A Transfiguration Theorem: Trading Boxes of Every Flavour Beans

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The Question: Exactly n Beans, Please!

After Prof. Lupin drove a dementor out of our compartment on the Hogwarts Express with *Expecto Patronum*, Ron wanted beans, not chocolate. The trolley sells Bertie Bott's Every Flavour Beans only in boxes of 6, 9, and 20 beans.



1. Which numbers of beans can Ron buy exactly?
2. In how many ways can he buy them?
3. How are the different ways related to each other?

Notation and First Definitions

We write $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

The Bertie Bott monoid is the set of numbers of beans Ron can buy:

$$\mathcal{B} := \{\, 6a + 9b + 20c : a, b, c \in \mathbb{N}_0 \,\}.$$

Example

The number $44 = 6 + 2 \cdot 9 + 20$ belongs to $\mathcal{B}$, but 43 does not.

A factorization of $n \in \mathcal{B}$ is a triple $(a, b, c) \in \mathbb{N}_0^3$ with $6a + 9b + 20c = n$, that is, a way to buy exactly n beans. Its length is $a + b + c$, the number of boxes.

Example

The factorizations of 44 are $(1, 2, 1)$ and $(4, 0, 1)$, of lengths 4 and 5.

Accio Beans! Which Numbers Can Ron Buy?

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	**43**	44	45	46	47	48	49

Shaded: Ron can buy it. White: he cannot.

Since $44, 45, \ldots, 49$ are in $\mathcal{B}$, adding boxes of 6 gives every number from 44 on.

The Frobenius number of $\mathcal{B}$ is the largest number not in $\mathcal{B}$.

Example

The Frobenius number of $\mathcal{B}$ is 43.

With only two box sizes p and q, where $\gcd(p, q) = 1$, the Frobenius number is $pq - p - q$ (Sylvester, 1884). For boxes of 9 and 20, it is 151.

Many Ways to Buy the Same Beans

The length set of $n \in \mathcal{B}$ is the set $L(n)$ of lengths of all factorizations of n.

n	factorizations (a, b, c)	$L(n)$
18	$(3,0,0),\ (0,2,0)$	$\{2,3\}$
36	$(6,0,0),\ (3,2,0),\ (0,4,0)$	$\{4,5,6\}$
44	$(4,0,1),\ (1,2,1)$	$\{4,5\}$
60	$(10,0,0),\ (7,2,0),\ (4,4,0),\ (1,6,0),\ (0,0,3)$	$\{3,7,8,9,10\}$

Example

The length set of 60 is $L(60) = \{3,7,8,9,10\}$: Hermione buys 60 beans in 3 boxes, while Ron, who likes opening boxes, buys them in 10.

Our Main Result: Two Spells Are Enough

Consider the following two transfiguration spells:

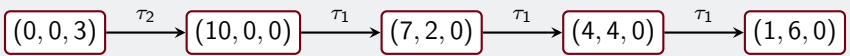
τ_1: three boxes of 6 $\longleftrightarrow$ two boxes of 9 (both hold 18 beans),

τ_2: three boxes of 20 $\longleftrightarrow$ ten boxes of 6 (both hold 60 beans).

Theorem (Granger, Potter, and Weasley, 2026)

Any two ways of buying the same number of beans can be transfigured into each other using only τ_1 and τ_2.

Example



All five factorizations of 60 are connected by our two spells.

Why Two Spells Are Enough: The Idea of the Proof

Recall

A factorization (a, b, c) of n is a way to buy exactly n beans, that is, $6a + 9b + 20c = n$.

Take two factorizations (a, b, c) and (a', b', c') of the same n.

1. Since $6a + 9b + 20c \equiv 2c \pmod 3$, we get $c \equiv c' \pmod 3$. Use τ_2 on the one with more boxes of 20 until $c = c'$.
2. Now $6a + 9b = 6a' + 9b'$, and reducing modulo 6 gives $b \equiv b' \pmod 2$. Use τ_1 until $b = b'$.
3. Then $6a = 6a'$, so $a = a'$: the two factorizations are the same.

Neither Spell Can Be Dropped

- The only ways to buy 18 beans are $(3, 0, 0)$ and $(0, 2, 0)$, and only τ_1 connects them.
- The way $(0, 0, 3)$ of buying 60 beans shares no box with the other four ways, and only τ_2 reaches it.

A set of spells is a minimal presentation of $\mathcal{B}$ if it connects any two factorizations of the same number and none of its spells can be dropped (Rosales and García-Sánchez, 2009).

Example

By our main result and the two observations above, $\{\tau_1, \tau_2\}$ is a minimal presentation of $\mathcal{B}$.

Back to Ron's Three Questions

1. **Which numbers of beans can Ron buy exactly?**
   All of them except 22 numbers, and the largest one he cannot buy is 43.
2. **In how many ways can he buy them?**
   Some in only one way, like 20; others in many ways, like 60, which he can buy in five ways, using from 3 to 10 boxes.
3. **How are the different ways related to each other?**
   Our two spells, τ_1 and τ_2, connect any two of them.

Hermione's Challenge for You

If the trolley also sold boxes of 7 beans, what would be the largest number of beans that Ron could not buy?

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Thank you for your attention!
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