide whether an element is a sum of squares (Proposition 5.4), how to bound its square length (Corollary 5.6), and how to decide whether its square length is at most 2 (Proposition 5.9). However, we have not been able to compute the square length of every element of ΣS^2. Given $s \in \Sigma S^2$ and a bound $D \in \mathbb{N}_0$, Lemma 5.8 allows us to find, by a finite search, the shortest representations of s whose square roots have denominators dividing 3^D. Therefore it would suffice to bound, in terms of s, the denominators of the square roots in some shortest representation of s. Proposition 5.9 provides such a bound for one or two squares, while Example 5.11 shows that, already for three squares, the required denominators can be much larger than the denominator of s. Hence the following question remains open for us.

Question 5.12. Is there an algorithm that, for each $s \in \Sigma S^2$, computes the square length $\ell_S(s)$ in the semidomain $S = \mathbb{N}_0[2/3]$?

Our proof of Proposition 5.1 relies on the identity $3q^2 = 2q$, which is very special to the generator $q = 2/3$. Thus, the following more general question remains open for us as well.

Question 5.13. Let $q \in \mathbb{Q}_{>0}$ such that $\mathsf{n}(q) > 1$ and $\mathsf{d}(q) > 1$. Is there an algorithm that, for each $f \in \mathbb{N}_0[X]$, decides whether $f(q) \in \Sigma S_q^2$? What can be said about the semidomain $\mathbb{N}_0[\alpha]$, where α is a positive algebraic number (see [7])?

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References

[1] S. Albizu-Campos, J. Bringas, and H. Polo, *On the atomic structure of exponential Puiseux monoids and semirings*, Comm. Algebra **49** (2021) 850–863.
[2] B. Babbling, *Ancient Runes Made Easy*, Flourish and Blotts, Diagon Alley, London, 1979.
[3] B. Bagshot, *A History of Magic*, Little Red Books, Godric's Hollow, 1947.
[4] B. C. Berndt and P. Moree, *Sums of two squares and the tau-function: Ramanujan's trail*, preprint (2024). Available on arXiv: https://arxiv.org/abs/2409.03428

[5] S. T. Chapman, F. Gotti, and M. Gotti, *Factorization invariants of Puiseux monoids generated by geometric sequences*, Comm. Algebra **48** (2020) 380–396.
[6] M. D. Choi, Z. D. Dai, T. Y. Lam, and B. Reznick, *The Pythagoras number of some affine algebras and local algebras*, J. Reine Angew. Math. **336** (1982) 45–82.
[7] J. Correa-Morris and F. Gotti, *On the additive structure of algebraic valuations of polynomial semirings*, J. Pure Appl. Algebra **226** (2022) 107104.
[8] A. Dumbledore, *A Transfiguration Primer: From Matchsticks to Every Flavour Beans*, Obscurus Books, London, 1962.
[9] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
[10] M. Goshawk, *The Standard Book of Spells (Grade 1)*, Obscurus Books, London, 1965.
[11] F. Gotti and H. Polo, *On the arithmetic of polynomial semidomains*, Forum Math. **35** (2023) 1179–1197.
[12] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th edition (revised by D. R. Heath-Brown and J. H. Silverman), Oxford University Press, Oxford, 2008.
[13] X. Lovegood, *Two thirds is a sum of three squares!*, The Quibbler **12** (1995) 4–5.
[14] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics Vol. 164, Springer-Verlag, New York, 1996.
[15] G. Ollivander, *Measuring Wands: Length, Flexibility, and Core*, Ollivanders: Makers of Fine Wands since 382 B.C., Diagon Alley, London, 1980.
[16] J. K. Rowling, *Harry Potter and the Sorcerer's Stone*, Arthur A. Levine Books, New York, 1998.
[17] J. K. Rowling, *Harry Potter and the Half-Blood Prince*, Arthur A. Levine Books, New York, 2005.
[18] N. Scamander, *Fantastic Beasts and Where to Find Them*, Obscurus Books, London, 1927.
[19] C. Vablatsky, *Unfogging the Future*, Flourish and Blotts, Diagon Alley, London, 1951.
[20] S. Vector, *On the arithmancy of sums of squares*, J. Hogwarts Arithmancy **7** (1997) 1–17.
[21] A. Waffling, *Magical Theory*, Obscurus Books, London, 1952.
[22] F. Weasley and G. Weasley, *Weasleys' Wizard Wheezes: Product Catalogue*, 93 Diagon Alley, London, 1996.

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